Chapter 6

Triangles and its Properties

Introduction to Triangles and Its Properties

Introduction

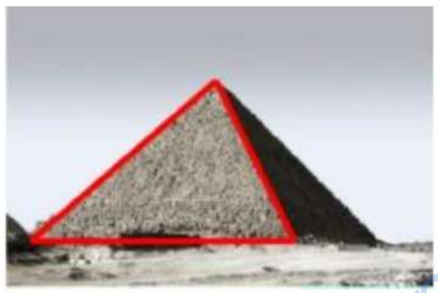
Do you know what triangles are, and where we see them?
A triangle is a unique figure; we see triangles all around us in different objects. In our daily life, we see different types of triangles are formed between the edges of different surfaces. Now, look at the following pictures. What do you see in them?

1. If we observe the sailing boat we see the sailing boat is in the shape of a triangle.

2. The traffic signal is in the shape of a triangle.

3. The slice of pizza is in the shape of a triangle.

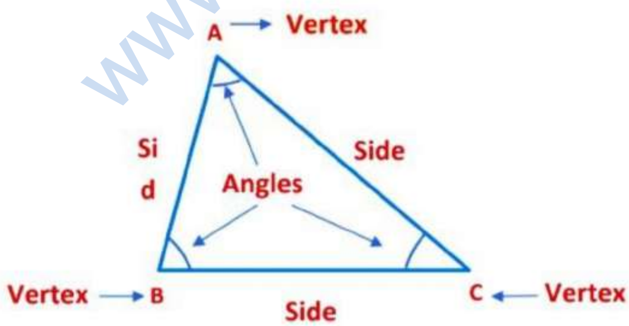
4. Pyramids are also in the shape of a triangle



So, in small-small things around us we see triangles. Triangle is a very common geometrical shape we use in day to day life.

What exactly is a triangle? How do the triangles look like?

A triangle is a simple closed curve bounded by three sides. A triangle has three sides, three angles, and three vertices. These three sides, three angles, and three vertices define a triangle.



Here,
In triangle ABC,
(i) AB, BC, and AC are the three sides of the triangle
(ii) ∠ABC, ∠BCA, ∠CAB are the three angles of the triangle and
(iii) A, B and C, are three vertices of the triangle

Also, consider some other things like

(1) Opposite side of a particular angle.

(i) Opposite side of ∠ABC AC
(ii) Opposite side of ∠BCA AB
(iii) Opposite side of ∠CAB BC

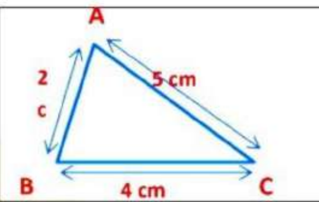
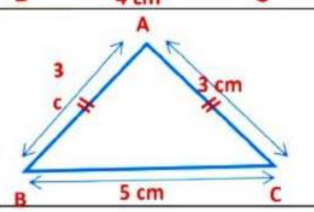
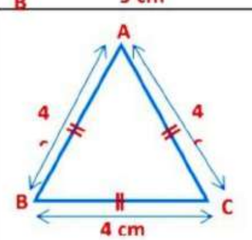
(2) Angle opposite to a particular side.

(i) Angle opposite to side AC ∠ABC
(ii) Angle opposite to side AB ∠BCA
(iii) Angle opposite side of BC ∠CAB

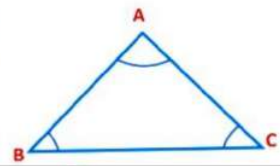
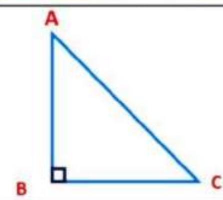
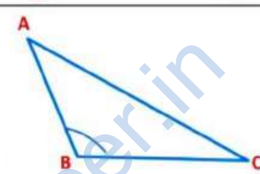
Classification of Triangles

Triangles can be classified on the basis of:
- Sides
- Angles

Classification of Triangles on the Basis of Their Sides

(i) Scalene triangle: In a scalene triangle, all three sides are unequal. The adjacent figure shows a scalene triangle where AB ≠ BC ≠ AC	
(ii) Isosceles triangle: An isosceles triangle is a triangle whose any two sides are equal. Adjacent figure shows an isosceles triangle where AB = AC	
(iii) Equilateral triangle: An equilateral triangle is a triangle whose all three sides are equal. The adjacent figure shows an equilateral triangle where AB = BC = AC	

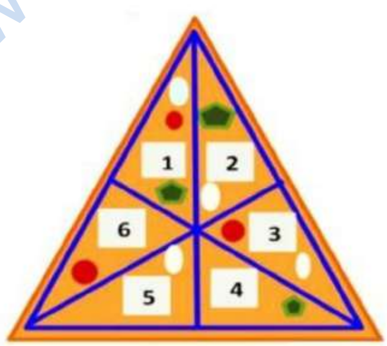
Classification of Triangles on the Basis of Their Angles

(i) Acute-angled triangle: If all the three angles of a triangle are less than 90^0 it is called an acute-angled triangle. In the adjacent figure, $\angle$ABC, $\angle$BCA, $\angle$CAB all are acute-angled triangle.	
(ii) Right-angled triangle: If the measure of one of the angles of a triangle is 90^0, it is called a right-angled triangle. In the adjacent figure, ΔABC= 90^0. Therefore, ΔABC is a right-angled triangle.	
(iii) Obtuse-angled triangle: If the measure of one of the angles of a triangle is greater than90^0, it is called an obtuse-angled triangle. In the adjacent figure, ΔABC$>$ 90^0 Therefore, ΔABC is an obtuse-angled triangle.	

Some Important Terms

Medians of a Triangle

There are 6 members in Kartik's family and he has only one triangular piece of pizza as shown in the figure below. Now, Kartik wants to divide this piece of pizza equally among all the family members. So, how will Kartik divide this piece of pizza?

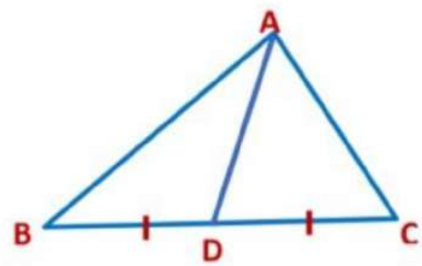


So, Kartik will divide the piece of pizza into six equal pieces by cutting along the lines as shown in the above figure. What are these lines called? These lines are nothing but the medians of a triangle.

So, what exactly is the median?

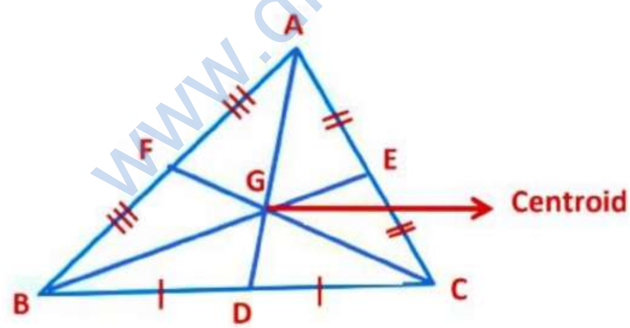
A line segment drawn from any vertex of the triangle to the mid-point of the opposite sides of a triangle is called a median.

Example:



In triangle ABC, segment AD is the median of the triangle because one of its endpoints A is the vertex of the triangle and his other endpoint D is the midpoint of its opposite side BC.

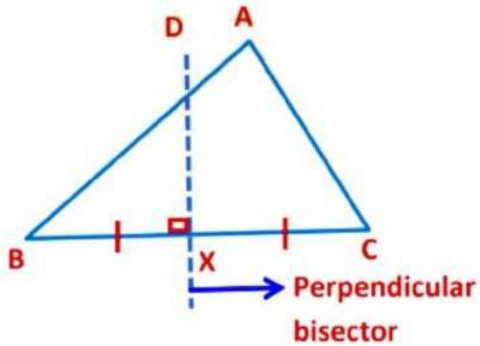
We know a triangle has three vertices. So, we can draw three medians to each triangle from each vertex as shown in the following figure and these all medians intersect is each other at a point is called the centroid of the triangle. These three medians of the triangle divide the triangle into six equal triangles.



Perpendicular bisector:

A perpendicular bisector is a segment, line, or ray that is perpendicular to a segment at its midpoint.

Example:

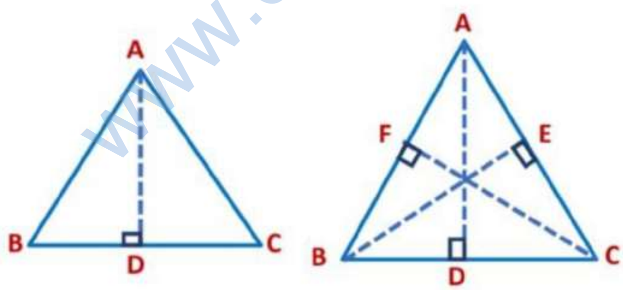


In triangle ABC, DX is the perpendicular bisector of segment BC because it is perpendicular to segment BC at its midpoint X.

Altitudes of a Triangle

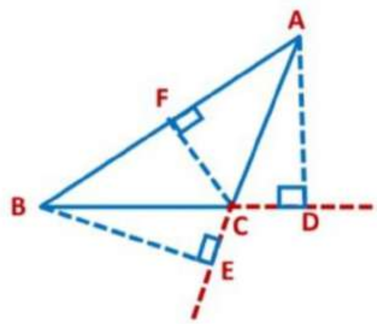
Altitude of a triangle is a perpendicular line segment that joins a vertex of the triangle to the opposite side.

Example: In triangle ABC, segment AD is an altitude because one of its endpoint A is the vertex of the triangle and its other endpoint D is on the opposite side of
the triangle. Such segment AD is the perpendicular to opposite side BC. We know there is a total of three vertexes in each triangle. So, we are able to draw a maximum of three altitudes in each triangle as shown in the adjacent figure.



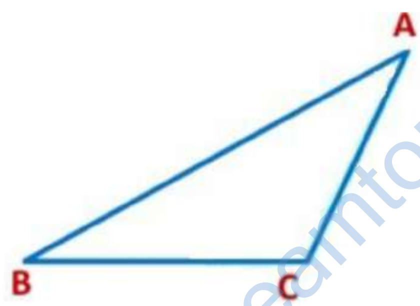
An altitude of a triangle can be inside or may lie outside the triangle.

Example: Suppose the triangle ABC, in which AD, BE, and CF are
the perpendicular drawn from the vertex A, B, and C respectively. These all perpendicular are nothing but the altitudes of the triangle.



Also, in the figure we see that altitude AD, BE lying outside the triangle and altitude CF will lie inside of the triangle.

Example: Draw the altitude and median in the given triangle from vertex A.

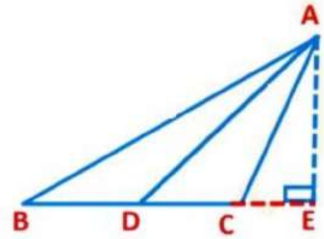


We know a line segment drawn from any vertex of the triangle to the mid-point of the opposite side of a triangle is called a median.

In the given triangle ABC to draw a median first we find the mid-point of segment BC then join it with vertex A

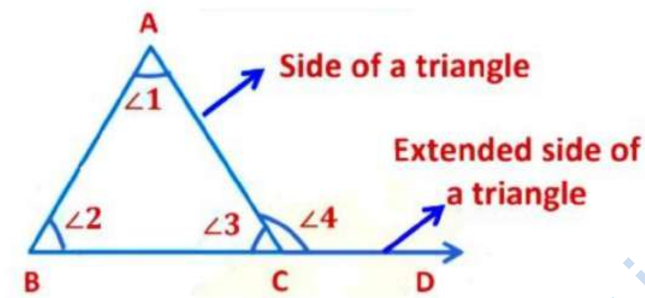
Also, we know the altitude of a triangle is a perpendicular line segment that joins a vertex of the triangle to the opposite side.

Hence, to draw altitude first we extend the side BC then we draw the perpendicular from vertex A.



Exterior Angle of Triangle And Its Properties

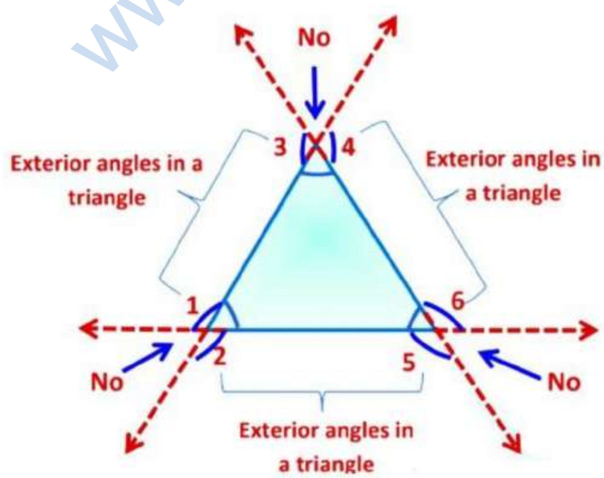
In the earlier chapter, we see angles that are present inside the triangle. Now, we talk about angles that are present outside of the triangle, and what are they called?



In the above ΔABC we see,∠1,∠2 and ∠3 are present inside the triangle so we called it as interior angles of a triangle. Since the triangle has interior angles it would have exterior angles. So, if we extend the BC to D observe ∠4 is formed at point C. This angle lies in the exterior of ΔABC. We call it the exterior angle of the ΔABC.

Also, we see the angle is formed between one side of a triangle and extended adjacent side so we say that

An exterior angle is an angle between one side of a triangle and an extended adjacent side.



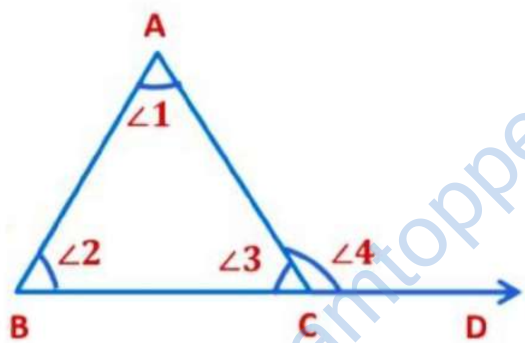
We know there are three sides of the triangle and when we extend these sides six exterior angles are formed.

Exterior Angle Theorem

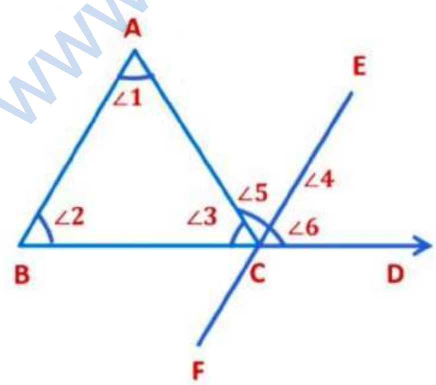
An exterior angle of a triangle is equal to the sum of its interior opposite angles.

To prove: $\angle 1 + \angle 2 = \angle 4$

In the figure we see, $\angle 3$ and $\angle 4$ are adjacent angles. $\angle 4$ is exterior angle and $\angle 1$ and $\angle 2$ are Interior opposite angles of $\angle 4$



Draw a line EF parallel to AB which is passing through C. Line EF divide $\angle 4$ into two angles $\angle ECA$ marked as $\angle 5$ and $\angle ECD$ marked as $\angle 6$



$\angle 5 + \angle 6 = \angle 4$ ... (i)

$\angle 1 = \angle 5$ AB||EF and AC is a transversal line

(Alternate interior angles are equal)... (ii)

$\angle 2 = \angle 6$ AB||EF and BD is a transversal line

(Corresponding angles are equal)... (iii)

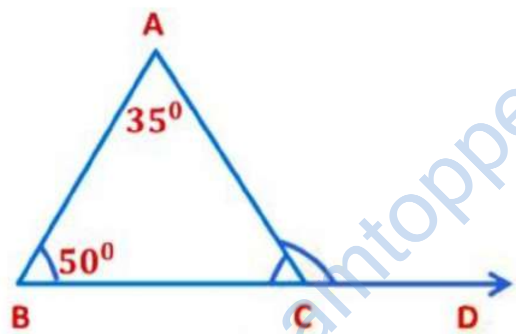
By adding (ii) and (iii) we get

$$\angle 1 + \angle 2 = \angle 5 + \angle 6$$

From equation (i)

$$\angle 1 + \angle 2 = \angle 4$$

Example: In the given figure, find $\angle ACD$



For $\Delta ABC, \angle ACD$, is an exterior angle. $\angle ABC$, $\angle BAC$areinterior opposite angles.

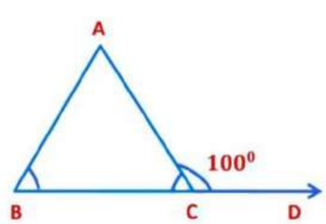
We know, an exterior angle of a triangle is equal to the sum of its interior opposite angles.

$$\therefore \angle ACD = \angle ABC + \angle BAC$$

$$\therefore \angle ACD = 50° + 35°$$

$$\therefore \angle ACD = 85°$$

Example: Exterior angle $\angle ACD = 100°$ and $\angle ABC$ and $\angle BAC$ are in the ratio of 3: 2. Find $\angle ABC$ and $\angle BAC$ of the triangle.



Let, $\angle ABC = 3x$ and $\angle BAC = 2x$

We know, an exterior angle of a triangle is equal to the sum of its interior opposite angles.

$\therefore \angle ACD = \angle ABC + \angle BAC$

$\therefore 100° = 3x + 2x$

$\therefore 100° = 5x$

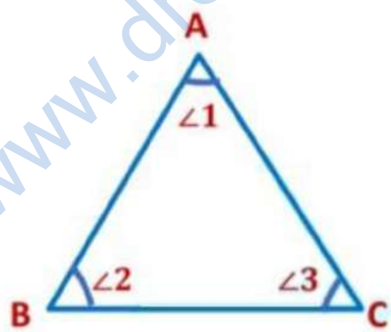
$\therefore x = \frac{100}{5}°$

$\therefore x = 20°$

$\rightarrow \angle ABC = 3x = 3 \times 20° = 60°$
$\rightarrow \angle BAC = 2x = 2 \times 20° = 40°$

Equilateral And Isosceles Triangle

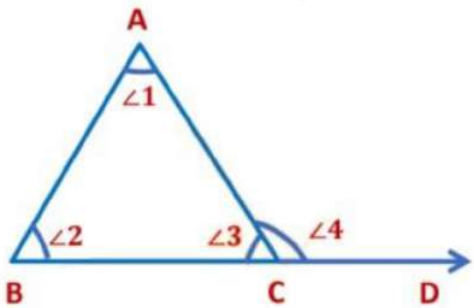
The sum of the interior angles of a triangle is 180°. This property of a triangle is true for all triangles.



In ΔABC, $\angle 1$, $\angle 2$, and $\angle 3$ are the interior angles of a triangle.

To prove: $\angle 1 + \angle 2 + \angle 3 = 180°$

Construction: Extend side BC to D we get exterior $\angle 4$



$\angle 1, \angle 2$ and $\angle 3$are angles of ΔABC. $\angle 4$is the exterior angle when BC is extended to D.

$\angle 1 + \angle 2 = \angle 4$ ... (By exterior angle property)

Adding $\angle 3$ to both the side

$$\angle 1 + \angle 2 + \angle 3 = \angle 4 + \angle 3$$

But, $\angle 4$ and $\angle 3$ form a linear pair

So,

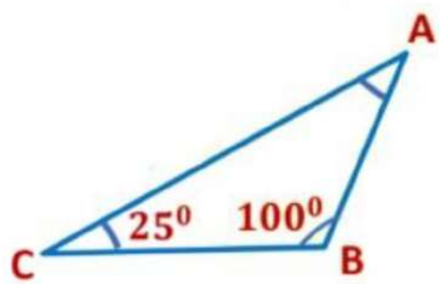
$\angle 4 + \angle 3 = 180^{o}$...(Linear pair)

Therefore,

$$\angle 1 + \angle 2 + \angle 3 = 180°$$

Hence, proved

Example: In ΔABC, find the measure of $\angle C$ if $\angle A = 25°$ and $\angle B = 65°$



We know that the sum of angles of a triangle is 180°

$$\therefore \angle A + \angle B + \angle C = 180°$$

$$\rightarrow 25° + 100° + \angle C = 180°$$

$$\rightarrow 125° + \angle C = 180°$$

$$\rightarrow 125° + \angle C - 125° = 180° - 125°$$

$$\rightarrow \angle C = 55°$$

Hence, the measure of $\angle C$ is 55°

.

Example: Find the measure of a triangle which are in the ratio of 2:1:1

Let the common ratio be x°

The measure of given angles of the triangle be 2x°, x°, and x° respectively.

We know that the sum of angles of a triangle is 180°

$$2x° + x° + x° = 180°$$

$$4x° = 180°$$

$$x° = \frac{180}{4}°$$

$$x° = 45°$$

So the measure of the angles ((2 × 45°), 45°, and 45°) i.e.,90°45°, and 45°

Example: If the angles of a triangle have measures $(3A + 40°)$,$(A + 20°)$, 2A.
Find the value of A

LetΔABC,

$\angle BAC = (3A + 40°)$, $\angle ACB = (A + 20°)$, and$\angle ABC = 2A$

We know that the sum of angles of a triangle is 180°

$$\therefore \angle BAC + \angle ACB + \angle ABC = 180°$$

$$\rightarrow (3A + 40°) + (A + 20°) + 2A = 180°$$

$$\rightarrow 3A + 40° + A + 20° + 2A = 180°$$

$$\rightarrow 3A + A + 2A + 40° + 20° = 180°$$

$$\rightarrow 6A + 60° = 180°$$

$$\rightarrow 6A + 60° - 60° = 180° - 60°$$

$$\rightarrow 6A = 120°$$

$$\rightarrow A = \frac{120}{6}°$$

$$\rightarrow A = 20°$$

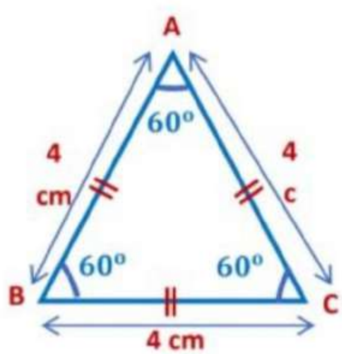
Hence, the value of A is 20°

Equilateral And Isosceles Triangle

Equilateral Triangle:

A triangle in which all three sides are of equal length and three angles are equal. Each angle is 60°

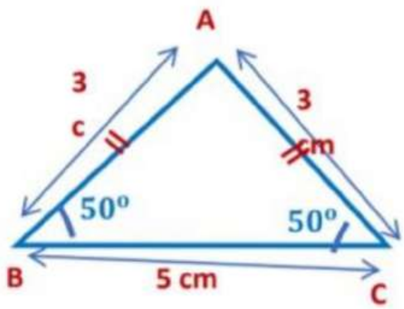
ΔABC is an equilateral triangle in which we see all the three sides of the triangle are of equal length and the measure of all the angles of the triangle is also equal.



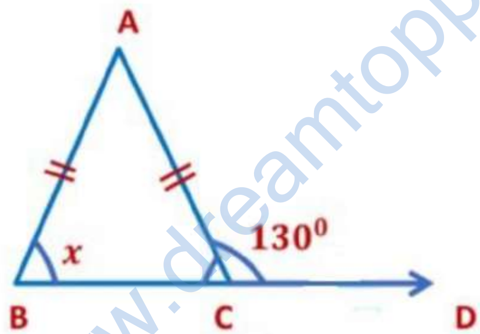
Isosceles Triangle:

A triangle in which two sides are of equal length and angles opposite to equal sides are equal

ΔABC is an isosceles triangle in which we see two sides of the triangle are of equal length and the angles opposite to equal sides are equal.



Find angle x in the following figure



ΔABCis an isosceles triangle.

So, AB = AC

$$\angle ABC = \angle ACB$$

$$x = \angle ACB$$

$\angle ACB + \angle ACD = 180°$...(Linear pair)

$$x + 130° = 180°$$

$$x + 130° - 130° = 180° - 130°$$

$$x = 150°$$

Find angle x in the following figure

ΔABC is an isosceles triangle.

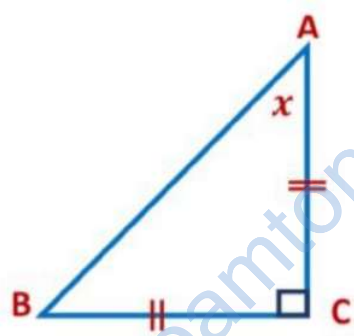
So, AB = BC

$$\angle ABC = \angle CAB$$

$$\angle CAB = x$$

Hence,

$$\angle ABC = \angle CAB = x$$



$\angle ABC + \angle BCA + \angle CAB = 180°$...(angle sum property)

$$x + 90° + x = 180°$$

$$2x + 90° = 180°$$

$$2x + 90° - 90° = 180° - 90°$$

$$2x = 90°$$

$$x = \frac{90}{2}°$$

$$x = 45°$$

Triangle Inequality

SUM OF THE LENGTHS OF TWO SIDES OF A TRIANGLE

OR

TRIANGLE INEQUALITY

In a school, Math's teacher tells students to draw a triangle. After some time teacher asked the students, have you drawn the triangle? Students: yes. So, firstly teacher asked Ben, how have you drawn the triangle? And what is the length of sides of the triangle you took to draw the triangle?

Ben said I have drawn a triangle in which I took the length of the sides of the triangle is 5 cm, 7 cm, and 12 cm.

So, the teacher said to students we just check we can draw a triangle from this length of the sides of the triangle or not? To check this we use triangle inequality theorem. Students asked

What is the triangle inequality theorem?

The teacher said Triangle inequality theorem is defined as

"The sum of any two sides of the triangle is greater than the length of the third side."

So, let's find out. We have 5 cm, 7 cm, and 12 cm three sides of the triangle. So,

Sum of the two sides		Third side
5 + 7	=	12

Here, we see the sum of the two sides is equal to third sides but to draw a triangle the sum of two sides has to be greater than the third side. It means the triangle with sides 5 cm, 7 cm, and 12 cm cannot be formed.

Now, the teacher asked to Nobi, how has she drawn the triangle?

She told that to draw a triangle she took the length of the sides of the triangle is 3 cm, 4 cm, and 5 cm. The teacher said let's check these sides are correct or not?

So, according to the theorem, we know the sum of any two sides of the triangle should be greater than the third side. Hence,

Sum of the two sides		Third side
3 + 4	>	5

Here, we see the sum of the two sides is greater than the third side. Can it tell us Nobita can draw a triangle from it? No, because this is only one case.

Now, we see the second combination,

Sum of the two sides		Third side
4 + 5	>	3

Now, we see the third combinations

Sum of the two sides		Third side
3 + 5	>	4

Here, all three cases can satisfy the condition of the triangle inequality. It means the triangle with sides 3 cm, 4 cm, and 5 cm can be formed.

Another way we can also find the given sides can form a triangle or not.

If the difference of any two sides of the triangle is less than the third side then we are able to draw a triangle.

Example: Let us consider the sides of the triangle be 5 cm, 10 cm, and 12 cm. So,

The difference of the two sides		Third side
10 - 5	<	12
12 - 10	<	5
12 - 5	<	10

Here, we see the difference of the two sides is less than the third side. It means according to the theorem, the triangle can be formed with sides 5 cm, 10 cm, and 12 cm.

Example: Is it possible to have a triangle with the following sides?

(i) 3 cm, 5 cm, and 7 cm
We know, the sum of two sides of a triangle is greater than the third side.
We check this property for the given sides of the triangle.

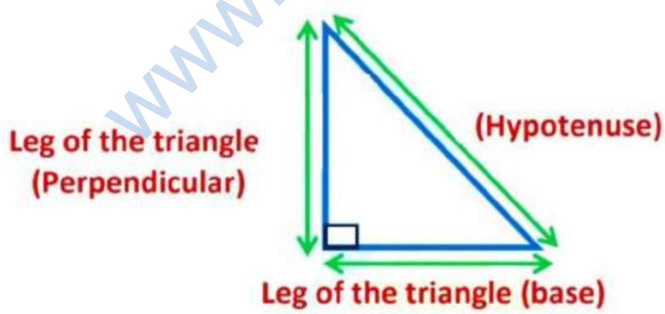
Sum of the two sides		Third side
3 + 5	>	7
5 + 7	>	3
3 + 7	>	5

Here, we see the sum of two sides of the triangle is greater than the third side. Hence, it follows triangle inequality. It means the triangle with sides 3 cm, 5 cm, and 7 cm can be formed.

Pythagoras Theorem

RIGHT-ANGLED TRIANGLES AND PYTHAGORAS THEOREM

Pythagoras theorem is related to the right-angled triangle. So, let us consider a right-angled triangle



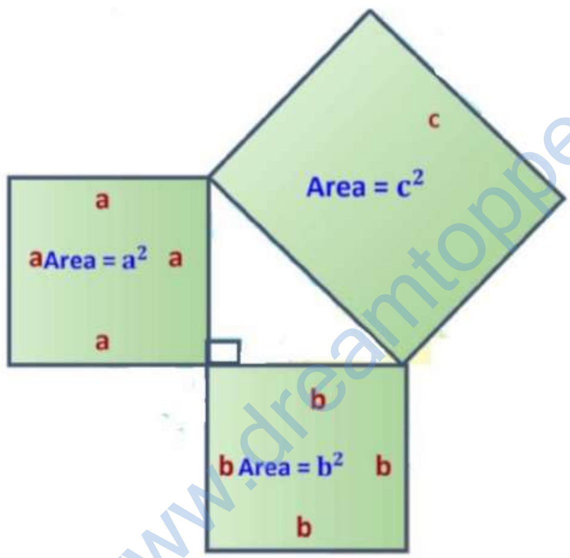
In a right-angled triangle, the side opposite to the right angle is called the hypotenuse, and the other two sides are known as legs of the right-angled triangle. So, perpendicular, base, and hypotenuse are the three sides of the right-angled triangle. Now, according to this theorem, if you have a right-angled triangle
in which
$c \rightarrow$Hypotenuse

b→ Leg of a triangle (Base side)
a→Leg of a triangle (Perpendicular side)

Then let's try this if we draw a square on the perpendicular side that means each side of the square would be equal to a and the area of that square is equal to a^2

If we draw a square on the base side that means each side of the square would be equal to b and the area of that square is equal to b^2.

In the same way, if we draw a square on hypotenuse side that means each side of the square would be equal to c and the area of that square is equal to c^2

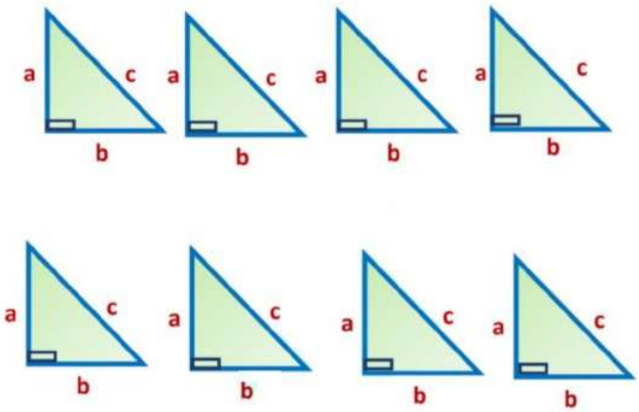


So, according to Pythagoras theorem,
In a right-angled triangle, the square of the hypotenuse equals the sum of the square of its two sides.
∴ $(\text{Perpendicular})^2 + (\text{Base})^2 = (\text{Hypotenuse})^2$

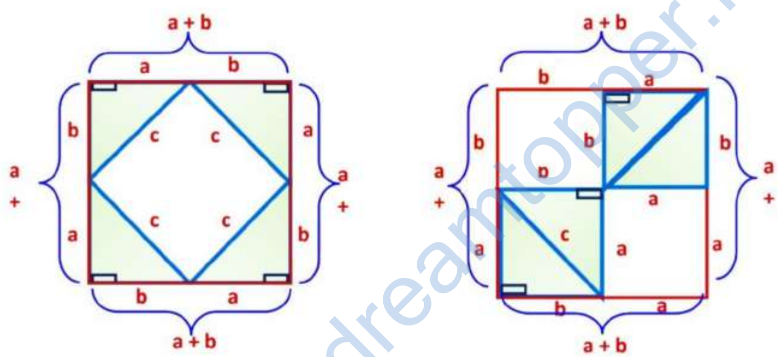
$$\therefore (a)^2 + (b)^2 = (c)^2$$

This is called the Pythagoras theorem.
There is another way we prove Pythagoras theorem

Suppose we have eight right angle triangles and two identical squares and the side of both squares is $(a + b)$.



Now, we place four triangles in the first square and remaining four triangles in the second square in a different way as shown below



Now, we know the squares are identical and the eight triangles inserted in the squares are also identical.

Uncovered area of 1st square = Uncovered area of 2nd square

Area of inner square of first square = The total area of two uncovered squares in 2nd square

$$(c)^2 = (a)^2 + (b)^2$$

Area of a square with side (a + b) = Area of inside square of side c + Area of four inside triangles

Area of a square with side (a + b) = $(a + b)^2$

Area of inside square of side (c) = $(c)^2$

Area of four inside triangles = 4 × Area of one triangle

Area of four inside triangles = $4 \times \frac{1}{2} \times ab = 2ab$

$$(a + b)^2 = c^2 + 2ab$$

$$a^2 + b^2 + 2ab = c^2 + 2ab$$

$$a^2 + b^2 + 2ab - 2ab = c^2 + 2ab - 2ab$$

$$a^2 + b^2 = c^2$$

Hence, Pythagoras theorem proved in another way.

Example: Verify Pythagoras theorem in triangle ABC
In ΔABC,

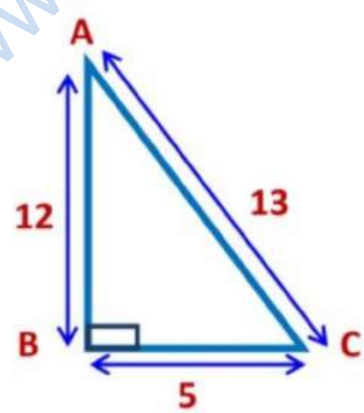
AC = 13 (Hypotenuse)

AB = 12 (Perpendicular)

BC = 5 (Base)

By Pythagoras theorem

$$(AB)^2 + (BC)^2 = (AC)^2$$



Take LHS: = $(AB)^2 + (BC)^2$

$= (12)^2 + (5)^2 = 144 + 25 = 169$

Take RHS: $= (AC)^2$S
$= (13)^2 = 169$

Here, LHS = RHS
Hence, the Pythagoras theorem is verified for this triangle.

Example: A 15 m long ladder is placed against a wall in such a way that the foot of the ladder is 6 m away from the wall. So, up to what height does the ladder reach the wall?
Consider BC be the wall and AC be the ladder.

Then, AB = 6 m and AC = 15 m

Now, ΔABC is a right-angled triangle.

Hence,

$$(AC)^2 = (AB)^2 + (BC)^2$$

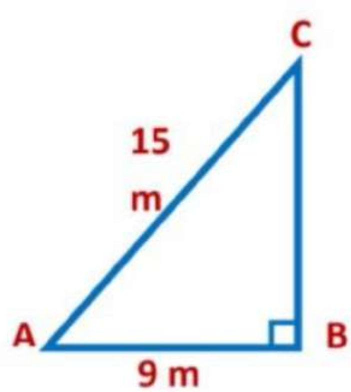
$$\Rightarrow (BC)^2 = (AC)^2 - (AB)^2$$

$$= (15)^2 - (6)^2$$

$$= 225 - 36 = 189$$

$$\Rightarrow BC = \sqrt{189} = 13 \text{ m}$$

Hence, the ladder reaches the wall up to 13 m.



Example: Find the perimeter of the rectangle whose length is 5 cm and its diagonal is 13 cm.

In ΔABC

$$(AC)^2 = (AB)^2 + (BC)^2$$

$$(13)^2 = (AB)^2 + (5)^2$$

$$169 = (AB)^2 + 25$$

$$169 - 25 = (AB)^2 + 25 - 25$$

$$144 = (AB)^2$$

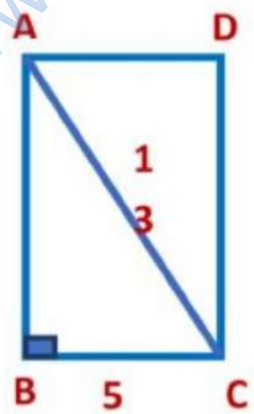
$$(12)^2 = (AB)^2$$

$$12 = AB$$

Now, we know the perimeter of the rectangle = 2 (l + b)

= 2 (5 + 12)
= 2 (17)
= 34 cm

Hence, the perimeter of the rectangle is 34 cm.



Example: An 9 m high pole is 7 m away from a building. If the distance between the top of the pole and top of the building is 25 m then find the height of the building.

Let AC be the height of the building. ED be the height of the pole. AE be the distance between their tops and CD be the distance between the foot of the pole and
the foot of the building. Now, from E draw EB ⊥ AC

$$AE = 25 \text{ m}$$

$$BE = CD = 9 \text{ m}$$

In the right-angled triangle ΔABE,

$$(AE)^2 = (AB)^2 + (BE)^2$$

$$(25)^2 = (AB)^2 + (9)^2$$

$$625 = (AB)^2 + 81$$

Subtracting 81 from both side

$$625 - 81 = (AB)^2 + 81 - 81$$

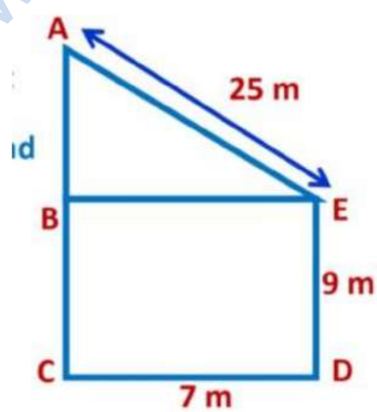
$$576 = (AB)^2$$

$$\therefore AB = 24 \text{ m}$$

Since AC = AB + BC and BC = ED = 9 m

$$\therefore AC = 24 + 9 = 33 \text{ m}$$

Hence, the height of the building is 33 m.



The converse of Pythagoras theorem

If any triangle obeys the Pythagoras theorem, then the triangle is right-angled.

Let's explain this theorem by using the example

Example: Determine whether the given triangle is right-angled.

According to the theorem, if any triangle obeys Pythagoras theorem, then the triangle is right-angled.
So, in ΔABC,

$$AB^2 = 32 = 3 \times 3 = 9$$

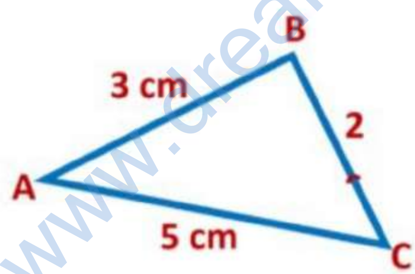
$$BC^2 = 42 = 4 \times 4 = 16$$

$$AC^2 = 52 = 5 \times 5 = 25$$

We find: $(AC)^2 = (AB)^2 + (BC)^2$

$$\Rightarrow 5^2 = 3^2 + 4^2$$

Therefore, the triangle is right-angled at B.



Example: Δ ABC is right-angled at C. If AC = 2.5 cm and BC = 6 cm find the length of AB

By Pythagoras theorem,

In ΔABC, $(AB)^2 = (AC)^2 + (BC)^2$

$$(AB)^2 = (2.5)^2 + (6)^2$$

$$(AB)^2 = 6.25 + 36$$

$$(AB)^2 = 42.25$$

$$\therefore AB = \sqrt{42.25} = 6.5 \text{ cm}$$

Hence, the length of AB is 6.5 cm