09

Sequence, generally means a collection, ordered in such a way that it has an indentified first, second and third members and so on. e.g. Amount of money deposited in bank over a number of years, population of human beings at different times etc. Sequences, following specific patterns are called progressions.

SEQUENCES AND SERIES

> **CHAPTER CHECKLIST**
> - Sequences and Series
> - Arithmetic Progression (AP) and Its General Terms
> - Sum of n Terms of an AP and Arithmetic Mean (AM)
> - Geometric Progression (GP)
> - Sum of First n Terms of a GP
> - Geometric Mean and Its Relation with Arithmetic Mean

|TOPIC 1|
Sequences and Series

SEQUENCE

A sequence is a succession of numbers or terms formed according to some rule. e.g. 3, 6, 9,... is a sequence.

The various numbers occurring in a sequence are called its **terms**. The images of $1, 2, 3, \ldots, n$ under a sequence $< a >$ are generally denoted by $a_1, a_2, a_3, \ldots, a_n$, respectively. Here, $a_1, a_2, a_3, \ldots, a_n$ are known as first term, second term, third term, ..., nth term, respectively of the sequence. The nth term is also called **general term** and it is denoted by 'a_n'.

e.g. 2, 6, 10, 14, ... is a sequence.

Here, $a_1 = 2$, $a_2 = 6$, $a_3 = 10$ and $a_4 = 14$.

Real Sequence A sequence whose range is a subset of the set of real numbers R, is called a real sequence.

In other words, we can say that, a real sequence is a function whose domain is the set of natural numbers N and the range is a subset of set of real numbers R.

Types of Sequence

1. FINITE SEQUENCE

A sequence containing finite number of terms, is called a finite sequence. e.g.

(i) 1, 3, 5, 7 is a finite sequence as it contains only 4 terms.

(ii) 2, 5, 8, 11, 14 is a finite sequence as it contains only 5 terms.

2. INFINITE SEQUENCE

A sequence which is not a finite sequence, is known as infinite sequence. i.e. in an infinite sequence, **number of terms never ends.** e.g.

(i) 1, 3, 5, 7,... is an infinite sequence, as it contains infinite number of terms.

(ii) The sequence of successive quotients obtain in the division of 10 by 3 i.e. 3, 3.3, 3.33, 3.333, …is an infinite sequence.

Representation of a Sequence

A real sequence can be represented by different ways. *Some ways of reprsentation are given below*

(i) A real sequence can be represented by listing its few terms till the rule for writing down other terms becomes clear.

e.g. 3, 5, 7, … is a sequence and rule for writing down other terms is $(2n+1)$.

(ii) A real sequence can be represented in terms of a rule or an algebraic formula of writing the nth term of the sequence.

e.g. The sequence 1, 3, 5, 7, … can be written as $a_n = 2n - 1$.

(iii) Sometimes the sequence i.e. an arrangement of numbers has no visible pattern but the sequence can be represented by the **recurrence relation.**

e.g. The sequence 1, 1, 2, 3, 5, 8, … has no visible pattern but its recurrence relation is $a_1 = a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}, n \geq 2$

This sequence is called **Fibonacci sequence.**

Sequence as a Function

A sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it of the type (1, 2, 3, …, k). Sometimes, we use the functional notation $a(n)$ for a_n.

EXAMPLE |1| Find the 20th term a_{20} of the sequence, whose nth term is **[NCERT]**

$$a_n = \frac{n(n-2)}{n+3}$$

Sol. We have, $a_n = \frac{n(n-2)}{n+3}$

On putting $n = 20$, we get $a_{20} = \frac{20(20-2)}{20+3}$

$$\Rightarrow \quad a_{20} = \frac{20 \times 18}{23} = \frac{360}{23}$$

EXAMPLE |2| Write first five terms of sequence

(i) $a_n = (-1)^{n-1} 5^{n+1}$ (ii) $a_n = 2n^2 - n + 1$ **[NCERT]**

Sol. (i) We have, $a_n = (-1)^{n-1} 5^{n+1}$

On putting $n = 1$, we get

$$a_1 = (-1)^{1-1} 5^{1+1} = (-1)^0 5^2 = 25$$

On putting $n = 2$, we get

$$a_2 = (-1)^{2-1} 5^{2+1} = (-1)^1 5^3 = -125$$

On putting $n = 3$, we get

$$a_3 = (-1)^{3-1} 5^{3+1} = (-1)^2 5^4 = 625$$

On putting $n = 4$, we get

$$a_4 = (-1)^{4-1} 5^{4+1} = (-1)^3 5^5 = -3125$$

On putting $n = 5$, we get

$$a_5 = (-1)^{5-1} 5^{5+1} = (-1)^4 5^6 = 15625$$

Hence, the first five terms of the given sequence are 25, − 125, 625, − 3125, 15625.

(ii) We have, $a_n = 2n^2 - n + 1$

On putting $n = 1$, we get

$$a_1 = 2(1)^2 - 1 + 1 = 2 - 1 + 1 = 2$$

On putting $n = 2$, we get

$$a_2 = 2(2)^2 - 2 + 1 = 8 - 2 + 1 = 7$$

On putting $n = 3$, we get

$$a_3 = 2(3)^2 - 3 + 1 = 18 - 3 + 1 = 16$$

On putting $n = 4$, we get

$$a_4 = 2(4)^2 - 4 + 1 = 32 - 4 + 1 = 29$$

On putting $n = 5$, we get

$$a_5 = 2(5)^2 - 5 + 1 = 50 - 5 + 1 = 46$$

Hence, the first five terms of the given sequence are 2, 7, 16, 29 and 46.

EXAMPLE |3| The Fibonacci sequence is defined by $1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}, n > 2$. Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$. **[NCERT]**

Sol. Given, $1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}, n > 2$

On putting $n = 3, 4, 5, 6$ respectively, we get

For $n = 3, a_3 = a_{3-1} + a_{3-2} = a_2 + a_1 = 1 + 1 = 2$

For $n = 4, a_4 = a_{4-1} + a_{4-2} = a_3 + a_2 = 2 + 1 = 3$

For $n = 5, a_5 = a_{5-1} + a_{5-2} = a_4 + a_3 = 3 + 2 = 5$

For $n = 6, a_6 = a_{6-1} + a_{6-2} = a_5 + a_4 = 5 + 3 = 8$

Now, $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$.

For $n = 1$, $\frac{a_2}{a_1} = \frac{1}{1} = 1$; For $n = 2$, $\frac{a_3}{a_2} = \frac{2}{1} = 2$

For $n = 3$, $\frac{a_4}{a_3} = \frac{3}{2}$; For $n = 4$, $\frac{a_5}{a_4} = \frac{5}{3}$

For $n = 5$, $\frac{a_6}{a_5} = \frac{8}{5}$

Hence, the required terms are 1, 2, $\frac{3}{2}$, $\frac{5}{3}$ and $\frac{8}{5}$.

SERIES

Let a sequence is $a_1, a_2, a_3, \ldots, a_n, \ldots$. Then, the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called the series associated with the given sequence. A series is **finite** or **infinite**, according as number of terms in the corresponding sequence is finite or infinite. Series can be represented in compact form, called **sigma notation.** It is denoted by the Greek letter 'Σ' (**sigma**) e.g. series $a_1 + a_2 + \ldots + a_n + \ldots$ can be represented in compact form as $\sum_{k=1}^{\infty} a_k$.

e.g. (i) $2+4+6+8+\ldots+102$ (ii) $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\ldots$

Note

When we use the term 'Sum of series', it means we add all these terms.

EXAMPLE |4| Let the sequence a_n is defined as follows $a_1 = 2, a_n = a_{n-1} + 3$ for $n \geq 2$.
Find first five terms and write corresponding series.

Sol. We have, $a_1 = 2$ and $a_n = a_{n-1} + 3$

On putting $n = 2$, we get $a_2 = a_1 + 3 = 2 + 3 = 5$

On putting $n = 3$, we get $a_3 = a_2 + 3 = 5 + 3 = 8$

On putting $n = 4$, we get $a_4 = a_3 + 3 = 8 + 3 = 11$

On putting $n = 5$, we get $a_5 = a_4 + 3 = 11 + 3 = 14$

Thus, first five terms of given sequence are 2, 5, 8, 11 and 14. Also, the corresponding series is $2+5+8+11+14+\ldots$

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The collection of objects listed in a sequence is
(a) random (b) ordered
(c) random or ordered (d) None of these

2 The general term of the sequence is denoted by
(a) a_n (b) a^n (c) n (d) $n \cdot a$

3 A sequence may be defined as a **[NCERT Exemplar]**
(a) relation, whose range $\subseteq N$ (natural number)
(b) function whose range $\subseteq N$
(c) function whose domain $\subseteq N$
(d) progression having real values

4 The series is finite or infinite according as the given sequence is ....
(a) infinite (b) finite
(c) finite or infinite (d) infinite or finite

5 Series are often represented in compact form, called
(a) beta notation (b) nano notation
(c) sigma notation (d) neu notation

VERY SHORT ANSWER Type Questions

6 Write the first five terms of each of the following sequence whose *n*th terms are given below.
(Each part carries 1 mark) **[NCERT]**

(i) $a_n = \frac{n}{n+1}$ (ii) $a_n = (-1)^{n+1} 3^{n+2}$

(iii) $b_n = \frac{n(n^2+5)}{4}$ (iv) $a_n = \frac{n^3+1}{2}$

(v) $a_1 = 1, a_n = a_{n-1} + 3$ for $n \geq 2$

7 Find the indicated terms in each of the sequence, whose *n*th terms are given.
(Each part carries 1 mark)

(i) $a_n = 5n - 3$, a_{12}, a_{15}

(ii) $b_n = (-1)^n (n^2 - 1), b_7, b_{13}$

(iii) $a_n = \frac{n^2}{2^n}, a_7$

8 Write the first five terms of each of the sequence and obtain the corresponding series.
(Each part carries 1 mark)

(i) $a_1 = 3, a_n = 3a_{n-1} + 2, \forall n > 1$

(ii) $a_1 = a_2 = 2,\ a_n = a_{n-1} - 1, n > 2$

(iii) $a_n = \frac{1}{6}(2n-3)$

(iv) $a_n = (-1)^{n^2}\left(\frac{2n^2+3}{2}\right)$

(v) $a_1 = -1,\ a_n = \frac{a_{n-1}}{n},\ n \geq 2$

HINTS & ANSWERS

1. (b) Collection of objects listed in a sequence is ordered in such a way that it has an identified first member, second member, third member and so on.

2. (a) The *n*th term of a sequence is known as general term and can be denoted by a_n.

3. (c) A sequence can be defined as a function having domain $\subseteq N$.

4. (c) The series is finite or infinite according as the given sequence is finite or infinite.

5. (c) Series are often represented in compact form called sigma notation.

6. (i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ and $\frac{5}{6}$ (ii) 27, −81, 243, −729, 2187

(iii) $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}$, 21, $\frac{75}{2}$ (iv) 1, $\frac{9}{2}$, 14, $\frac{65}{2}$, 63

(v) 1, 4, 7, 10, 13,

7. (i) $a_{12} = 57, a_{15} = 72$ (ii) $b_7 = -48, b_{13} = -168$

(iii) $a_7 = \frac{49}{128}$

8. (i) We have, $a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$.

On putting $n = 2$, we get $a_2 = 3a_{2-1} + 2 = 11$

On putting $n = 3$, we get $a_3 = 3a_{3-1} + 2 = 35$

On putting $n = 4$, we get $a_4 = 3a_{4-1} + 2 = 107$

On putting $n = 5$, we get $a_5 = 3a_{5-1} + 2 = 323$

∴ Sequence is 3, 11, 35, 107, 323, ...

∴ Series is $3 + 11 + 35 + 107 + 323 + \dots$

(ii) Sequence is 2, 2, 1, 0, −1, ...

Series is $2 + 2 + 1 + 0 + (-1) + \dots$

(iii) Sequence is $-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \dots$

Series is $-\frac{1}{6} + \frac{1}{6} + \frac{1}{2}, \frac{5}{6} + \frac{7}{6} + \dots$

(iv) Sequence is $-\frac{5}{2}, \frac{11}{2}, -\frac{21}{2}, \frac{35}{2}, -\frac{53}{2}, \dots$

Series is $-\frac{5}{2} + \frac{11}{2} - \frac{21}{2} + \frac{35}{2} - \frac{53}{2} + \dots$

(v) Sequence is $-1, -\frac{1}{2}, -\frac{1}{6}, -\frac{1}{24}, -\frac{1}{120}, \dots$

Series is $-1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{24} - \frac{1}{120} - \dots$

|TOPIC 2|

Arithmetic Progression (AP) and Its General Terms

PROGRESSIONS

It is not necessary that the terms of a sequence always follow a certain pattern. Those sequence whose terms follow a certain pattern or rule, are called progression.

e.g. 2, 4, 6, 8,..., 100

We observe that each term in above example is formed by adding 2 to the preceding term.

ARITHMETIC PROGRESSION (AP)

A sequence whose terms **increases or decreases** by a fixed number, is called an Arithmetic Progression (AP).

In other words, we can say that, a sequence is called an arithmetic progression if the difference of a term and the previous term is always same i.e.

$$a_{n+1} - a_n = \text{constant for all } n.$$

This constant or same difference is called the **common difference** of an AP and it is denoted by ***d***.

In an AP, we usually denote the first term by a, common difference by d and the nth term by a_n or T_n and then AP is $a, a + d, a + 2d, \dots$, e.g., 1, 4, 7, 10, ... is an AP.

Here, first term = 1 and common difference = 4 − 1 = 3.

To check whether a given sequence is an AP or not, *we use the following steps*

***Step* I** Find nth term a_n.

***Step* II** Replace n by $n + 1$ to get $(n + 1)$th term i.e. a_{n+1}.

***Step* III** Find difference between a_n and a_{n+1} *i.e.* calculate $a_{n+1} - a_n$.

***Step* IV** If value obtained in step III is independent of n i.e. does not contain n, then given sequence is an AP otherwise not.

EXAMPLE |1| Show that the sequence a_n defined by $a_n = 2n^2 + 1$, is not an AP.

Sol. We have, $a_n = 2n^2 + 1$

On replacing n by $n + 1$, we get

$$a_{n+1} = 2(n+1)^2 + 1$$
$$= 2(n^2 + 1 + 2n) + 1$$
$$= 2n^2 + 2 + 4n + 1$$
$$= 2n^2 + 4n + 3$$

Here, $a_{n+1} - a_n = (2n^2 + 4n + 3) - (2n^2 + 1)$
$$= 2n^2 + 4n + 3 - 2n^2 - 1$$
$$= 4n + 2$$

It is not independent of n. So, given sequence is not an AP.

EXAMPLE |2| The nth term of a sequence is $3n - 2$. Is the sequence an AP? If it is an AP, then find its 6th term.

Sol. Given, $a_n = 3n - 2$

Replacing n by $n + 1$ in a_n, we get

$$a_{n+1} = 3(n + 1) - 2 = 3n + 3 - 2 = 3n + 1$$

Then, $d = a_{n+1} - a_n = (3n + 1) - (3n - 2)$
$$= 3n + 1 - 3n + 2 = 3$$

It is independent of n, so given sequence is an AP and its 6th term, $a_6 = 3 \times 6 - 2 = 18 - 2 = 16$

EXAMPLE |3| Write down the next term of the sequence $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \ldots$.

Sol. Given sequence is $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \ldots$

Here, $a_1 = \frac{1}{6}, a_2 = \frac{1}{3}, a_3 = \frac{1}{2}$

Then $a_2 - a_1 = \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}$

$a_3 - a_2 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$

Since, common difference is same, so it is an AP.

Then, $d = \frac{1}{6}$

$\therefore \quad a_4 = a_3 + d = \frac{1}{2} + \frac{1}{6} = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3}$

Hence, the next term of the given sequence is $\frac{2}{3}$.

EXAMPLE |4| If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP whose common difference is d, show that

$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

Sol. Given, $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP.

$\Rightarrow \quad \theta_2 - \theta_1 = \theta_3 - \theta_2 = \cdots = \theta_n - \theta_{n-1} = d \quad \ldots(i)$

Now, we have to prove

$$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \cdots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}$$

or it can be written as $\sin d[\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \cdots + \sec\theta_{n-1} \sec\theta_n] = \tan\theta_n - \tan\theta_1$

Now, taking only first term of LHS,

$$\sin d \sec\theta_1 \sec\theta_2 = \frac{\sin d}{\cos\theta_1 \cos\theta_2}$$

$$= \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1 \cos\theta_2} \quad \text{[from Eq. (i)]}$$

$$= \frac{\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1}{\cos\theta_1 \cos\theta_2}$$

$[\because \sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B]$

$$= \frac{\sin\theta_2 \cos\theta_1}{\cos\theta_1 \cos\theta_2} - \frac{\cos\theta_2 \sin\theta_1}{\cos\theta_1 \cos\theta_2}$$

$$= \tan\theta_2 - \tan\theta_1$$

Similarly, we can solve other terms which will be $\tan\theta_3 - \tan\theta_2, \tan\theta_4 - \tan\theta_3, \cdots$

$\therefore$ LHS $= \tan\theta_2 - \tan\theta_1 + \tan\theta_3 - \tan\theta_2 + \cdots + \tan\theta_n - \tan\theta_{n-1}$

$= -\tan\theta_1 + \tan\theta_n = \tan\theta_n - \tan\theta_1$

= RHS **Hence proved.**

General Term of an AP

Let a be the first term and d be the common difference of an AP. Then, its nth term is denoted by a_n or T_n and defined as

$$T_n = a_n = a + (n-1)d$$

Also, $l = a + (n-1)d$, where l is the last term of the sequence.

EXAMPLE |5| Find the indicated term of given AP.

$a = 3, d = 2; T_n, T_{10}$

Sol. Given, $a = 3$ and $d = 2$

$\because \quad T_n = a + (n-1)d \quad \ldots(i)$

On putting the values of a and d in Eq. (i) we get

$T_n = 3 + (n-1)2 = 3 + 2n - 2$

$\Rightarrow \quad T_n = 1 + 2n$

and $\quad T_{10} = a + (10-1)d = a + 9d \quad$ [from Eq. (i)]

$= 3 + 9(2) \quad [\because a = 3, d = 2]$

$= 3 + 18 = 21$

EXAMPLE |6| Is 667 a term of an AP 11, 18, 25,...?

Sol. Given AP is 11, 18, 25,...

Here, $a = 11$, $d = 18 - 11 = 7$ and $a_n = 667$ (say)

$\because \quad a + (n-1)d = a_n$

$\therefore 11 + (n-1)7 = 667 \Rightarrow 11 + 7n - 7 = 667$

$\Rightarrow \quad 7n + 4 = 667 \Rightarrow 7n = 667 - 4$

$\Rightarrow \quad 7n = 663$

$\Rightarrow \quad n = \frac{663}{7}$, which is not a whole number.

Hence, 667 is not a term of an AP.

EXAMPLE |7| How many terms are there in AP 20, 25, 30,..., 100?

Sol. Given AP is 20, 25, 30,..., 100.

Here, $a = 20$, $d = 25 - 20 = 5$ and $l = 100$

$\because$ Last term, $\quad l = 100$

$\Rightarrow \quad a + (n-1)d = 100$

$\Rightarrow \quad 20 + (n-1)5 = 100 \quad [\because a = 20 \text{ and } d = 5]$

$\Rightarrow \quad 20 + 5n - 5 = 100 \Rightarrow 15 + 5n = 100$

$\Rightarrow \quad 5n = 100 - 15 = 85 \Rightarrow n = 17$

Hence, there are 17 terms in given AP.

EXAMPLE |8| Which term of the sequence 72, 70, 68, 66,... is 40?

Sol. Given seqeunce is 72, 70, 68, 66, ...

Clearly, the successive difference of the terms is same. So, the above sequence forms an AP with first term, $a = 72$ and common difference, $d = 70 - 72 = -2$.

Let nth term, $\quad T_n = 40$

and $\qquad T_n = a + (n-1)d$

$\therefore \qquad 40 = 72 + (n-1)(-2)$

$\Rightarrow \qquad 40 = 72 - 2n + 2$

$\Rightarrow \qquad 2n = 72 - 40 + 2$

$\Rightarrow \qquad 2n = 34$

$\Rightarrow \qquad n = 17$

Hence, 17th term of the given sequence is 40.

EXAMPLE |9| The 10th term of an AP is 41 and 18th term is 73. Find AP.

Sol. Given, $a_{10} = 41$ and $a_{18} = 73$

We know that,

nth term, $a_n = a + (n-1)d$

$\therefore \qquad a_{10} = a + (10-1)d = 41$

and $\qquad a_{18} = a + (18-1)d = 73$

$\Rightarrow \quad a + 9d = 41$ and $a + 17d = 73$

On solving above two equations, we get

$a = 5$ and $d = 4$

Hence, the required AP is $5, 5+4, 5+2(4), \ldots$

i.e. $5, 9, 13, \ldots$ $\qquad$ [$\because$ AP is $a, a+d, a+2d, \ldots$]

EXAMPLE |10| Which term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ is the first negative term?

Sol. The given sequence is $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$

Clearly, the successive difference of the terms is same. So, the above sequence forms an AP with first term, $a = 20$ and common difference, $d = 19\frac{1}{4} - 20 = -\frac{3}{4}$.

Let the n term of the given AP be the first negative term.

Then, $a_n < 0 \Rightarrow a + (n-1)d < 0$

$\Rightarrow \quad 20 + (n-1) \times (-3/4) < 0$

$\Rightarrow \quad \frac{83}{4} - \frac{3n}{4} < 0 \Rightarrow \frac{3n}{4} > \frac{83}{4}$

$\Rightarrow \qquad 3n > 83 \Rightarrow n > 27\frac{2}{3}$

Since 28 is just greater than $27\frac{2}{3}$. So, $n = 28$. Thus, 28th term of the given sequence will be first negative term.

EXAMPLE |11| How many three-digit numbers are divisible by 7?

First, write a number series of three-digit numbers divisible by 7 i.e. 105, 112, 119,..., 994 and then use the formula $a_n = a + (n-1)d$.

Sol. The series of numbers divisible by 7 having three digits is 105, 112, 119, ...,994.

Clearly, the successive difference of the terms is same. So, the above list of numbers forms an AP with first term, $a = 105$, common difference, $d = 112 - 105 = 7$

and last term, $a_n = 994$.

$\because \quad a + (n-1)d = a_n \qquad \Rightarrow 105 + (n-1)7 = 994$

$\Rightarrow \qquad (n-1)7 = 994 - 105 \Rightarrow 7n - 7 = 889$

$\Rightarrow \qquad 7n = 889 + 7 \Rightarrow 7n = 896 \Rightarrow n = 128$

Hence, there are 128 three-digit numbers which are divisible by 7.

EXAMPLE |12| In the arithmetic progressions 2, 5, 8, ... upto 50 terms and 3, 5, 7, 9, ... upto 60 terms, find how many terms are identical.

Sol. Given, first AP is 2, 5, 8, ... upto 50 terms, we have $a_1 = 2$, $d_1 = 5 - 2 = 3$ and second AP is 3, 5, 7, 9... upto 60 terms,.

Here $a_2 = 3$, $d_2 = 5 - 3 = 2$

Let the mth term of the first AP be equal to the nth term of the second AP.

Then, $2 + (m-1) \times 3 = 3 + (n-1) \times 2$

$\Rightarrow \qquad 3m - 1 = 2n + 1$

$\Rightarrow \qquad 3m = 2n + 2$

$\Rightarrow \qquad \frac{m}{2} = \frac{n+1}{3} = k$ (say)

$\Rightarrow \qquad m = 2k$ and $n = 3k - 1$

$\Rightarrow \qquad 2k \le 50$ and $3k - 1 \le 60$ $\quad$ [$\because m \le 50$ and $n \le 60$]

$\Rightarrow \qquad k \le 25$ and $k \le 20\frac{1}{3}$

$\Rightarrow \qquad k \le 20$ $\qquad$ [$\because k$ is a natural number]

$\therefore \qquad k = 1, 2, 3, \ldots, 20$

Corresponding to each value of k, we get a pair of identical terms. Hence, there are 20 identical terms in the two AP's.

EXAMPLE |13| If the mth term of a given AP is n and its nth term is m, then show that its pth term is $(n + m - p)$. **[NCERT]**

Sol. In the given AP, let the first term $= a$ and the common difference $= d$.

According to the question,

mth term of a given AP $= n \Rightarrow a_m = n$

and nth term of a given AP $= m \Rightarrow a_n = m$

$\therefore \qquad a + (m-1)d = n \qquad$ ...(i)

and $\qquad a + (n-1)d = m \qquad$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$(m-n)d = (n-m) \Rightarrow d = -1$

On putting $d = -1$ in Eq. (i), we get $a = (n + m - 1)$.

$\therefore$ pth term $= a + (p-1)d = (n + m - 1) + (p-1)(-1)$

[$\because a = (n + m - 1)$ and $d = -1$]

$= (n + m - p)$

Hence, the pth term of the given AP is $(n + m - p)$.

EXAMPLE |14| If the mth term of an AP be $1/n$ and its nth term be $1/m$, then show that its mnth term is 1.

Sol. In the given AP, let the first term = a and the common difference = d.

According to the question, $a_m = \frac{1}{n}$ and $a_n = \frac{1}{m}$

$\therefore \quad a + (m-1)d = 1/n$ ...(i)

and $\quad a + (n-1)d = \frac{1}{m}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(m-n)d = \left(\frac{1}{n} - \frac{1}{m}\right) \Rightarrow d = \frac{1}{mn}$$

On putting $d = \frac{1}{mn}$ in Eq. (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\} = \frac{1}{mn}.$$

Now, mnth term $= a + (mn-1)d$

$$= \left\{\frac{1}{mn} + \frac{(mn-1)}{mn}\right\} = \frac{mn}{mn} = 1$$

Hence, the mnth term of the given AP is 1.

EXAMPLE |15| The pth, qth and rth terms of an AP are a, b, c, respectively. Show that

$(q-r)a + (r-p)b + (p-q)c = 0$. **[NCERT]**

Sol. In the given AP, let the first term = A and the common difference = D.

Given, pth term $= A + (p-1)D = a$ ...(i)

qth term $= A + (q-1)D = b$ ...(ii)

rth term $= A + (r-1)D = c$ ...(iii)

Now, we have to prove

$$(q-r)a + (r-p)b + (p-q)c = 0$$

On taking LHS $= (q-r)a + (r-p)b + (p-q)c$ ... (iv)

On putting the values of a, b and c from Eqs. (i), (ii) and (iii) in Eq. (iv), we get

$$\text{LHS} = (q-r)[A + (p-1)D] + (r-p)[A + (q-1)D] + (p-q)[A + (r-1)D]$$

$$= (q-r)A + (q-r)(p-1)D + (r-p)A + (r-p)(q-1)D + (p-q)A + (p-q)(r-1)D$$

$$= A(q-r+r-p+p-q) + D[(q-r)(p-1) + (r-p)(q-1) + (p-q)(r-1)]$$

$$= A(0) + D[qp - q - rp + r + rq - r - pq + p + pr - p - qr + q]$$

$$= 0 + 0 = 0 = \text{RHS}$$

Hence, $(q-r)a + (r-p)b + (p-q)c = 0$

mth Term of an AP from the End

Let a be the first term and d be the common difference of an AP having n terms. Then, mth term from the end is $(n-m+1)$ th term from the beginning.

$\therefore$ mth term from the end, a_{n-m+1}

$$= a + (n-m+1-1)d$$

$$= a + (n-m)d, \text{ where } n > m$$

Also, mth term from the end

$$= a_n + (m-1)(-d)$$

$$= a_n - (m-1)d \quad \text{or} \quad l - (m-1)d$$

where, a_n (or l) is the last term of a sequence.

EXAMPLE |16| Find the 6th term from the end of the sequence 9, 12, 15, ..., 20th term.

Sol. Given sequence is 9, 12, 15, ..., 20th term.

Clearly, the successive difference of the terms is same. So, the above sequence forms an AP with first term, $a = 9$ and common difference, $d = 12 - 9 = 3$.

We know that, if a sequence has n term, then mth term from end is equal to $(n-m+1)$th term from the beginning.

Here, $n = 20$, $m = 6$, $a = 9$ and $d = 3$

Now, 6th term from the end of sequence $= (20-6+1)$ i.e. 15th term from the beginning.

$\therefore \quad T_{15} = 9 + (15-1)3$ $\quad [\because T_n = a + (n-1)d]$

$= 9 + (14)3$

$= 9 + 42 = 51$

Hence, the 6th term from the end of the sequence is 51.

EXAMPLE |17| Determine the number of terms in the AP 3, 7, 11, ..., 407. Also, find the 18th term from the end.

Sol. Given AP is 3, 7, 11, ..., 407

Here, first term, $a = 3$ and common difference, $d = 4$ and nth term, $T_n = 407$

Also, $\quad T_n = a + (n-1)d$

$\therefore \quad 407 = 3 + (n-1)4$

$\Rightarrow \quad 407 = 3 + 4n - 4$

$\Rightarrow \quad 4n = 407 - 3 + 4$

$\Rightarrow \quad 4n = 408$

$\Rightarrow \quad n = 102$

Hence, there are 102 terms in the given AP.

Now, 18th term from the end $= [n - m + 1]$th term from the beginning $= [102 - 18 + 1]$th term from the beginning $=$ 85th term from the beginning $= 3 + (85-1)4 = 339$

Selection of Terms in an AP

Sometimes, we have to select certain terms in an AP. *The convenient ways of selecting terms are given below*

Number of terms	Terms	Common difference
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

From this table, it is clear that if number of terms is odd, then middle term is a and the common difference is d and if number of terms is even, then middle terms are $a-d$, $a+d$ and the common difference is $2d$.

Note

If the terms of an AP are not given, then the terms are chosen as $a, a+d, a+2d, \ldots$

EXAMPLE |18| The sum of three consecutive terms of an AP is 15 and their product is 105. Find the numbers.

Sol. Let three numbers in AP be $a-d$, a and $a+d$.

According to the question,

Sum of three consecutive terms = 15

$\therefore (a-d)+a+(a+d)=15 \Rightarrow 3a=15 \Rightarrow a=5$

and the product of three consecutive terms = 105

$\therefore \quad (a-d)(a)(a+d)=105$

$\Rightarrow \quad (5-d)(5)(5+d)=105$ [put $a=5$]

$\Rightarrow \quad (25-d^2)5=105$ [$\because (A-B)(A+B)=A^2-B^2$]

$\Rightarrow \quad 25-d^2=21$ [dividing both sides by 5]

$\Rightarrow \quad d^2=4 \therefore d=\pm 2$

When $d=2$, then terms in AP are $5-2$, 5 and $5+2$ i.e. 3, 5, 7.

When $d=-2$, then terms in AP are $5+2$, 5, $5-3$ i.e. 7, 5, 3.

EXAMPLE |19| The product of three numbers in AP is 224, and the largest number is 7 times the smallest. Find the numbers.

Sol. Let the three numbers in AP be $a-d$, a, $a+d$. [$\because d<0$]

Then, according to the question,

$$(a-d)a(a+d)=224$$

$$\Rightarrow \quad a(a^2-d^2)=224 \quad \ldots(i)$$

Also, the largest number is 7 times the smallest,

i.e. $\quad a+d=7(a-d) \Rightarrow 8d=6a \Rightarrow d=\dfrac{3a}{4}$

On substituting this value of d in Eq. (i), we get

$$a\left(a^2-\frac{9a^2}{16}\right)=224$$

$$\Rightarrow \quad \frac{7a^3}{16}=224 \Rightarrow a^3=512 \Rightarrow a=8$$

Then, $\quad d=\dfrac{3a}{4}=\dfrac{3}{4}\times 8=6$

Hence, the three numbers are $8-6$, 8, $8+6$ i.e. 2, 8, 14.

PROPERTIES OF AN ARITHMETIC PROGRESSION

(i) If a **constant is added** to each term of an AP, then the resulting sequence is also an AP with the same common difference.

(ii) If a **constant is subtracted** from each term of an AP, then the resulting sequence is also an AP with the same common difference.

(iii) If each term of an AP is **multiplied by a constant** say k, then the resulting sequence is also an AP with common difference kd, where d is the common difference of the given AP.

(iv) If each term of an AP is **divided by a non-zero constant** say k, then the resulting sequence is also an AP with common difference d/k, where d is the common difference of the given AP.

(v) In an AP, the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last term.

i.e. $\quad a_1+a_n=a_2+a_{n-1}=a_3+a_{n-2}=\ldots$

(vi) The resulting sequence formed by adding or subtracting the corresponding terms of two AP's is also an AP.

(vii) A sequence is an AP, iff its nth term is a linear expression in n i.e. $a_n=An+B$, where A and B are constants and coefficient of n is the common difference of the AP.

EXAMPLE |20| If $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in AP, prove that a, b, c are in AP. **[NCERT]**

While doing this question, we should keep all the properties of AP, in mind given in Important Results.

Sol. Given, $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in AP.

$\Rightarrow \quad a\left(\dfrac{b+c}{bc}\right), b\left(\dfrac{a+c}{ac}\right), c\left(\dfrac{b+a}{ab}\right)$ are in AP.

$\Rightarrow \quad \dfrac{ab+ac}{bc}, \dfrac{ba+bc}{ac}, \dfrac{cb+ca}{ab}$ are in AP.

On adding 1 to each term, we get

$\dfrac{ab+ac}{bc}+1, \dfrac{ba+bc}{ac}+1, \dfrac{cb+ca}{ab}+1$ are in AP.

$$\Rightarrow \frac{ab+ac+bc}{bc}, \frac{ba+bc+ac}{ac}, \frac{bc+ac+ab}{ab} \text{ are in AP.}$$

On dividing each term by $ab + bc + ac$, we get

$\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in AP.

On multiplying each term by abc, we get

$\frac{abc}{bc}, \frac{abc}{ac}, \frac{abc}{ab}$ are in AP $\Rightarrow$ a, b, c are in AP.

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 A sequence $a_1, a_2, a_3, ..., a_n, ...$ is called arithmetic sequence or arithmetic progression, if

(a) $a_{n+1} = a_{n-1} + d, n \in N$, where a_1, d are first term and common differenece respectively

(b) $a_{n-1} = a_{n+1} + d, n \in N$, where a_1, d are first term and common term respectively

(c) $a_{n+1} = a_n + d, n \in N$, where a_1, d are first term and common differenece respectively

(d) $a_{n+1} = a_n + d, n \in N$, where a_1, d are first term and common term respectively

2 If the first term is a and the common difference is d, then the arithmetic progression is

(a) $a+d, a+2d, a+3d, ...$ (b) $a, a+d, a+2d, a+3d, ...$

(c) $a-d, a-2d, a-3d, ...$ (d) $a, a-d, a-2d, a-3d, ...$

3 If the first term is a and common difference is d, then the n^{th} term (general term) of A.P. is

(a) $a_n = a+(n-1)d$ (b) $a_n = a+(n+1)d$

(c) $a_n = a+nd$ (d) $a_n = a-nd$

4 Which of the following is incorrect?

(a) If a constant is added to each term of an A.P., the resulting sequence is also an A.P.

(b) If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P.

(c) If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.

(d) If each term of an A.P. is divided by a constant, then the resulting sequence is also an A.P.

5 A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalment by ₹ 5 every month, then the amount he will pay in the 30^{th} instalment is

(a) ₹ 241 (b) ₹ 250 (c) ₹ 245 (d) ₹ 265

VERY SHORT ANSWER Type Questions

6 Show that each of the following sequence is an AP. And also write three more terms in each case. (Each part carries 1 mark)

(i) $-1, \frac{1}{4}, \frac{3}{2}, \frac{11}{4}$

(ii) $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, ...$

7 Find the 23rd term of the sequence 7, 5, 3, 1, ...

8 Find the nth term of the sequence 8, 3, −2, −7, ...

9 Which term of the sequence (Each part carries 1 mark)

(i) 5, 8, 11, 14, ... is 320.

(ii) 84, 80, 76,..., is 0?

10 (i) Is 68 a term of the AP 7, 10, 13, ... ?

(ii) Is 302 a term of the AP 3, 8, 13, ... ? (Each part carries 1 mark)

11 How many terms are there in the AP

$$-1, -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, ..., \frac{10}{3}?$$

12 The first term of an AP is 5, the common difference is 3 and the last term is 80, find the number of terms.

SHORT ANSWER Type I Questions

13 The 6th and 17th terms of an AP are 19 and 41 respectively, find the 40th term.

14 If 9 times the 9th term of an AP is equal to 13 times the 13th term, then find the 22nd term of the AP. **[NCERT Exemplar]**

15 The nth term of a sequence is given by $a_n = 2n + 7$. Show that it is an AP. Also, find its 8th term.

16 The 5th and 13th terms of an AP are 5 and −3. respectively. Find the AP and obtain its 16th term.

17 If the 9th term of an AP is 0, prove that its 29th term is double the 19th term.

18 If 7 times the 7th term of an AP is equal to 11 times its 11th term, show that its 18th term is 0.

19 Which term of the progression $19, 18\frac{1}{5}, 17\frac{2}{5}, ...$ is negative term?

SHORT ANSWER Type II Questions

20 The 2nd, 31st and last term of an AP are $7\frac{2}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$, respectively. Find the first term and the number of terms.

21 Find the number of terms common to the two AP's 3, 7, 11, ... 407 and 2, 9, 16, ... 709.

22 Find the middle terms in the AP 20, 16, 12, ..., –176.

23 Find the 15th term from the end of the AP 3, 5, 7, 9, ..., 201.

24 Three numbers are in AP. If their sum is 27 and the product 648, find the numbers.

25 Find four numbers in AP whose sum is 20 and the sum of whose squares is 120.

26 In an AP, the first term is 2 and the sum of the first five terms is one-fourth of the next five terms, show that 20th terms is – 112. **[NCERT]**

27 If the nth term of a progression is a linear expression in n, then show that it is an AP.

LONG ANSWER Type Questions

28 If a, b, c are in AP, then prove that
$(b+c)^2 - a^2, (c+a)^2 - b^2, (a+b)^2 - c^2$ are also in AP.

29 If $a_1, a_2, a_3, \ldots, a_n$ be an AP of non-zero terms, prove that

$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{n-1}a_n} = \frac{n-1}{a_1a_n}.$$

30 Let a, b, c be respectively the pth, qth, rth terms of an AP.
Prove that **[NCERT]**
(i) $a(q-r) + b(r-p) + c(p-q) = 9$
(ii) $(a-b)r + (b-c)p + (c-a)q = 0$

31 If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{a_{n-1}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$ **[NCERT Exemplar]**

32 If $a_1, a_2, \ldots, a_n$ are in AP with common difference d (where $d \neq 0$); then prove that sum of the series $\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ... + cosec a_{n-1} coseca_n) equal to $\cot a_1 - \cot a_n$.

HINTS & ANSWERS

1. (c) A sequence $a_1, a_2, a_3, \ldots a_n, \ldots$ is called arithmetic sequence or arithmetic progression if $a_{n+1} = a_n + d$, $n \in N$, where a_1 is the first term and the constant term d is the common difference of the AP.

2. (b) If the first term is a and common difference is d, then the arithmetic progression is
$a, a+d, a+2d, \ldots$

3. (a) If the first term is a and common difference is d, then the n^{th} term is $a_n = a + (n-1)d$.

4. (d) If each term of an A.P. is divided by a non-zero constant, then the resulting sequence is also an A.P. . But in option (d), non-zero constant is not given.

5. (c) Given, $a = 100$, $d = 5$
$\because T_n = a + (n-1)d$
$\therefore T_{30} = 100 + (30-1)5 = 100 + 29 \times 5 = 100 + 145 = 245$

6. (i) $4, \frac{21}{4}, \frac{13}{2}$ (ii) $9\sqrt{2}, 11\sqrt{2}, 13\sqrt{3}$

7. – 37

8. $13 - 5n$

9. (i) $320 = 5 + (n-1)3$ **Ans.** 106
(ii) $0 = 84 + (n-1)(-4)$ **Ans.** 22

10. (i) No (ii) No

11. $\frac{10}{3} = -1 + (n-1)\frac{1}{6}$ **Ans.** 27

12. $80 = 5 + (n-1)3$ **Ans.** 26

13. $19 = a + (6-1)d$ and $41 = a + (17-1)d$ **Ans.** 87

14. $9T_9 = 13T_{13} \Rightarrow 9(a+8d) = 13(a+12d)$
$\Rightarrow a = -21d$ **Ans.** 0

15. 23

16. 9, 8, 7, 6, ... and $a_{16} = -6$

17. Let a and d be the first term and common difference, respectively of the given AP.
According to the question,
$a_q = 0 \Rightarrow a + 8d = 0 \Rightarrow a = -8d$ ...(i)
and $a_{29} = a + 28d = a + 28d + 8d - 8d$
$= a + 39d + a$ [from Eq. (i)]
$= 2a + 36d = 2(a + 18d)$
$= 2a_{19}$ **Hence proved.**

18. $7(a+6d) = 11(a+10d) \Rightarrow a = -17d$

19. Let $T_n < 0$
$\Rightarrow \left[19 + (n-1)\left(-\frac{4}{5}\right)\right] < 0 \Rightarrow n > 24\frac{3}{4} =$ 25th term
Ans. 25th

20. $T_2 = 7\frac{3}{2} \Rightarrow a + d = \frac{31}{4}$ and $T_{31} = \frac{1}{2} \Rightarrow a + 30d = \frac{1}{2}$
On solving, we get $d = \frac{-1}{4}$, $a = 8$
Now, $8 + (n-1)\left(-\frac{1}{4}\right) = \frac{-13}{2}$ **Ans.** 8, 59

21. $407 = 3 + (m-1) \times 4$ and $709 = 2 + (n-1) \times 7$

$\Rightarrow \quad m = 102$ and $n = 102$

Let pth term of first AP = qth term of second AP.

$\Rightarrow 3 + (p-1) \times 4 = 2 + (q-1) \times 7$

$\Rightarrow \quad \frac{p+1}{7} = \frac{q}{4} = k$ (say) $\Rightarrow p = 7k - 1$ and $q = 4k$

Since, each AP consists of 102 terms.

$\therefore \quad p \le 102$ and $q \le 102$

$\Rightarrow \quad k \le \frac{103}{7}$ and $k \le \frac{102}{4}$ **Ans.** 14 term

22. $-176 = 20 + (n-1)(-4) \Rightarrow n = 50$

Then, middle term = T_{25} **Ans.** -76

23. 173 **24.** 6, 9, 12 or 12, 9, 6

25. $(a-3d) + (a-d) + (a+d) + (a+3d) = 20 \Rightarrow a = 5$

and $(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a-3d)^2 = 120$

$\Rightarrow \quad 4a^2 + 20d^2 = 120 \Rightarrow 25 + 5d^2 = 30 \Rightarrow d = \pm 1$

Ans. 2, 4, 6, 8 or 8, 6, 4, 2

26. $a + (a+d) + (a+2d) + (a+3d) + a + 4d$

$$= \frac{1}{4}[a + 5d + a + 6d + a + 7d + a + 8d + a + 9d]$$

$\Rightarrow \quad d = -6$

27. Let $T_n = an + b$, where a and b are constants.

Then, $T_{n-1} = a(n-1) + b$

$\therefore T_n - T_{n-1} = (an + b) - [a(n-1) + b]$

$= an - an + b + a - b = a$, which is constant.

28. $(b+c)^2 - a^2, (c+a)^2 - b^2, (a+b)^2 - c^2$ are in AP,

$\Rightarrow (b+c+a)(b+c-a), (c+a+b)(c+a-b),$

$(a+b+c)(a+b-c)$ are in AP.

$\Rightarrow (b+c-a), (c+a-b), (a+b-c)$ are in AP.

$\Rightarrow (b+c-a) - (a+b+c)$

$(c+a-b) - (a+b+c), (a+b-c) - (a+b+c)$ are in AP.

$\Rightarrow -2a, -2b, -2c$ are in AP.

$\Rightarrow a, b, c$ are in AP, which is true.

29. Let $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$ (say)

Now, $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \ldots + \frac{1}{a_{n-1} a_n}$

$$= \frac{1}{d}\left\{\frac{d}{a_1 a_2} + \frac{d}{a_2 a_3} + \frac{d}{a_3 a_4} + \ldots + \frac{d}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\frac{(a_2 - a_1)}{a_1 a_2} + \frac{(a_3 - a_2)}{a_2 a_3} + \frac{(a_4 - a_3)}{a_3 a_4} + \ldots + \frac{(a_n - a_{n-1})}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a_1} - \frac{1}{a_n}\right\} = \frac{1}{d}\left\{\frac{a_n - a_1}{a_1 a_n}\right\} = \frac{1}{d}\left\{\frac{a_1 + (n-1)d - a_1}{a_1 a_n}\right\}$$

30. Let A be the first term and d be the common difference.

Since, $\quad T_p = a \Rightarrow A + (p-1)d = a \quad$ ...(i)

$\quad T_q = b \Rightarrow A + (q-1)d = b \quad$ ...(ii)

and $\quad T_r = c \Rightarrow A + (r-1)d = c \quad$ ...(iii)

(i) On multiplying Eq. (i) by $(q-r)$, Eq. (ii) by $(r-p)$ and Eq. (iii) by $(p-q)$, we get

$(q-r)A + (p-1)(q-r)d = a(q-r) \quad$ ...(iv)

$(r-p)A + (q-1)(r-p)d = b(r-p) \quad$ ...(v)

and $(p-q)A + (r-1)(p-q)d = c(p-q) \quad$ ...(vi)

On adding Eqs. (iv), (v) and Eq. (vi), we get

$a(q-r) + b(r-p) + c(p-q) = 9$

(ii) On subtracting Eq. (ii) from Eq. (i), Eq. (iii) from Eqs (ii) and Eq. (i) from Eq. (iii), we get

$a - b = (p-q)d \quad$ ...(vii)

$b - c = (q-r)d \quad$ ...(viii)

and $\quad c - a = (r-p)d \quad$ ...(ix)

On multiplying Eq. (vii) by r, Eq. (viii) by p and Eq. (ix) by q and then adding, we get

$r(a-b) + p(b-c) + q(c-a) = 0$

31. $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$ (say)

If $a_2 - a_1 = d$, then $(\sqrt{a_2})^2 - (\sqrt{a_1})^2 = d$

$\Rightarrow \quad (\sqrt{a_2} - \sqrt{a_1})(\sqrt{a_2} + \sqrt{a_1}) = d$

$\Rightarrow \quad \frac{1}{\sqrt{a_1} + \sqrt{a_2}} = \frac{\sqrt{a_2} - \sqrt{a_1}}{d}$

On adding, we get

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$$

$$= \frac{1}{d}[\sqrt{a_2} - \sqrt{a_1} + \sqrt{a_3} - \sqrt{a_2} + \ldots + \sqrt{a_n} - \sqrt{a_{n-1}}]$$

$$= \frac{1}{d}[\sqrt{a_n} - \sqrt{a_1}] \quad \text{...(i)}$$

Also, $\quad a_n - a_1 = (n-1)d$

$\Rightarrow (\sqrt{a_n})^2 - (\sqrt{a_1})^2 = (n-1)d$

$\Rightarrow \quad \sqrt{a_n} - \sqrt{a_1} = \frac{(n-1)d}{\sqrt{a_n} + \sqrt{a_1}}$

32. $\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3

$+ \ldots +$ cosec a_{n-1})

$$= \sin d\left[\frac{1}{\sin a_1 \sin a_2} + \frac{1}{\sin a_2 \sin a_3} + \ldots + \frac{1}{\sin a_{n-1} \sin a_n}\right]$$

$$= \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3 - a_2)}{\sin a_2 \sin a_3} + \ldots + \frac{\sin(a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$= \frac{(\sin a_2 \cos a_1 - \cos a_2 \sin a_1)}{\sin a_1 \sin a_2} + \frac{(\sin a_3 \cos a_2 - \cos a_3 \sin a_2)}{\sin a_2 \sin a_3} + \ldots + \frac{(\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3)$

$+ \ldots + (\cot a_{n-1} - \cot a_n) = \cot a_1 - \cot a_n$

|TOPIC 3|

Sum of n Terms of an AP and Arithmetic Mean (AM)

SUM OF n TERMS OF AN AP

If a be the first term and d be the common difference of an AP, then the sum of n terms, S_n of this AP is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Also, if a be the first term and l be the last term, then the sum of n terms of this AP is

$$S_n = \frac{n}{2}(a + l)$$

Note

If sum of n terms, S_n of an AP is given, then nth term, a_n of that AP can be determined by the formula, $a_n = S_n - S_{n-1}$, where $a_1 = S_1$.

EXAMPLE |1| Find the rth term of an AP, sum of whose first n terms is $2n + 3n^2$. **[NCERT Exemplar]**

Sol. Given that, sum of n terms of an AP,

$$S_n = 2n + 3n^2$$

On replacing $(n-1)$ by n, we get

$$S_{n-1} = 2(n-1) + 3(n-1)^2$$

Then, $T_n = S_n - S_{n-1}$

$$= (2n + 3n^2) - [2(n-1) + 3(n-1)^2]$$

$$= (2n + 3n^2) - [2n - 2 + 3(n^2 + 1 - 2n)]$$

$$= (2n + 3n^2) - (2n - 2 + 3n^2 + 3 - 6n)$$

$$= 2n + 3n^2 - 2n + 2 - 3n^2 - 3 + 6n$$

$$= 6n - 1$$

$\therefore$ rth term, $T_r = 6r - 1$

Problems Based on Sum of n Terms of an AP

There are many types of problems which can be solved directly or indirectly by using the sum of n terms.

Some types are given below

TYPE I

PROBLEMS BASED ON FINDING THE SUM OF GIVEN NUMBER OF TERMS

In this type of problems, an AP is given and we have to find sum of given number of terms directly.

EXAMPLE |2| Find the sum of the series $45 + 47 + 49 + \ldots + 99$.

Sol. Given series is $45 + 47 + 49 + \ldots + 99$.

Here, $a = 45$ and $d = 47 - 45 = 2$ and $l = a_n = 99$

$\because$ $a_n = 99 \Rightarrow a + (n-1)d = 99$

$\Rightarrow$ $45 + (n-1)2 = 99$

$\Rightarrow$ $45 + 2n - 2 = 99 \Rightarrow 2n = 56$

$\Rightarrow$ $n = 28$

$\because$ $S_n = \frac{n}{2}(a + l)$

$\therefore$ $S_{28} = \frac{28}{2}(45 + 99) = 14(144) = 2016$

EXAMPLE |3| Find the sum of 20 terms of an AP, whose first term is 3 and last term is 57.

Sol. We have, $a = 3$, $l = 57$ and $n = 20$

$\because$ $S_n = \frac{n}{2}[a + l]$

$\therefore$ $S_{20} = \frac{20}{2}[3 + 57] = 10 \times 60 = 600$

EXAMPLE |4| Find the sum of indicated number of terms in the following AP.

$x + y,\ x - y,\ x - 3y, \ldots,$ 22 terms

Sol. Given series is $x + y,\ x - y,\ x - 3y, \ldots,$ 22 terms.

Here, $a = x + y$

$d = a_2 - a_1 = (x - y) - (x + y) = -2y$

and $n = 22$

$\because$ $S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore$ $S_{22} = \frac{22}{2}[2 \times (x + y) + (22 - 1)(-2y)]$

$$= 11[2x + 2y + (21)(-2y)]$$

$$= 11[2x + 2y - 42y]$$

$$= 11[2x - 40y] = 22[x - 20y]$$

EXAMPLE |5| If the first term of an AP is a and the sum of the first p terms is zero, then find the sum of its next q terms. **[NCERT Exemplar]**

Sol. Let the common difference be d.

Given, sum of first p terms $= 0$

$\Rightarrow$ $S_p = 0$

$$\Rightarrow \frac{p}{2}[2a+(p-1)d]=0 \quad \text{[given, } a = \text{first term of an AP]}$$

$$\Rightarrow \quad 2a+(p-1)d=0$$

$$\Rightarrow \quad d=-\frac{2a}{p-1} \quad \text{...(i)}$$

$\therefore$ Required sum of next q terms

$= \text{Sum of first } (p+q) \text{ terms} - \text{Sum of first } p \text{ terms}$

$= S_{p+q} - S_p$

$$= \frac{p+q}{2}[2a+(p+q-1)d]-0 \quad [\because S_p=0]$$

$$= \frac{p+q}{2}\left[2a+(p+q-1)\left(\frac{-2a}{p-1}\right)\right] \quad \text{[from Eq. (i)]}$$

$$= \frac{p+q}{2}\times 2a\left[\frac{1}{1}-\frac{p+q-1}{p-1}\right]$$

$$= (p+q)\,a\left[\frac{p-1-p-q+1}{p-1}\right]$$

$$= \frac{-a(p+q)q}{p-1}$$

EXAMPLE |6| Find the sum to n terms of the sequence $\log a$, $\log ar$, $\log ar^2$, ...

Sol. Given sequence is $\log a$, $\log ar$, $\log ar^2$,...

Above sequence can be expressed as

$\log a$, $(\log a + \log r)$, $(\log a + 2\log r)$,...

$[\because \log mn = \log m + \log n \text{ and } \log a^x = x\log a]$

which is clearly an AP with first term, $A = \log a$ and common difference, $D = \log r$.

We know that,

Sum of n terms,

$$S_n = \frac{n}{2}[2A+(n-1)D]$$

$$\therefore \quad S_n = \frac{n}{2}[2\log a+(n-1)\log r]$$

$$= \frac{n}{2}[\log a^2 + \log r^{n-1}] \quad [\because x\log a = \log a^x]$$

$$= \frac{n}{2}[\log a^2 r^{n-1}] \quad [\because \log m + \log n = \log mn]$$

EXAMPLE |7| Find the sum of all numbers between 200 and 400 which are divisible by 7. **[NCERT]**

Sol. The numbers which are divisible by 7 between 200 and 400 are 203, 210, 217,..., 399.

Clearly, they form an AP.

where, $a = 203$, $d = 7$ and $T_n = 399$

Now, $T_n = a+(n-1)d$

$\Rightarrow \quad 399 = 203+(n-1)7$

$\Rightarrow (n-1)7 = 399-203$

$\Rightarrow (n-1)7 = 196 \Rightarrow n-1 = \frac{196}{7}$

$\Rightarrow \quad n-1 = 28 \Rightarrow n = 29$

Now, $S_n = \frac{n}{2}(a+T_n)$ $\quad [\because T_n = \text{last term of given AP}]$

$$\therefore \quad S_{29} = \frac{29}{2}[203+399]$$

$$= \frac{29}{2}\times 602 = 29\times 301 = 8729$$

Note

Students should remember that numbers 200 and 400 are excluded.

EXAMPLE |8| If the first term of an AP is 100 and the sum of first six terms is five times the sum of the next six terms, then find the common difference.

Sol. Given, $a = 100$

Let d be the common difference.

According to the question,

$$a_1 + a_2 + \ldots + a_6 = 5(a_7 + a_8 + \ldots + a_{12})$$

$$\Rightarrow \frac{6}{2}[2a+(6-1)d]$$

$$= 5[(a+6d)+(a+7d)+\ldots+(a+11d)]$$

$[\because a_7 = a+6d, \ldots, a_{12} = a+11d]$

$$\Rightarrow \quad 3[2a+5d]$$

$$= 5[6a+(6d+7d+\ldots+11d)]$$

$$= 5\left[6a+\left\{\frac{6}{2}[2\times 6d+(6-1)d]\right\}\right]$$

$\left[\because S_n = \frac{n}{2}\{2a+(n-1)d\}\right]$

$$= 5[6a+\{3(12d+5d)\}]$$

$$= 5[6a+36d+15d] = 5[6a+51d]$$

$\Rightarrow 3(2a+5d) = 5\times 3(2a+17d)$

$\Rightarrow \quad 2a+5d = 10a+85d$

$\Rightarrow \quad -80d = 8a$

$\Rightarrow \quad -80d = 8(100)$ $\quad [\because a = 100]$

$\Rightarrow \quad d = -10$

TYPE II

PROBLEMS BASED ON FINDING THE NUMBER OF TERMS, WHEN SUM OF ITS TERMS IS GIVEN

In this type of problems, sum of certain number of terms is given and we have to find that certain number of terms with the help of sum.

EXAMPLE |9| How many terms in the AP $-9, -6, -3, \ldots$ must be added together so that the sum may be 66?

Sol. Let 66 be the sum of n terms.

We have, $\quad a = -9$

and $\quad d = -6-(-9) = -6+9 = 3$

$\because \quad S_n = \frac{n}{2}[2a+(n-1)d]$

$\therefore \quad 66 = \frac{n}{2}[2(-9)+(n-1)3]$

$\Rightarrow \quad 66 = \frac{n}{2}[-18+3n-3] \Rightarrow 132 = n[3n-21]$

$\Rightarrow \quad 44 = n[n-7]$ [dividing both sides by 3]

$\Rightarrow \quad n^2 - 7n - 44 = 0$

$\Rightarrow \quad (n-11)(n+4) = 0$

$\Rightarrow \quad n = 11$ or -4

Rejecting $n = -4$ because number of terms cannot be negative.

$\therefore \quad n = 11$

EXAMPLE |10| Find the number of terms of the sequence 54, 51, 48, ..., when there sum is 513.

Sol. Given series is 54, 51, 48, ...

Clearly, the successive difference of the terms is same. So, the above sequence forms an AP, with first term, $a = 54$ and common difference, $d = 51-54 = -3$.

Let number of terms be n, then

Their sum $(S_n) = 513$

$\therefore \quad \frac{n}{2}[2a+(n-1)d] = 513$

$\Rightarrow \quad \frac{n}{2}[2\times 54+(n-1)(-3)] = 513$

$\Rightarrow \quad n(108-3n+3) = 513\times 2$

$\Rightarrow \quad -3n^2+111n = 1026$

$\Rightarrow \quad -(3n^2-111n+1026) = 0$

$\Rightarrow \quad n^2-37n+342 = 0$ [dividing both sides by (-3)]

$\Rightarrow \quad (n-18)(n-19) = 0 \Rightarrow n = 18$ or 19

Here, the common difference is negative.

$\therefore$ 19th term, $T_{19} = 54+(19-1)(-3) = 0$

So, the sum of 18 terms as well as that of 19 terms is 513.

EXAMPLE |11| How many terms of the AP $-6, -\frac{11}{2}, -5$ ..... are needed to give sum -25? **[NCERT]**

Sol. Given that the sequence $-6, -\frac{11}{2}, -5, \ldots$ is in AP.

Here, $a = -6$, $d = -\frac{11}{2}-(-6) = -\frac{11}{2}+6$

$$d = \frac{-11+12}{2} = \frac{1}{2}$$

Now, $\quad S_n = \frac{n}{2}[2a+(n-1)d]$

$\Rightarrow \quad -25 = \frac{n}{2}\left[2(-6)+(n-1)\frac{1}{2}\right]$

$\Rightarrow \quad -25\times 2 = n\left[\frac{-12}{1}+\frac{(n-1)}{2}\right]$

$\Rightarrow \quad -50 = n\left[\frac{-24+n-1}{2}\right]$

$\Rightarrow \quad -50\times 2 = n(n-25)$

$\Rightarrow \quad -100 = n^2-25n \Rightarrow n^2-25n+100 = 0$

$\Rightarrow \quad n^2-20n-5n+100 = 0$

$\Rightarrow \quad n(n-20)-5(n-20) = 0$

$\Rightarrow \quad (n-20)(n-5) = 0$

$\Rightarrow \quad n = 5$ or 20

TYPE III

PROVING RESULTS RELATED TO THE SUM OF n TERMS OF AN AP

In this type of problems, some basic terms related to given AP as its sum of finite number of terms i.e. nth term etc are given and we have to prove result based on sum of the n terms of AP.

EXAMPLE |12| If S_n denotes the sum of n terms of an AP and $S_1 = 6$ and $S_7 = 105$, then show that $S_n : S_{n-3} = (n+3) : (n-3)$.

Sol. Given, $S_1 = 6$ and $S_7 = 105$

We know that, $\quad a_1 = S_1$

$\therefore \quad a_1 = 6$

Again, $\quad S_7 = 105$

$\Rightarrow \quad \frac{7}{2}[2(6)+(7-1)d] = 105 \quad \left[\because S_n = \frac{n}{2}\{2a+(n-1)d\}\right]$

$\Rightarrow \quad \frac{7}{2}[12+6d] = 105 \Rightarrow 7(6+3d) = 105$

$\Rightarrow \quad 6+3d = 15 \Rightarrow 3d = 9 \Rightarrow d = 3$

Now, $\quad S_n = \frac{n}{2}[2(6)+(n-1)3] = \frac{3n}{2}[n+3] \quad$ ...(i)

On replacing n by $n-3$ in Eq. (i), we get

$$S_{n-3} = \frac{3(n-3)(n)}{2} \quad \text{...(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{S_n}{S_{n-3}} = \frac{n+3}{n-3}$$

or $\quad S_n : S_{n-3} = (n+3):(n-3)$

EXAMPLE |13| The first term of an AP is a, the second term is b and the last term is c. Show that the sum of the AP is $\dfrac{(b+c-2a)(c+a)}{2(b-a)}$

Sol. Let d be the common difference and n be the number of terms of the AP.

Since, the first term is a and the second term is b.

Therefore, $d = b - a$

Also, last term, $l = c$

$\because \quad l = a + (n-1)d$

$\therefore \quad c = a + (n-1)(b-a) \qquad [\because d = b - a]$

$\Rightarrow \quad n - 1 = \dfrac{c-a}{b-a}$

$\Rightarrow \quad n = 1 + \dfrac{c-a}{b-a} = \dfrac{b-a+c-a}{b-a} = \dfrac{b+c-2a}{b-a}$

Therefore, $S_n = \dfrac{n}{2}(a+l) = \dfrac{(b+c-2a)}{2(b-a)}(a+c)$

EXAMPLE |14| If the sum of p terms of an AP is q and the sum of q terms is p, then show that the sum of $p+q$ terms is $-(p+q)$. Also, find the sum of first $p-q$ terms (where, $p > q$). **[NCERT Exemplar]**

Sol. Let first term and common difference of the AP be a and d, respectively.

Then, $S_p = q$

$\Rightarrow \quad \dfrac{p}{2}[2a + (p-1)d] = q \quad \left[\because S_n = \dfrac{n}{2}\{2a + (n-1)d\}\right]$

$\Rightarrow \quad 2a + (p-1)d = \dfrac{2q}{p} \qquad \text{...(i)}$

and $\quad S_q = p \Rightarrow \dfrac{q}{2}[2a + (q-1)d] = p$

$\Rightarrow \quad 2a + (q-1)d = \dfrac{2p}{q} \qquad \text{...(ii)}$

On subtracting Eq. (ii) from Eq. (i), we get

$2a + (p-1)d - 2a - (q-1)d = \dfrac{2q}{p} - \dfrac{2p}{q}$

$\Rightarrow \quad [(p-1)-(q-1)]\, d = \dfrac{2q^2 - 2p^2}{pq}$

$\Rightarrow \quad [p - 1 - q + 1]\, d = \dfrac{2(q^2 - p^2)}{pq}$

$\Rightarrow \quad (p-q)\, d = \dfrac{2(q^2 - p^2)}{pq}$

$\therefore \quad d = \dfrac{-2(p+q)}{pq} \qquad \text{...(iii)}$

On substituting the value of d in Eq. (i), we get

$2a + (p-1)\left(\dfrac{-2(p+q)}{pq}\right) = \dfrac{2q}{p}$

$\Rightarrow \quad 2a = \dfrac{2q}{p} + \dfrac{2(p+q)(p-1)}{pq}$

$\Rightarrow \quad a = \left[\dfrac{q}{p} + \dfrac{(p+q)(p-1)}{pq}\right] \qquad \text{...(iv)}$

Now, $S_{p+q} = \dfrac{p+q}{2}[2a + (p+q-1)d]$

$= \dfrac{p+q}{2}\left[\dfrac{2q}{p} + \dfrac{2(p+q)(p-1)}{pq} - \dfrac{(p+q-1)2(p+q)}{pq}\right]$

$= (p+q)\left[\dfrac{q}{p} + \dfrac{(p+q)(p-1) - (p+q-1)(p+q)}{pq}\right]$

$= (p+q)\left[\dfrac{q}{p} + \dfrac{(p+q)(p-1-p-q+1)}{pq}\right]$

$= (p+q)\left[\dfrac{q}{p} - \dfrac{p+q}{p}\right] = (p+q)\left[\dfrac{q-p-q}{p}\right]$

$\Rightarrow \quad S_{p+q} = -(p+q)$

Now, $S_{p-q} = \dfrac{p-q}{2}[2a + (p-q-1)d]$

$= \dfrac{p-q}{2}\left[\dfrac{2q}{p} + \dfrac{2(p+q)(p-1)}{pq} - \dfrac{(p-q-1)2(p+q)}{pq}\right]$

$= (p-q)\left[\dfrac{q}{p} + \dfrac{(p+q)(p-1-p+q+1)}{pq}\right]$

$= (p-q)\left[\dfrac{q}{p} + \dfrac{(p+q)q}{pq}\right]$

$= (p-q)\left[\dfrac{q}{p} + \dfrac{p+q}{p}\right] = (p-q)\dfrac{(p+2q)}{p}$

EXAMPLE |15| The pth term of an AP is a and qth term is b. Prove that sum of its $(p+q)$th term is $\dfrac{p+q}{2}\left[a + b + \dfrac{a-b}{p-q}\right]$. **[NCERT Exemplar]**

Sol. Let A be the first term and D be the common difference of the given AP. Then,

$T_p = a \Rightarrow A + (p-1)D = a \qquad \text{...(i)}$

and $T_q = b \Rightarrow A + (q-1)D = b \qquad \text{...(ii)}$

On subtracting Eq. (ii) from Eq. (i), we get

$(p-q)D = a - b \Rightarrow D = \dfrac{a-b}{p-q} \qquad \text{...(iii)}$

On adding Eqs. (i) and (ii), we get

$2A + (p+q-2)D = a + b$

$\Rightarrow \quad 2A + pD + qD - 2D = a + b$

$\Rightarrow \quad 2A + pD + qD - D = a + b + D$

$\Rightarrow \quad 2A + (p+q-1)D = a + b + D$

$\Rightarrow \quad 2A + (p+q-1)D = a + b + \left(\dfrac{a-b}{p-q}\right)$

[from Eq. (iii)] ...(iv)

Now, $S_{p+q} = \frac{p+q}{2}[2A + (p+q-1)D]$

$$= \frac{p+q}{2}\left[a + b + \frac{a-b}{p-q}\right] \text{ [from Eq. (iv)]}$$

Hence proved.

EXAMPLE |16| If the sum of m terms of an AP is equal to the sum of either the next n terms or the next p terms; then prove that

$$(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

Sol. Let the AP be $a, a+d, a+2d \ldots$.

We are given

$$a_1 + a_2 + \ldots + a_m = a_{m+1} + a_{m+2} + \ldots + a_{m+n} \quad \ldots\text{(i)}$$

Adding $a_1 + a_2 + \ldots + a_m$ on both sides of Eq. (i), we get

$$2[a_1 + a_2 + \ldots + a_m] = a_1 + a_2 + \ldots + a_m + a_{m+1} + \ldots + a_{m+n}$$

$$\Rightarrow \quad 2S_m = S_{m+n}$$

$$\Rightarrow \quad 2\frac{m}{2}\{2a + (m-1)d\} = \frac{m+n}{2}\{2a + (m+n-1)d\}$$

On putting $2a + (m-1)d = x$ in the above equation, we get

$$mx = \frac{m+n}{2}(x + nd)$$

$$\Rightarrow \quad (2m - m - n)x = (m+n)nd$$

$$\Rightarrow \quad (m-n)x = (m+n)nd \quad \ldots\text{(ii)}$$

Similarly,

$$a_1 + a_2 + \ldots + a_m = a_{m+1} + a_{m+2} + \ldots + a_{m+p}$$

On adding $a_1 + a_2 + \ldots + a_m$ on both sides, we get

$$2(a_1 + a_2 + \ldots + a_m) = a_1 + a_2 + \ldots + a_{m+1} + \ldots + a_{m+p}$$

$$\Rightarrow \quad 2S_m = S_{m+p}$$

$$\Rightarrow 2\left[\frac{m}{2}\{2a + (m-1)d\}\right] = \frac{m+p}{2}\{2a + (m+p-1)d\}$$

$$\Rightarrow mx = \frac{(m+p)}{2}(x + pd)$$

$$\Rightarrow 2mx = mx + mpd + px + p^2d$$

$$\Rightarrow \quad (m-p)x = (m+p)pd \quad \ldots\text{(iii)}$$

On dividing Eq. (ii) by Eq. (iii), we get

$$\frac{(m-n)x}{(m-p)x} = \frac{(m+n)nd}{(m+p)pd}$$

$$\Rightarrow (m-n)(m+p)p = (m-p)(m+n)n$$

On dividing both sides by mnp, we get

$$(m+p)\left(\frac{1}{n} - \frac{1}{m}\right) = (m+n)\left(\frac{1}{p} - \frac{1}{m}\right)$$

$$\text{or } (m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

EXAMPLE |17| If the ratio between the sums of n terms of two arithmetic progressions is $(7n+1):(4n+27)$, find the ratio of their 11th terms.

Sol. Let a_1, a_2 be the first terms and d_1, d_2 be the common differences of the first and second arithmetic progressions, respectively. Then, the sums of n terms of these progressions are given by

$$S_n = \frac{n}{2}\cdot\{2a_1 + (n-1)d_1\} \quad \ldots\text{(i)}$$

and

$$S_n' = \frac{n}{2}\cdot\{2a_2 + (n-1)d_2\}. \quad \ldots\text{(ii)}$$

On dividing Eq. (i) by (ii), we get

$$\frac{\frac{n}{2}\cdot\{2a_1 + (n-1)d_1\}}{\frac{n}{2}\cdot\{2a_2 + (n-1)d_2\}} = \frac{S_n}{S_n'} = \frac{7n+1}{4n+27} \quad \text{[given]}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27} \quad \ldots\text{(iii)}$$

Ratio of 11th terms $= \frac{a_1 + 10d_1}{a_2 + 10d_2}$

$$= \frac{2a_1 + 20d_1}{2a_2 + 20d_2} = \frac{2a_1 + (21-1)d_2}{2a_2 + (21-1)d_2}$$

$$= \frac{7 \times 21 + 1}{4 \times 21 + 27} = \frac{148}{111}$$

[putting $n = 21$ in Eq. (iii)].

Hence, the required ratio is 148 : 111.

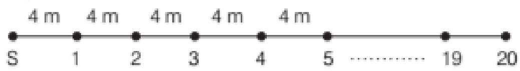
Applications of an AP

Sometimes we can solve the some real life problems by using the concept of AP. *Some to them are given below*

EXAMPLE |18| In a potato race, 20 potatoes are placed in a line at intervals of 4 m with the first potato 24 m from the starting point. A contestant is required to put the potatoes back to the starting place on that time. How far would he run in bringing back all the potatoes?

[NCERT Exemplar]

Sol. As per the given information we have the following diagram



Starting point = S

Distance travelled to bring the first potato

$$= 24 + 24 = 48 \text{ m}$$

Distance travelled to bring the second potato

$$= 2(24 + 4) = 56 \text{ m}$$

Distance travelled to bring the third potato

$$= 2(24 + 4 + 4) = 64 \text{ m}$$

Therefore, the series will be 48, 56, 64, …

which is clearly an AP in which $a = 48$, $d = 56 - 48 = 8$

We have to find the total distance to bring all the potatoes back, so, $n = 20$

$\because \quad S_n = \frac{n}{2}[2n + (n-1)d]$

$\therefore \quad S_{20} = \frac{20}{2}[2 \times 48 + (20-1)8] = 10[96 + 152]$

$= 10 \times 248 = 2480$ m

Hence, the required distance is 2480 m.

EXAMPLE |19| A man arranges to pay-off a debt of ₹ 3600 by 40 annual instalments, which are in AP, when 30 of the instalments are paid, he dies leaving one-third of the debt unpaid. Find the 8th instalment. **[NCERT]**

Sol. Here, total debt, $S =$ ₹ 3600

and total instalments, $n = 40$

Let a and d be the first instalment and increment in instalment, respectively.

Thus, we get an AP here.

By using sum of n terms, we get

$$3600 = \frac{40}{2}[2a + (40-1)d]$$

$\Rightarrow \quad 180 = 2a + 39d \quad$ ...(i)

Now, after 30 instalments, one-third of the debt is unpaid. i.e. $\frac{3600}{3} =$ ₹ 1200 is unpaid and then paid money $= 3600 - 1200 =$ ₹ 2400

So, again by using sum of n terms, we get

$$S_{30} = 2400 = \frac{30}{2}[2a + (30-1)d]$$

$\Rightarrow \quad 160 = 2a + 29d \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$a = 51$, $d = 2$

We know that, nth term $(T_n) = a + (n-1)d$

$\therefore$ 8th instalment, $T_8 = a + (8-1)d = 51 + 7 \times 2$

$= 51 + 14 =$ ₹ 65

Hence, the 8th instalment is ₹ 65.

EXAMPLE |20| Shamshad Ali buys a scooter for ₹22000. He pays ₹ 4000 cash and agrees to pay the balance in annual instalments of ₹ 1000 plus 10% interest on the unpaid amount. How much will scooter cost him?

Sol. Given, cost of scooter = ₹ 22000,

Down payment = ₹ 4000, balance payment = ₹ 18000

Now, interest on Ist instalment $= \frac{18000 \times 10 \times 1}{100} =$ ₹ 1800

$\left[\because I = \frac{P \times R \times T}{100}\right]$

Unpaid amount $= 18000 - 1000 =$ ₹ 17000

Interest on IInd instalment $= \frac{17000 \times 10 \times 1}{100} =$ ₹ 1700

Unpaid amount $= 17000 - 1000 =$ ₹ 16000

Interest on IInd instalment $= \frac{16000 \times 10 \times 1}{100} =$ ₹ 1600

....

....

$\therefore$ Total interest paid by him

$= 1800 + 1700 + 1600 + \ldots + 18$ terms

which is clearly an AP with $a = 1800$, $d = 1700 - 1800 = -100$

Therefore, total interest $= \frac{18}{2}[2 \times 1800 + (18-1)(-100)]$

$\left[\because \text{Sum of an AP, } S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$

$= 9(3600 - 1700) = 9 \times 1900 = 17100$

Hence, total amount or actual cost $= 22000 + 17100$

$=$ ₹ 39100

EXAMPLE |21| A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹ 320 in the very next month and each month thereafter.

(i) Find his salary for the tenth month. **[NCERT]**

(ii) What is his total earnings during the first year?

Sol. Here, the man get a fixed increment of ₹ 320 each month.

Therefore, this forms an AP whose first term = 5200 and common difference $(d) = 320$

(i) Salary for tenth month i.e. for $n = 10$,

$a_{10} = a + (n-1)d$

$\Rightarrow \quad a_{10} = 5200 + (10-1) \times 320$

$\Rightarrow \quad a_{10} = 5200 + 9 \times 320$

$\Rightarrow \quad a_{10} = 5200 + 2880$

$\therefore \quad a_{10} = 8080$

(ii) In a year there are 12 month i.e. $n = 12$,

Total earning during the first year.

$$S_{12} = \frac{12}{2}[2 \times 5200 + (12-1)320]$$

$= 6[10400 + 11 \times 320]$

$= 6[10400 + 3520] = 6 \times 13920 =$ ₹ 83520

EXAMPLE |22| The digit of a three-digit number are in AP and their sum is 21. The number obtained by reversing the digit is 396 less than the original number. Find the number.

Sol. Let the hundred's, ten's and unit's digits be $(a+d)$, a, $(a-d)$.

Then, $(a+d) + a + (a-d) = 21$

$\Rightarrow \quad 3a = 21$

$\Rightarrow \quad a = 7$

∴ Original three-digit number

$$= (7 + d) \times 100 + 7 \times 10 + (7 - d)$$

According to the given condition,

$$(7 + d) \times 100 + 7 \times 10 + (7 - d) - \{(7 - d) \times 100 + 7 \times 10 + (7 + d)\} = 396$$

$\Rightarrow$ $693 + 99d - \{693 - 99d\} = 396$

$\Rightarrow$ $198d = 396 \Rightarrow d = 2$

∴ Original three digit numbers

$$= (7 + 2) \times 100 + 7 \times 10 + (7 - 2)$$
$$= 900 + 70 + 5 = 975$$

EXAMPLE |23| The interior angles of a polygon are in AP. The smallest angle is 120° and the common difference is 5°. Find the number of sides of polygon.

Sol. Let n be the number of sides of the polygon. We know that, sum of interior angles of a polygon of n sides

$$= (2n - 4) \times 90 \quad \text{...(i)}$$

Given, $a = 120°$ and $d = 5°$, $n = n$

∴ Sum of interior angles of polygon of n sides

$$= \frac{n}{2}[2(120) + (n - 1)5] \quad \text{...(ii)}$$

Now, from Eqs. (i) and (ii), we get

$$\frac{n}{2}[2(120) + (n - 1)5] = (2n - 4)90$$

$\Rightarrow$ $n(240 + 5n - 5) = 180(2n - 4)$

$\Rightarrow$ $n(235 + 5n) = 360n - 720$

$\Rightarrow$ $5n^2 + 235n = 360n - 720$

$\Rightarrow$ $5n^2 - 125n + 720 = 0$

$\Rightarrow$ $n^2 - 25n + 144 = 0$ [divide both sides by 5]

$\Rightarrow$ $(n - 9)(n - 16) = 0$

∴ $n = 9$ or 16

But, if $n = 16$, last (greatest) angle of the polygon $= 120 + (16 - 1)5 = 195°$.

No, interior angle of the polygon can be greater than or equal to 180°. So, $n \neq 16$. Therefore, $n = 9$

Hence, the number of sides of polygon is 9.

Arithmetic Mean (AM)

Let a and b be any two numbers and A be a number between them so that a, A, b are in AP, then such a number A is called an **arithmetic mean.**

Thus, a, A, b are in AP.

∴ $A - a = b - A$ [common difference]

$\Rightarrow$ $2A = a + b \Rightarrow A = \frac{a + b}{2}$

Thus, Arithmetic mean (or average) between two given numbers is equal to half of their sum.

i.e. $$\text{AM} = \frac{a + b}{2}$$

EXAMPLE |24| Find AM between 16 and 20.

Sol. $\text{AM} = \frac{a + b}{2} = \frac{16 + 20}{2} = \frac{36}{2} = 18$

INSERTING n AM'S BETWEEN TWO POSITIVE NUMBERS

Let a and b be any two positive numbers and $A_1, A_2, A_3, \ldots, A_n$ be the n AM's between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ are in AP and total number of terms $= n + 2$.

Here, b is $(n + 2)$th term.

i.e. $b = T_{n+2} = a + (n + 2 - 1)d$

$\Rightarrow$ $b = a + (n + 1)d$

$\Rightarrow$ $d = \frac{b - a}{n + 1}$

Thus, the n arithmetic means between a and b are given below

$$A_1 = a + d = a + \frac{b - a}{n + 1}$$
$$A_2 = a + 2d = a + \frac{2(b - a)}{n + 1}$$
$$A_3 = a + 3d = a + \frac{3(b - a)}{n + 1}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$A_n = a + nd = a + \frac{n(b - a)}{n + 1}$$

Note

The sum of n AM's between a and b is equal to n times the single arithmetic mean between a and b

i.e. $A_1 + A_2 + \ldots + A_n = n\left(\frac{a + b}{2}\right)$.

EXAMPLE |25| Insert 6 arithmetic means between 3 and 24.

Sol. Let A_1, A_2, A_3, A_4, A_5 and A_6 be six arithmetic means between 3 and 24.

Then, $3, A_1, A_2, A_3, A_4, A_5, A_6, 24$ are in AP and number of terms is 8.

∴ $a = 3$, and $T_8 = 24$

$\Rightarrow$ $a + (8 - 1)d = 24$ [$\because T_n = a + (n - 1)d$]

$\Rightarrow$ $a + 7d = 24$

$\Rightarrow$ $3 + 7d = 24$

$\Rightarrow$ $7d = 21 \Rightarrow d = 3$

Now, $A_1 = a + d = 3 + 3 = 6$

$A_2 = a + 2d = 3 + 2(3) = 9$

$A_3 = a + 3d = 3 + 3(3) = 12$

$A_4 = a + 4d = 3 + 4(3) = 15$

$A_5 = a + 5d = 3 + 5(3) = 18$

and $A_6 = a + 6d = 3 + 6(3) = 21$

Hence, 6 arithmetic means between 3 and 24 are 6, 9, 12, 15, 18 and 21.

EXAMPLE |26| Prove that the sum of n arithmetic means between two numbers is n times the single AM between them.

Sol. Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an AP with common difference d given by $d = \frac{b-a}{n+1}$.

Now, $A_1 + A_2 + \ldots + A_n = \frac{n}{2}(A_1 + A_n)$ $\left[\because S_n = \frac{n}{2}(a+l)\right]$

$= \frac{n}{2}(a+b)$

[$\because a, A_1, A_2, \ldots, A_n, b$ is an AP, so $a + b = A_1 + A_n$]

$= n\left(\frac{a+b}{2}\right) = n \times$ (AM between a and b)

EXAMPLE |27| If the AM between pth and qth terms of an AP be equal to the AM between rth and sth terms of the AP, then show that $p + q = r + s$.

Sol. Let a be the first term and d be the common difference of the given AP.

Then, $a_p = p\text{th term} = a + (p-1)d$;

$a_q = q\text{th term} = a + (q-1)d$

$a_r = r\text{th term} = a + (r-1)d$

and $a_s = s\text{th term} = a + (s-1)d$

It is given that,

AM between a_p and a_q = AM between a_r and a_s

$\Rightarrow \frac{1}{2}(a_p + a_q) = \frac{1}{2}(a_r + a_s)$

$\Rightarrow a_p + a_q = a_r + a_s$

$\Rightarrow \{a + (p-1)d\} + \{a + (q-1)d\}$

$= \{a + (r-1)d\} + \{a + (s-1)d\}$

$\Rightarrow (p + q - 2)d = (r + s - 2)d$

$\therefore p + q = r + s$

EXAMPLE |28| If $\frac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is AM between a and b, then find the value of n.

We know that, arithmetic mean between two numbers a and b is $\frac{a+b}{2}$. Put it equal to the given arithmetic mean and solve it. **[NCERT]**

Sol. We know that AM between a and b is $\frac{a+b}{2}$.

$\therefore \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \frac{a+b}{2}$

$\Rightarrow 2a^n + 2b^n = a^n + ab^{n-1} + ba^{n-1} + b^n$

$\Rightarrow 2a^n - a^n + 2b^n - b^n = ab^{n-1} + ba^{n-1}$

$\Rightarrow a^n + b^n = ab^{n-1} + ba^{n-1}$

$\Rightarrow a^n - ba^{n-1} = ab^{n-1} - b^n$

$\Rightarrow a^{n-1}[a-b] = b^{n-1}[a-b]$

$\Rightarrow a^{n-1} = b^{n-1}$ $[\because a \neq b]$

$\Rightarrow \left(\frac{a}{b}\right)^{n-1} = 1 = \left(\frac{a}{b}\right)^0$ $[\because x^0 = 1]$

On comparing the exponential powers, we get

$n - 1 = 0 \Rightarrow n = 1$

EXAMPLE |29| Suppose x and y are two real numbers such that the rth mean between x and $2y$ is equal to the rth mean between $2x$ and y when n arithmetic means are inserted between them in both the cases. Show that $\frac{n+1}{r} - \frac{y}{x} = 1$.

Sol. Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between x and $2y$. Then, $x, A_1, A_2, \ldots, A_n, 2y$ are in AP with common difference d_1 given by $d_1 = \frac{2y-x}{n+1}$.

$\therefore$ rth mean $= A_r = x + rd_1 = x + r\left(\frac{2y-x}{n+1}\right)$

$\therefore$ Let $A'_1, A'_2, \ldots, A'_n$ be n arithmetic means between $2x$ and y. Then, $2x, A_1', A_2', \ldots, A_n', y$ are in AP with common difference d_2 given by $d_2 = \left(\frac{y-2x}{n+1}\right)$

$\therefore$ rth mean, $A'_r = 2x + rd_2$

$= 2x + r\left(\frac{y-2x}{n+1}\right)$

It is given that, $A_r = A_r'$

$\Rightarrow x + r\left(\frac{2y-x}{n+1}\right) = 2x + r\left(\frac{y-2x}{n+1}\right)$

$\Rightarrow (n+1)x + r(2y-x) = (n+1)2x + r(y-2x)$

$\Rightarrow (n+1)x - ry = rx$

$\therefore \frac{n+1}{r} - \frac{y}{x} = 1$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 Let an AP $a, a+d, a+2d, \ldots, l$ has n terms, then $S_n = \frac{n}{2}[2a+(n-1)d]$ can also be written as

(a) $S_n = \frac{n}{2}[a-l]$ (b) $S_n = n[a+l]$

(c) $S_n = \frac{n}{2}[a+l]$ (d) $S_n = n[a-l]$

2 The income of a person is ₹ 300000 in the first year and he receives an increase of ₹ 10000 to his income per year for the next 19 yr. Then, the total amount he received in 20 yr, is

(a) 7900000 (b) 6000000

(c) 8000000 (d) 6900000

3 If A is the arithmetic mean of the numbers a and b, then

(a) A, a, b, are in AP (b) a, A, b, are in AP

(c) A, b, a, are in AP (d) None of these

4 If A is the arithmetic mean of the numbers a and b, then which of the following is not correct?

(a) $A = \frac{a+b}{2}$ (b) $A - a = b - A$

(c) $a - A = A - b$ (d) $A = \frac{a-b}{2}$

5 If x and y are inserted between 4 and 16, so that the resulting sequence becomes an AP, then

(a) $x = 8, y = 12$ (b) $x = 10, y = 12$

(c) $x = 8, y = 10$ (d) $x = 10, y = 14$

VERY SHORT ANSWER Type Questions

6 If the sum of first n terms of a progression is a quadratic expression in n, then show that it is an AP.

7 If the sum of n terms of an AP is given by $S_n = 3n + 2n^2$, then find the common difference of the AP. **[NCERT Exemplar]**

8 Find the sum of 24 terms of an AP 1, 3, 5, 7, ...

9 Find the sum of 10 terms of an AP 4, $5\frac{1}{3}$, $4\frac{2}{3}$, ...

10 Find the sum of 20 terms of the sequence $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots$

11 Find the sum of 100 terms of the series $0.7 + 0.71 + 0.72 + \ldots$

12 Find the sum of $-0.5, -1.0, -1.5, \ldots$ upto 10 terms.

13 Find the value of x, when $1 + 6 + 11 + \ldots + x = 148$.

14 Determine the sum of first 35 terms of an AP, if its second term is 2 and seventh term is 22.

15 Find the sum of odd integers from 1 to 2001.

16 Find the arithmetic mean between the following (Each part carries 1 mark)

(i) $(a-b)$ and $(a+b)$ (ii) 12 and 22.

17 Insert five numbers bewteen 8 and 26 such that the resulting sequence is an AP. **[NCERT]**

18 Insert the three numbers between 3 and 19 such that the resulting sequence is an AP.

SHORT ANSWER Type I Questions

19 The sum of first 7 terms of an AP is 10 and that of next 7 terms is 17. Find the AP.

20 Find the sum of first 20 terms of an AP, in which 3rd terms is 7 and 7th term is two more than thrice of its 3rd term.

21 Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.

22 Find the sum of all two digit numbers which when divided by 4, yield 1 as remainder. **[NCERT]**

23 Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

24 If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference. **[NCERT]**

25 If the sum of n terms of an AP is $3n^2 + 5n$ and its mth term is 164, find the value of m.

26 On the first day strike of physicians in a hospital, the attendence of the OPD was 1500 patients. As the strike continued, the attendence declined by 100 patients every day. Find from which day of the strike, the OPD would have no patient?

27 The gate receipts at the show of 'Comedy Nights' amounted ₹ 9500 on the first night and showed a drop of ₹ 250 every succeeding night. If the operational expenses of the show are ₹ 2000 a day, then find on which night, the show ceases to be profitable?

28 A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job? [NCERT Exemplar]

SHORT ANSWER Type II Questions

29 How many terms of the AP 18, 16, 14, 12, ... are needed to give the sum 78? Explain the double answer?

30 The sum of n terms of two arithmetic progressions are in the ratio $(7n - 5) : (5n + 17)$. Show that their 6th terms are equal.

31 If in an AP, $S_n = qn^2$ and $S_m = qm^2$, where S_r denotes the sum of r terms of the AP, then find S_q. [NCERT Exemplar]

32 Let S_n denote the sum of the first n terms of an AP. If $S_{2n} = 3S_n$, then find $S_{3n} : S_n$.

[NCERT Exemplar]

33 In an AP, if the pth term is $\frac{1}{q}$ and qth term is $\frac{1}{p}$.

Prove that the sum of first pq terms is $\frac{1}{2}(pq + 1)$ where $p \neq q$. [NCERT]

34 If the sum of first p terms of an AP is equal to the sum of first q terms, then find the sum of first $(p + q)$ terms. [NCERT]

35 Let the sum of n, $2n$, $3n$ terms of an AP be S_1, S_2 and S_3, respectively. Show that $S_3 = 3(S_2 - S_1)$.

36 Sum of the first p, q and r terms of an AP are a, b and c, respectively. Prove that

$$\frac{a}{p}(q - r) + \frac{b}{q}(r - p) + \frac{c}{r}(p - q) = 0.$$

37 If the first term of an AP is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, then show that the 20th term is –112. [NCERT]

38 The sum of interior angles of a triangle is 180°. Show that the sum of the interior angles of polygons with 3, 4, 5, 6, ... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon.

39 The sum of two numbers is $\frac{13}{6}$. An even number of AM's are being inserted between them. The sum of means inserted exceeds the number of means by 1. Find the number of AM's inserted.

40 Insert four AM's between $\frac{1}{2}$ and 3 and prove that $A_1 + A_2 + A_3 + A_4 = 7$.

41 Between 1 and 31, m AM's have been inserted in such a way that the ratio of the 7th and $(m - 1)$th means is 5 : 9. Find the value of m. [NCERT]

42 If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of n.

LONG ANSWER Type Questions

43 The digits of a positive number having three digits are in AP and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.

44 A farmer buys a used tractor of ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalment of ₹ 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

45 A man deposited ₹ 10000 in a bank at the rate of 5% simple interest to annually. Find the amount in 15th yr, since he deposited the amount and also calculate the amount after 20 yr.

46 Find the sum of integers from 1 to 100 that are divisible by 2 or 5. [NCERT]

HINTS & ANSWERS

1. (c) We know, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$= \frac{n}{2}[a + a + (n-1)d]$$

$$= \frac{n}{2}[a + l] \qquad [\because l = a + (n-1)d]$$

2. (a) Here, we have an A.P. with $a = 300000$

$d = 10000$ and $n = 20$.

Using the sum formula, we get

$$S_{20} = \frac{20}{2}[600000 + 19 \times 10000] = 7900000$$

3. (b) If A is the arithmetic mean of the numbers a and b, then a, A, b are in A.P. .

4. (d) If A is the arithmetic mean of the numbers a and b, then

$$A - a = b - A$$

$\Rightarrow \quad a - A = A - b$

Also, $\quad A = \frac{a+b}{2}$

5. (a) Since, 4, 8, 12, 16 forms an AP with common difference 4.

$\therefore \; x = 8$ and $y = 12$

6. Let $S_n = an^2 + bn + c$ ...(i)

where a, b, c are constant and $a \neq 0$.

On replacing n by $(n-1)$ in Eq. (i) we get

$$S_{n-1} = a(n-1)^2 + b(n-1) + c$$

$$= a(n^2 + 1 - 2n) + b(n-1) + c \quad ...(ii)$$

Now, $a_n = S_n - S_{n-1}$

$$= (an^2 + bn + c) - [a(n^2 + 1 - 2n) + b(n-1) + c]$$

$$= 2an + (b - a)$$

$\therefore \quad a_{n-1} = 2a(n-1) + (b-a) \quad$ [replacing n by $(n-1)$]

$\Rightarrow a_n - a_{n-1} = 2a$, which is constant.

Hence, the given progression is an AP.

7. 4

8. $S_{24} = \frac{24}{2}[2 \times 1 + (24-1) \times 2]$ **Ans.** 576

9. $S_{10} = \frac{10}{2}\left[2 \times 6 + (10-1)\left(\frac{-2}{3}\right)\right]$ **Ans.** 30

10. $S_{20} = \frac{20}{2}[2 \times \sqrt{2} + (20-1)\sqrt{2}]$ **Ans.** $210\sqrt{2}$

11. $S_{100} = \frac{100}{2}[2 \times 0.7 + (100-1)(0.01)]$ **Ans.** 119.5

12. – 27.5

13. $a = 1, d = 5, S_n = 148$ **Ans.** $x = 36$

14. 2310

15. 1002001

16. (i) a (ii) 17

17. Solve as Example 25. **Ans.** 11, 14, 17, 21, 24

18. Solve as Example 25. **Ans.** 7, 11, 15

19. 1, $1\frac{1}{7}$, $1\frac{2}{7}$, ...

20. 740

21. 156375

22. 1210

23. 98450

24. Common difference = $2q$

25. $m = 27$

26. The 16th day of strike, the OPD will have no patient.

27. 1400

28. 12

29. 6, 13

31. $S_q = q^3$

32. 6 : 1

34. $S_{p+q} = 0$

38. We know that, sum of interior angles of a polygon of side $n = (2n-4) \times 90° = (n-2) \times 180°$

The sequence will be 180°, 360° 540°, 720°, 900°, ...

We have to find the sum of interior angles of a 21 sides polygon. It means, we have to find the 19th term of the above series. [since, 180° is the sum of three angle]

Ans. 3420

39. Let a and b be two numbers and $A_1, A_2, \ldots A_{2n}$ be $2n$ (even numbers) AMs. Then, $a + b = \frac{13}{6}$

and $A_1 + A_2 + \ldots + A_{2n} = 2n\left(\frac{a+b}{2}\right) = \left(\frac{13}{6}\right)n$

According to the question,

$A_1 + A_2 + \ldots + A_{2n} = 2n + 1 \Rightarrow n = 6$ **Ans.** 12

40. Solve as Example 25. **Ans.** 1, $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}$

41. Let $A_1, A_2, A_3, A_4, \ldots, A_m$ be m AM's between 1 and 31.

Therefore, 1, $A_1, A_2, A_3, \ldots, A_m$, 31 are in AP.

Here, the total number of terms is $m + 2$ and $T_{m+2} = 31$

$\Rightarrow \quad 1 + (m + 2 - 1)d = 31 \Rightarrow (m+1)d = 30$

$$\Rightarrow \quad d = \frac{30}{m+1} \quad ...(i)$$

$\therefore \quad A_7 = T_8 = a + 7d$

$$= 1 + 7 \times \frac{30}{m+1} = \frac{m + 211}{m+1} \quad \text{[from Eq. (i)]}$$

$$A_{m-1} = T_m = 1 + (m-1)d = 1 + (m-1)\frac{30}{m+1}$$

$$= \frac{m + 1 + 30m - 30}{m+1} \quad \text{[from Eq. (i)]}$$

$$= \frac{31m - 29}{m+1}$$

$$\therefore \quad \frac{A_7}{A_{m-1}} = \frac{(m + 211)/(m+1)}{(31m - 29)/m+1} = \frac{m + 211}{31m - 29} \Rightarrow m = 14$$

42. Let $A_1, A_2, \ldots, A_n$ be n arithmetic means betwenn 20 and 80 and d be the common difference of the AP

$20, A_1, A_2, \ldots, A_n, 80$. Here, $a = 20, b = 80$

Then, $d = \dfrac{b-a}{n+1} = \dfrac{60}{n+1}$

Now, $A_1 = a + d = 20 + \dfrac{60}{n+1} = \dfrac{20(n+4)}{n+1}$

and $A_n = a + nd = 20 + \dfrac{60n}{n+1} = \dfrac{20(4n+1)}{n+1}$

According to the question, $A_1 : A_n = 1:3$

$$\Rightarrow \frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{\frac{20(n+4)}{n+1}}{\frac{20(4n+1)}{n+1}} = \frac{1}{3} \Rightarrow \frac{n+4}{4n+1} = \frac{1}{3}$$

$\Rightarrow 3n + 12 = 4n + 1 \Rightarrow 4n - 3n = 12 - 1 \Rightarrow n = 11$

43. Here, $a - d + a + a + d = 15 \Rightarrow 3a = 15 \Rightarrow a = 5$

Original number $= 100(a+d) + 10a + (a-d)$
$= 111a + 99d$ ...(i)

Number after reversing the digits
$= 100(a-d) + 10a + a + d = 111a - 99d$ ...(ii)

According to the question, $111a + 99d - 594 = 111a - 99d$

$\Rightarrow d = 3$ **Ans.** 852

44. ₹ 16680

45. Interest on ₹ 10000 after 1 yr

$$= \frac{10000 \times 5 \times 1}{100} = ₹500$$

Now, interest on ₹ 10000 after 2 yr

$$= \frac{10000 \times 5 \times 2}{100} = ₹1000$$

$\therefore$ Amount after 2 yr $= 10000 + 1000 = ₹11000$

Hence, the amout in the account of man in first, second and third years are ₹10000, 10500, 11000, ...

It is an AP, where $a = 10000$ and $d = 500$.

Ans. T_{15} = ₹17000 and T_{20} = ₹ 20000

46. The numbers from 1 to 100 which are divisible by 2 are 2, 4, 6, 8, ..., 100. Then, $100 = 2 + (n-1)\,2 \Rightarrow n = 50$

$$S_{50} = \frac{50}{2}(2 \times 2 + (50-1)2) \Rightarrow S_{50} = 2550$$

The numbers from 1 to 100 which are divisible by 5 are 5, 10, 15, 20, ...,100.

$\therefore$ $100 = 5 + (n-1)5 \Rightarrow n = 20$

and $S_{20} = \dfrac{20}{2}[2 \times 5 + (20-1)5] \Rightarrow S_{20} = 1050$

The numbers from 1 to 100 which are divisible by 10 are 10, 20, 30, ..., 100.

Then, $n = 10$ and $S_{10} = \dfrac{10}{2}[2 \times 10 + (10-1)10] = 550$

Hence, required sum of integers from 1 to 100 which are divisible by 2 or 5 = 2550 + 1050 − 550
= 3600 − 550 = 3050 **Ans.** 3050

|TOPIC 4|

Geometric Progression (GP)

A sequence of non-zero numbers is said to be a geometric progression, if the ratio of each term, except the first one, by its preceding term is always constant.

In other words, we can say that a sequence $a_1, a_2, \ldots, a_n$ is called geometric progression (geometric sequence), if it follows the relation $\dfrac{a_{k+1}}{a_k} = r$ (constant) for all $k \in N$.

The constant ratio is called **common ratio** of the GP and it is denoted by r. In a GP, we usually denote the first term by a, the **nth term** by T_n or a_n.

Thus, a GP can be written as $a, ar, ar^2, ar^3, \ldots$ and so on.

e.g. The sequence 4, 12, 36, 108, ... is a GP.

Here, $a = 4$, $r = \dfrac{12}{4} = \dfrac{36}{12} = \dfrac{108}{36} = \ldots = 3$

Geometric Series

Let $a_1, a_2, a_3, \ldots, a_n$ be a GP. Then, the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a geometric series. Geometric series is finite, if the corresponding GP is finite otherwise it is an infinite geometric series.

Note

To show that given numbers a, b, c, d and so on are in GP, we have to show that $\dfrac{b}{a} = \dfrac{c}{b}$.

EXAMPLE |1| For what values of k, the numbers $-\dfrac{2}{7}, k, -\dfrac{7}{2}$ are in GP? **[NCERT]**

Sol. If $-\dfrac{2}{7}, k, -\dfrac{7}{2}$ are in GP.

Then, $\dfrac{a_2}{a_1} = \dfrac{a_3}{a_2}$ $\left[\because \text{common ratio } (r) = \dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \dfrac{a_4}{a_3} = \ldots\right]$

$\therefore$ $\dfrac{k}{-2/7} = \dfrac{-7/2}{k}$

$\Rightarrow$ $\dfrac{7}{-2}k = \dfrac{-7}{2} \times \dfrac{1}{k}$

$\Rightarrow$ $7k \times 2k = -7 \times (-2)$

$\Rightarrow$ $14k^2 = 14$

$\Rightarrow$ $k^2 = 1 \Rightarrow k = \pm 1$

EXAMPLE |2| If a, b and c are in GP, then find the value of $\frac{a-b}{b-c}$.

Sol. Given that, a, b and c are in GP.

Then, $\frac{b}{a}=\frac{c}{b}=r$ (constant)

$\Rightarrow \quad b=ar,\ c=br$

Now, $\frac{a-b}{b-c}=\frac{a-ar}{ar-br}$

$=\frac{a(1-r)}{r(a-b)}=\frac{a(1-r)}{r(a-ar)}=\frac{a(1-r)}{ar(1-r)}=\frac{1}{r}$

$\Rightarrow \quad \frac{a-b}{b-c}=\frac{1}{r}=\frac{a}{b}$ or $\frac{b}{c}$

EXAMPLE |3| If a and b are the roots of $x^2-3x+p=0$ and c and d are roots of $x^2-12x+q=0$, where a, b, c, d form a GP prove that $(q+p):(q-p)=17:15$.

Sol. Given, a and b are roots of $x^2-3x+p=0$.

$\therefore$ Sum of roots, $a+b=-\frac{(-3)}{1}=3$

$\left[\because \text{sum of roots} = -\frac{(\text{coefficient of } x)}{(\text{coefficient of } x^2)}\right]$

$\Rightarrow \quad a+b=3 \quad \ldots(i)$

and product of roots, $ab=p \quad \ldots(ii)$

Again, given that c and d are roots of $x^2-12x+q=0$

$\therefore$ Sum of roots, $c+d=-\frac{(-12)}{1}=12$

$\Rightarrow \quad c+d=12 \quad \ldots(iii)$

and product of roots, $cd=q \quad \ldots(iv)$

$\left[\because \text{product of roots} = \frac{\text{constant term}}{\text{coefficient of } x^2}\right]$

Again, it is given that a, b, c, d are in GP.

$\Rightarrow \quad b=ar,\ c=ar^2,\ d=ar^3$

On putting these values in Eqs. (i) and (iii), then dividing Eq. (i) by Eq. (iii), we get

$\frac{a+ar}{ar^2+ar^3}=\frac{3}{12} \Rightarrow \frac{a(1+r)}{ar^2(1+r)}=\frac{1}{4}$

$\Rightarrow \quad \frac{1}{r^2}=\frac{1}{4} \Rightarrow r^2=4 \Rightarrow r=\pm 2$

When $r=2$, then from Eq. (i), we get

$a+ar=3 \Rightarrow a+2a=3$

$\Rightarrow \quad 3a=3 \Rightarrow a=1$

So, GP is

$a=1,\ b=ar=1\times 2=2,\quad c=ar^2=1\times 2^2=4,$

$d=ar^3=1\times 2^3=8$

From Eq. (ii), $p=ab=1\times 2=2$

From Eq. (iv), $q=cd=4\times 8=32$

Now, $\frac{q+p}{q-p}=\frac{32+2}{32-2}=\frac{34}{30}=\frac{17}{15}$

Hence, $(q+p):(q-p)=17:15$

When $r=-2$, then from Eq. (i), $a+ar=3$

$\Rightarrow \quad a-2a=3$

$\Rightarrow \quad a=-3$

So, GP is $a=-3,\ b=ar=(-3)(-2)=6$

$c=ar^2=(-3)(-2)^2=-12$

$d=ar^3=(-3)(-2)^3=24$

From Eq. (ii), $p=ab=(-3)\cdot(+6)=-18$

From Eq. (iv), $q=cd=(-12)(24)=-288$

$\therefore \quad \frac{q+p}{q-p}=\frac{-288-18}{-288+18}=\frac{-306}{-270}=\frac{17}{15}$

Hence, $(q+p):(q-p)=17:15$

General Term of a GP

If a is the first term of a GP and its common ratio is r, then general term or nth term, $T_n=ar^{n-1}$

or $l=ar^{n-1}$, where l is the last term.

mth Term of a Finite GP from the End

Let a be the first term and r be the common ratio of a GP having n terms. Then, mth term from the end is $(n-m+1)$th term from the beginning.

$\therefore \quad m$th term from the end $=ar^{n-m+1-1}=ar^{n-m}$

where, $n>m$. Also, mth term from the end $=l\left(\frac{1}{r}\right)^{n-1}$,

where, l is the last term of the finite GP.

Problems Based on General Term of a GP

Some problems can be solved directly or indirectly with the help of general term.

Some types are given below

TYPE I

PROBLEMS BASED ON FINDING THE INDICATED TERM

In this type of problems, first term and common ratio of a GP are given and we have to find the value of indicated term.

EXAMPLE |4| Find the nth term and the 12th term of the sequence $-6, 18, -54, \ldots$.

Sol. Given sequence is $-6, 18, -54, \ldots$.

Clearly, the successive ratio of the terms is same.
So, the given sequence forms a GP with first term, $a=-6$ and common ratio, $r=\frac{18}{-6}=-3$.

$$\therefore \quad T_n = ar^{n-1} = -6(-3)^{n-1} = (-1)^n 6\cdot 3^{n-1}$$

and
$$T_{12} = (-1)^{12} 6\cdot 3^{12-1} = (1)\cdot 6\cdot 3^{11} = 2\cdot 3\cdot 3^{11} = 2\cdot 3^{12}$$

Hence, the nth term of given GP is $(-1)^n 6\cdot 3^{n-1}$ and 12th term is $2\cdot 3^{12}$.

EXAMPLE |5| If the pth and qth terms of a GP are q and p respectively, then show that its $(p+q)$th term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

Sol. Let the first term and common ratio of GP be a and r, respectively.
According to the question, pth term $= q$

$$\Rightarrow \quad a\cdot r^{p-1} = q \quad \ldots(i)$$

and qth term $= p$

$$\Rightarrow \quad ar^{q-1} = p \quad \ldots(ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{ar^{p-1}}{ar^{q-1}} = \frac{q}{p} \Rightarrow r^{p-1-q+1} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-q} = \frac{q}{p} \Rightarrow r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

On substituting the value of r in Eq. (i), we get

$$a\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}} = q \Rightarrow a = \frac{q}{\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}}} = q\cdot\left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

$\therefore$ $(p+q)$th term, $T_{p+q} = a\cdot r^{p+q-1}$

$$= q\cdot\left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}\cdot(r)^{p+q-1} = q\cdot\left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}\times\left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= \frac{q^{1-\frac{p-1}{p-q}+\frac{p+q-1}{p-q}}}{p^{\frac{p+q-1}{p-q}-\frac{p-1}{p-q}}} = \frac{q^{\frac{p-q-p+1+p+q-1}{p-q}}}{p^{\frac{p+q-1-p+1}{p-q}}}$$

$$= \frac{q^{\frac{p}{p-q}}}{p^{\frac{q}{p-q}}} = \left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$$

EXAMPLE |6| The $(m+n)$th and $(m-n)$th terms of a GP are p and q respectively. Show that the mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{m/2n}$, respectively.

Sol. Let a be the first term and r be the common ratio. then,

$$a_{m+n} = p \text{ and } a_{m-n} = q$$

$$\Rightarrow \quad ar^{m+n-1} = p \text{ and } ar^{m-n-1} = q$$

$$\Rightarrow \quad \frac{ar^{m+n-1}}{ar^{m-n-1}} = \frac{p}{q} \Rightarrow r^{2n} = \frac{p}{q}$$

$$\Rightarrow \quad r = \left(\frac{p}{q}\right)^{1/2n} \Rightarrow \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}$$

Now, $\quad a_m = ar^{m-1}$

$$\Rightarrow \quad a_m = ar^{(m+n-1)}\left(\frac{1}{r}\right)^n$$

$$\Rightarrow \quad a_m = a_{m+n}\left(\frac{1}{r}\right)^n \quad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{n/2n} \quad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{1/2} = \sqrt{pq}$$

and $\quad a_n = ar^{n-1}$

$$\Rightarrow \quad a_n = ar^{m+n-1}\left(\frac{1}{r}\right)^m = a_{m+n}\left(\frac{1}{r}\right)^m \quad [\because a_{m+n} = ar^{m+n-1}]$$

$$\therefore \quad a_n = p\left(\frac{q}{p}\right)^{m/2n} \quad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

TYPE II

PROBLEMS BASED ON FINDING THE POSITION OF A GIVEN TERM

In this type of problems, a GP is given from which we get first term and common ratio. Then, assume that the given term is nth term and find the value of n. This value of n gives its position in given GP.

EXAMPLE |7| Which term of the GP 5, 10, 20, 40, ... is 5120?

Sol. Given GP is 5, 10, 20, 40, ...

Here, $\quad a = 5$ and $r = \frac{10}{5} = 2$

Let nth term of given GP $= 5120$ i.e. $T_n = 5120$

Now, $\quad T_n = ar^{n-1} = 5120$

$$\Rightarrow \quad 5(2)^{n-1} = 5120 \quad [\because a = 5, r = 2]$$

$\Rightarrow \quad 2^{n-1} = \frac{5120}{5} = 1024$

$\Rightarrow \quad 2^{n-1} = 1024$

$\Rightarrow \quad 2^{n-1} = 2^{10}$

On equating the powers, we get

$n - 1 = 10$

$\Rightarrow \quad n = 10 + 1 = 11$

Hence, 11th term of given GP is 5120.

Note

We can equate the powers from both sides only when the bases are same i.e. if $b^n = b^m$, then we can equate the powers (here, b is base).

TYPE III

PROBLEMS BASED ON FINDING THE GENERAL TERM FROM END

In this type of problems, a finite GP is given from which we can find a and r and then we find a general term or mth term from the end of given finite GP.

EXAMPLE |8| Find the 8th term from the end of the sequence 3, 6, 12, ... 25th term.

Sol. We know that if a sequence has n terms, then mth term from end is equal to $(n - m + 1)$th term from the beginning.

Here, $a = 3$, $r = \frac{6}{3} = 2$, $m = 8$ and $n = 25$.

Now, 8th term from the end of sequence is equal to the $(25 - 8 + 1)$. i.e. 18th term from the beginning.

$\because \quad T_n = ar^{n-1}$

$\therefore \quad T_{18} = ar^{18-1} = 3(2)^{18-1} \qquad [\because a = 3, r = 2]$

$= 3(2)^{17} = 3 \times 131072 = 393216$

Hence, the 8th term from the end is 393216.

TYPE IV

PROBLEMS BASED ON FINDING THE GP

In this type of problem, two or more than two terms are given and we have to find the GP by using these terms. It can be understand with the help of following example.

EXAMPLE |9| If the 4th and 9th term of a GP are 54 and 13122 respectively, then find the GP.

Sol. Let a be the first term and r be the common ratio of GP.

Given, 4th term, $T_4 = 54$

$\Rightarrow \quad ar^{4-1} = 54 \Rightarrow ar^3 = 54 \qquad \text{...(i)}$

and 9th term, $T_9 = 13122$

$\Rightarrow \quad ar^{9-1} = 13122$

$\Rightarrow \quad ar^8 = 13122 \qquad \text{...(ii)}$

On dividing Eq. (ii) by Eq. (i), we get

$\frac{ar^8}{ar^3} = \frac{13122}{54} \Rightarrow r^5 = 243 \Rightarrow r^5 = (3)^5$

$\therefore \quad r = 3$

On putting the value of r in Eq. (i), we get

$a(3)^3 = 54 \Rightarrow 27a = 54 \Rightarrow a = \frac{54}{27} = 2$

$\therefore$ Required GP is $a, ar, ar^2, ar^3, \ldots$

i.e. 2, 6, 18, 54, ...

EXAMPLE |10| Find four numbers forming a GP in which the third term is greater than the first term by 9 and the second term is greater than 4th by 18.

Sol. Let the GP is $a, ar, ar^2, ar^3, \ldots$

Given, third term = first term +9

$\Rightarrow \quad T_3 = a + 9 \Rightarrow ar^2 = a + 9$

$\Rightarrow \quad ar^2 - a = 9 \qquad \text{...(i)}$

Again, second term = fourth term +18

$T_2 = T_4 + 18 \Rightarrow ar = ar^3 + 18$

$\Rightarrow \quad ar - ar^3 = 18 \qquad \text{...(ii)}$

On dividing Eq. (i) by Eq. (ii), we get

$\frac{ar^2 - a}{ar - ar^3} = \frac{9}{18} \Rightarrow \frac{a(r^2 - 1)}{ar(1 - r^2)} = \frac{1}{2}$

$\Rightarrow \quad \frac{-1}{r}\frac{(1 - r^2)}{(1 - r^2)} = \frac{1}{2} \Rightarrow -\frac{1}{r} = \frac{1}{2} \Rightarrow r = -2$

On putting $r = -2$ in Eq. (ii)

$a(-2) - a(-2)^3 = 18 \Rightarrow -2a + 8a = 18$

$\Rightarrow \quad 6a = 18 \Rightarrow a = 3$

$\therefore$ GP is $3, 3(-2), 3(-2)^2, 3(-2)^3, \ldots$

i.e. $\quad 3, -6, 12, -24, \ldots$

Note Here, greater than word stands for excess in term.

Selection of Terms in GP

Sometimes we have to select certain number of terms in GP. *The convenient method of selecting terms is given below*

Number of terms	Terms	Common ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

It is clear from the above table that, if number of terms is odd, then the middle term is a and common difference is r and if number of terms is even, then the middle terms are $\frac{a}{r}$, ar and the common ratio is r^2.

Note

If the terms of the GP are not given, then the terms are chosen as a, $ar, ar^2, ar^3, \dots$.

EXAMPLE |11| Find four numbers in GP, whose sum is 85 and product is 4096.

Sol. Let the four numbers in GP be

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3 \quad \dots \text{(i)}$$

Product of four numbers = 4096 [given]

$$\Rightarrow \left(\frac{a}{r^3}\right)\left(\frac{a}{r}\right)(ar)(ar^3) = 4096$$

$$\Rightarrow \quad a^4 = 4096 \Rightarrow a^4 = 8^4$$

On comparing the base of the power 4, we get

$$a = 8$$

and sum of four numbers = 85 [given]

$$\Rightarrow \frac{a}{r^3} + \frac{a}{r} + ar + ar^3 = 85 \Rightarrow a\left[\frac{1}{r^3} + \frac{1}{r} + r + r^3\right] = 85$$

$$\Rightarrow 8\left[r^3 + \frac{1}{r^3}\right] + 8\left[r + \frac{1}{r}\right] = 85 \quad [\because a = 8]$$

$$\Rightarrow 8\left[\left(r + \frac{1}{r}\right)^3 - 3\left(r + \frac{1}{r}\right)\right] + 8\left(r + \frac{1}{r}\right) = 85$$

$$[\because a^3 + b^3 = (a+b)^3 - 3(a+b)]$$

$$\Rightarrow 8\left(r + \frac{1}{r}\right)^3 - 16\left(r + \frac{1}{r}\right) - 85 = 0 \quad \dots \text{(ii)}$$

On putting $\left(r + \frac{1}{r}\right) = x$ in Eq. (ii), we get

$$8x^3 - 16x - 85 = 0$$

$$\Rightarrow (2x - 5)(4x^2 + 10x + 17) = 0$$

$$\Rightarrow 2x - 5 = 0 \quad [\because 4x^2 + 10x + 17 = 0 \text{ has imaginary roots}]$$

$$\Rightarrow \quad x = \frac{5}{2} \Rightarrow r + \frac{1}{r} = \frac{5}{2} \quad \left[\text{put } x = r + \frac{1}{r}\right]$$

$$\Rightarrow \quad 2r^2 - 5r + 2 = 0$$

$$\Rightarrow \quad (r-2)(2r-1) = 0 \Rightarrow r = 2 \text{ or } r = \frac{1}{2}$$

On putting $a = 8$ and $r = 2$ or $r = \frac{1}{2}$ in Eq. (i), we obtain four numbers as

$$\frac{8}{2^3}, \frac{8}{2}, 8 \times 2, 8 \times 2^3 \text{ or } \frac{8}{(1/2)^3}, \frac{8}{(1/2)}, 8 \times \frac{1}{2}, 8 \times \left(\frac{1}{2}\right)^3$$

i.e. 1, 4, 16, 64 or 64, 16, 4, 1.

EXAMPLE |12| Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a GP. Then, find $P^2R^3 : S^3$.

Sol. Let us take a GP with three terms $\frac{a}{r}, a, ar$.

Then, $$S = \frac{a}{r} + a + ar = \frac{a(r^2 + r + 1)}{r}$$

$$P = \left(\frac{a}{r}\right)(a)(ar) = a^3 \text{ and } R = \frac{r}{a} + \frac{1}{a} + \frac{1}{ar} = \frac{1}{a}\left(\frac{r^2 + r + 1}{r}\right)$$

Now, $$\frac{P^2R^3}{S^3} = \frac{a^6 \cdot \frac{1}{a^3}\left(\frac{r^2 + r + 1}{r}\right)^3}{a^3\left(\frac{r^2 + r + 1}{r}\right)^3} = 1$$

Therefore, the required ratio is 1 : 1.

PROPERTIES OF GEOMETRIC PROGRESSION

(i) If all the terms of a GP are multiplied by the same quantity, then the resulting sequence is also a GP with the same common ratio.

(ii) The reciprocals of the terms of a given GP also form a GP.

(iii) If each term of a GP is raised to same power, then the resulting sequence is also a GP.

(iv) In a finite GP, the product of the terms equidistant from the beginning and from the end is always same and is equal to the product of the first and the last terms.

(v) The resulting sequence formed by taking the product of the corresponding terms of two GP's is also a GP.

(vi) The resulting sequence formed by dividing the terms of a GP by the corresponding terms of another GP is also a GP.

(vii) If the terms of a given GP are chosen at regular intervals, then the new sequence forms a GP.

EXAMPLE |13| If a, b, c and d are in GP, then show that $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$. **[NCERT]**

Sol. Given, a, b, c, d are in GP.

$$\therefore \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r \text{ (say)}$$

$$\Rightarrow \quad b = ar, c = br, d = cr$$

$$\Rightarrow \quad b = ar, c = (ar)r, d = (br)r$$

$$\Rightarrow \quad b = ar, c = ar^2, d = br^2$$

$$\Rightarrow \quad b = ar, c = ar^2, d = (ar)r^2 = ar^3 \quad \dots \text{(i)}$$

Now, we have to prove that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$$

$$\text{LHS} = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$
$= a^2(1 + r^2 + r^4)a^2r^2(1 + r^2 + r^4)$
$= a^4r^2(1 + r^2 + r^4)^2 = [a^2r(1 + r^2 + r^4)]^2$
$= [a^2r + a^2r^3 + a^2r^5]^2$
$= [a \cdot ar + ar \cdot ar^2 + ar^2 \cdot ar^3]^2$
$= [ab + bc + cd]^2$ [from Eq. (i)]
= RHS

EXAMPLE |14| If a, b, c are respectively the pth, qth and rth terms of a GP, show that
$(q - r) \log a + (r - p) \log b + (p - q) \log c = 0.$

Sol. Let A be the first term and R the common ratio of the given GP.

Then, $a = p\text{th term} \Rightarrow a = AR^{p-1}$

$\Rightarrow \log a = \log A + (p - 1) \log R$ ...(i)

$b = q\text{th term} \Rightarrow b = AR^{q-1}$

$\Rightarrow \log b = \log A + (q - 1) \log R$ ...(ii)

$c = r\text{th term} \Rightarrow c = AR^{r-1}$

$\Rightarrow \log c = \log A + (r - 1) \log R$ ...(iii)

Now, LHS $= (q - r) \log a + (r - p) \log b + (p - q) \log c$

On substituting the values of $\log a$, $\log b$ and $\log c$, we get

$$\text{LHS} = (q - r)\{\log A + (p - 1)\log R\} + (r - p)\{\log A + (q - 1)\log R\} + (p - q)\{\log A + (r - 1)\log R\}$$
$$= \log A\{(q - r) + (r - p) + (p - q)\} + \log R\{(p - 1)(q - r) + (q - 1)(r - p) + (r - 1)(p - q)\}$$
$$= (\log A) 0 + \{p(q - r) + q(r - p) + r(p - q) - (q - r) - (r - p) - (p - q)\} \log R$$
$$= (\log A) 0 + (\log R) 0 = 0 = \text{RHS}$$

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1. If every term in a progression except the first term bears a constant ratio to the term immediately preceding it, then such progression is called
(a) arithmetic progression
(b) geometric progression
(c) harmonic progression
(d) None of the above

2. If $a, ar, ar^2, ar^3, \ldots$ is a geometric progression, then a and r are respectively called
(a) common ratio, first term
(b) common difference, first term
(c) first term, common difference
(d) first term, common ratio

3. Which of the following is not a geometric sequence?
(a) 2, 4, 8, 16, ... (b) $\frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243}, \ldots$
(c) 0.01, .0001, .000001, ... (d) None of these

4. Which of the following is correct?
(a) $a, ar, ar^2, ar^3, \ldots\ldots, ar^{n-1}$ is a finite G.P.
(b) $a, r, r^2, r^3, \ldots\ldots, r^{n-1}, \ldots$ is a finite G.P.
(c) $a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ is a finite G.P.
(d) $a + r + r^2 + \ldots\ldots + r^{n-1} + \ldots$ is a finite G.P.

5. The n^{th} term of a G.P. 5, 25, 125, ... is
(a) 5^n (b) 5^{n-1} (c) 5^{n+1} (d) 5^{n-2}

VERY SHORT ANSWER Type Questions

6 Which term of the following sequences
(i) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ? (ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?
(Each part carries 1 mark)

7 For what value of x are the numbers $(x + 9)$, $(x - 6)$ and 4 are is GP?

8 Show that the following progressions is a GP. Also, find the common ratio in each case. (Each part carries 1 mark)
(i) $4, -2, 1, -1/2, \ldots$ (ii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \ldots$

9 Find the following. (Each part carries 1 mark)
(i) 11th term of the GP 3, 6, 12, 24, ...
(ii) 10th term of the G.P $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$
(iii) 17th term of the GP $2, 2\sqrt{2}, 4, 8\sqrt{2}, \ldots$
(iv) 8th term of the GP 0.3, 0.06, 0.012, ...
(v) 12th term of the GP $\frac{1}{a^3x^3}, ax, a^5x^5, \ldots$

SHORT ANSWERT Type I Questions

10 Find the 10th term of the GP $5 + 25 + 125 + \ldots$. Also, find its nth term.

11 Find 12th term of a GP, whose 8th term is 192 and common ratio is 2.

12 Find the 6th term from the end of the GP $8, 4, 2, \ldots, \frac{1}{1024}$

13 Find the geometric series whose 5th and 8th terms are 80 and 640, respectively.

SHORT ANSWER Type II Questions

14 If $a, b, c\ d$ are in GP, then prove that $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are also in GP

15 Find the 20th and nth terms of the GP; $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \dots$

16 Prove that in a finite GP the product of the terms equidistant form the beginning and the end is always same and equal to the product of first and last term.

17 Find a GP for which sum of the first two terms is -4 and fifth term is 4 times the third term.

18 The first term of a GP is 1. the sum of the third term and fifth term is 90. Find the common ratio of the GP.

19 The sum of first three terms of a GP is $\frac{39}{10}$ and their produt is 1. Find the common ratio and the terms

20 The sum of three numbers in GP is 21 and the sum of their squares is 189. Find the numbers.

21 If the pth, qth and rth terms of a GP are a, b and c respectively, then prove that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$.

22 If the first and nth terms of a GP are a and b respectively and P is the product of n terms, then prove that $p^2 = (ab)^n$.

23 The sum of first three terms of a GP is $\frac{13}{12}$ and their product is -1. Find the terms.

24 Find the three numbers in GP, whose sum is 19 and product is 216.

4. (a) $a, ar, ar^2, ar^3, \dots, ar^{n-1}$ is a finite GP.

5. (a) Here, $a = 5$ and $r = 5$

Thus, $a_n = ar^{n-1} = 5 \times (5)^{n-1} = 5^n$

6. (i) Here, $a = \sqrt{3}$, $r = \frac{3}{\sqrt{3}} = \sqrt{3}$, $T_n = 729$

Now, apply the formula $T_n = ar^{n-1}$ **Ans.** 12

(ii) Here, $a = \frac{1}{3}$, $r = \frac{1/9}{1/3} = \frac{3}{9} = \frac{1}{3}$ and $T_n = \frac{1}{19683}$

Now, apply the formula, $T_n = ar^{n-1}$. **Ans.** 9

7. 0 or 16

8. (i) $-\frac{1}{2}$ (ii) $\frac{3a}{4}$

9. (i) 3072 (ii) $\frac{4}{6561}$ (iii) 512

(iv) $(0.3)(0.2)^7$ (v) $(ax)^{41}$

10. $5^{10}; 5^n$

11. Here, $T_8 = 192$ and $r = 2$

$\because \quad T_8 = ar^{8-1} = ar^7$

$\therefore \quad a = \frac{3}{2}$

Now, $\quad T_{12} = 3072$

12. $\frac{1}{32}$

13. $5 + 10 + 20 + 40 + \dots$

14. $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = r$

$\Rightarrow b = ar,\ c = br = ar^2,\ d = cr = ar^3$

Now, $\quad a^2 - b^2 = a^2 - a^2r^2 = a^2(1 - r^2)$

$b^2 - c^2 = a^2r^2 - a^2r^4 = a^2r^2(1 - r^2)$

and $\quad c^2 - d^2 = a^2r^4 - a^2r^6 = a^2r^4(1 - r^2)$

Therefore, $\frac{b^2 - c^2}{a^2 - b^2} = \frac{c^2 - d^2}{b^2 - c^2} = r^2$

HINTS & ANSWERS

1. (b) If every term in a progression except the first term bears a constant ratio to the term immediately preceding it. Then, such progression is called geometric progression.

2. (d) If $a, ar, ar^2, ar^3, \dots$, is geometric progression, then a and r are called first term and common ratio.

3. (d) 2, 4, 8, 16, ... is a geometric sequence with common ratio 2.

$\frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243}, \dots$ is a geometric sequence with common ratio $\frac{-1}{3}$.

.01, .0001, .000001, ... is a geometric sequence with common ratio .01.

15. Here, a = first term $= \frac{5}{2}$ and

common ratio $(r) = \frac{5/4}{5/2} = \frac{2}{4} = \frac{1}{2}$

$T_{20} = \frac{5}{2^{20}}$ and $T_n = \frac{5}{2^n}$

16. a_k = kth term from the beginning $= a_1 r^{k-1}$

a_{n-k+1} = kth term from the end $= a_n \left(\frac{1}{r}\right)^{k-1}$

where $1 < k < n$

$a_k\, a_{n-k+1} = (a_1 r^{k-1})\, a_n \left(\frac{1}{r}\right)^{k-1} = a_1 a_n$

for all k satisfying $1 < k < n$.

17. Given, $a + ar = -4$ ...(i)

and $T_5 = 4T_3 \Rightarrow ar^{5-1} = 4ar^{3-1} \Rightarrow r = \pm 2$

If $r = 2$, then from Eq. (i),

$$a + a(2) = -4 \Rightarrow 3a = -4$$

$$\therefore \quad a = -\frac{4}{3}$$

If $r = -2$, then from Eq. (i),

$$a + a(-2) = -4 \Rightarrow -a = -4$$

$$\therefore \quad a = 4$$

Hence, GP is $\ldots -\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or $4, -8, 16, \ldots$

18. Let the GP be $a, ar, ar^2, ar^3, \ldots$

Given, $a = 1$ and $T_3 + T_5 = 90$

$$\therefore \quad ar^2 + ar^4 = 90 \Rightarrow r^2 + r^4 = 90$$

$$r^4 + r^2 - 90 = 0 \quad [\text{as } a = 1]$$

$$(r^2 + 10)(r^2 - 9) = 0 \Rightarrow r = \pm 3$$

$[\because r^2 = -10$ is a complex number$]$

19. Here, $\frac{a}{r} + a + ar = \frac{39}{10}$ and $\left(\frac{a}{r}\right) \times (a) \times (ar) = 1$

$$\Rightarrow \quad a^3 = 1 \Rightarrow a = 1$$

On putting the value of $a = 1$ in Eq. (i) we get

$$\frac{1}{r} + 1 + r = \frac{39}{10} \Rightarrow \frac{1 + r + r^2}{r} = \frac{39}{10}$$

$$\Rightarrow \quad 10r^2 - 29r + 10 = 0 \Rightarrow (5r - 2)(2r - 5) = 0$$

$$\therefore \quad r = \frac{2}{5} \text{ or } r = \frac{5}{2}$$

When $a = 1$ and $= \frac{2}{5}$, then the numbers are $\frac{5}{2}, 1, \frac{2}{5}$.

When $a = 1$ and $r = \frac{5}{2}$, then the numbers are $\frac{2}{5}, 1, \frac{5}{2}$

20. $a + ar + ar^2 = 21 \Rightarrow a^2 + (ar)^2 + (ar^2)^2 = 189$

Ans. (3, 6, 12) or (12, 6, 3)

21. Given, $T_p = a \Rightarrow AR^{p-1} = a$ ...(i)

$$T_q = b \Rightarrow Ar^{q-1} = b \quad \text{...(ii)}$$

and $T_r = c \Rightarrow AR^{r-1} = c$ ...(iii)

$$\text{LHS} = a^{q-r} b^{r-p} b^{p-q}$$

22. Let the GP be $A, AR, AR^2, AR^3 \ldots$

Given, first term,

$$A = a \quad \text{...(i)}$$

and nth term, $AR^{n-1} = b$ ...(ii)

Now, p = Product of n terms

$$\Rightarrow \quad p = A \times AR^1 \times AR^2 \times AR^3 \times \ldots \times n \text{ terms}$$

$$\Rightarrow \quad p = A^{1+1+1+\ldots+n \text{ terms}} \; R^{1+2+3+\ldots+(n-1)}$$

$$\Rightarrow \quad p = A^n R^{\frac{n(n-1)}{2}}$$

$$\Rightarrow \quad p^2 = A^n A^n R^{n(n-1)} = A^n (AR^{n-1})^n$$

$$\Rightarrow \quad p^2 = a^n b^n \quad [\text{using Eqs. (i) and (ii)}]$$

23. Here, sum $= \frac{a}{r} + a + ar = \frac{13}{12}$

$$\Rightarrow \quad a(1 + r + r^2) = \frac{13}{12} r \quad \text{...(i)}$$

Their product $= \frac{a}{r} \cdot a \, ar = -1$

$$\Rightarrow \quad a^3 = -1$$

$$\Rightarrow \quad a = -1 \quad [\text{taking cube root both sides}] \text{ ...(ii)}$$

On putting the value of a in Eq. (i), we get

$$(-1)[1 + r + r^2] = \frac{13}{12} r$$

$$\Rightarrow \quad 12r^2 + 25r + 12 = 0$$

$$\therefore \quad r = -\frac{4}{3} \Rightarrow r = \frac{-3}{4}$$

When $a = -1$ and $r = -\frac{4}{3}$, then the numbers are $\frac{3}{4}, -1, \frac{4}{3}$.

When $a = -1$ and $r = -\frac{3}{4}$, then the numbers are $\frac{4}{3}, -1, \frac{3}{4}$.

24. Product of three numbers $= \frac{a}{r} \cdot a \cdot ar = 216$

$$\therefore \quad a = 6$$

And sum of three numbers $= \frac{a}{r} + a + ar = 19$

$$\Rightarrow \quad 6 + 6r + 6r^2 = 19r \Rightarrow 6r^2 - 13r + 6 = 0$$

$$\therefore \quad r = \frac{13 \pm \sqrt{(-13)^2 - 4 \cdot 6 \cdot 6}}{2 \cdot 6} = \frac{13 \pm 5}{12} = \frac{3}{2} \text{ or } \frac{2}{3}$$

When $r = \frac{3}{2}$, then the numbers are 4, 6, 9.

When $r = \frac{2}{3}$, then the numbers are 9, 6, 4.

|TOPIC 5|

Sum of First n Terms of a GP

If a and r are the first term and common ratio of a GP respectively, then sum of n terms of this GP is given by

$$S_n = \frac{a(1-r^n)}{1-r}, \text{ where } r < 1$$

or $$S_n = \frac{a(r^n - 1)}{r-1}, \text{ where } r > 1 \; (r \neq 1)$$

(i) If $r = 1$, then $S_n = a + a + \dots n$ terms $= na$

(ii) Suppose a GP contains n terms with first term $= a$, common ratio $= r$ and last term $= l$. The sum of GP

$$S_n = \frac{a - lr}{1-r}, \text{ where } r < 1 \text{ or } S_n = \frac{lr - a}{r-1}, \text{ where } r > 1$$

Sum of an Infinite GP

If a is the first term and r is the common ratio of a GP such that $|r| < 1$, then sum of an infinite GP is given by

$$S = \frac{a}{1-r}$$

Problems Based on Sum of Terms of a GP

There are so many problems which can be solved by using the sum of terms of a finite GP or infinite GP directly or indirectly i.e. we can find the values of other terms related to GP with the help of sum formula.

Some types of these problems are given below

TYPE I

PROBLEMS BASED ON FINDING THE SUM OF GIVEN NUMBER OF TERMS OF FINITE GP

In this type of problems, a finite GP is given and we have to find sum of given number of terms.

EXAMPLE |1| Find the sum of the following sequences

(i) $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \dots$ upto 10 terms.

(ii) $2, -\frac{1}{2}, \frac{1}{8}, \dots$, upto 12 terms.

Sol. (i) We have, sequence $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \dots$

Clearly, the successive ratio of the terms is same. So, the above sequence forms a GP, with first term, $a = \frac{1}{2}$

and common ratio, $r = \frac{3}{2} \div \frac{1}{2} = 3$.

$$\therefore \quad S_{10} = \frac{\frac{1}{2}[3^{10} - 1]}{3-1} \quad \left[\because S_n = \frac{a(r^n - 1)}{r-1} \text{ as } r > 1\right]$$

$$= \frac{1}{2}\frac{(3^{10} - 1)}{2} = \frac{1}{4}[3^{10} - 1] = 14762$$

(ii) We have, sequence $2, -\frac{1}{2}, \frac{1}{8}, \dots$

Clearly, the successive ratio of the terms is same. So, the above sequence forms a GP with first term, $a = 2$ and common ratio, $r = -\frac{1}{2} \div 2 = \frac{-1}{4}$.

$$\therefore \quad S_{12} = \frac{2\left[1 - \left(\frac{-1}{4}\right)^{12}\right]}{1 - \left(\frac{-1}{4}\right)} \quad \left[\because S_n = \frac{a(1-r^n)}{1-r} \text{ as } r < 1\right]$$

$$= \frac{2\left[1 - \frac{1}{(4)^{12}}\right]}{1 + \frac{1}{4}} = \frac{2\left[1 - \left(\frac{1}{4}\right)^{12}\right]}{\left(\frac{5}{4}\right)} = \frac{8}{5}\left[1 - \left(\frac{1}{4}\right)^{12}\right]$$

EXAMPLE |2| Find the sum of the series $2 + 6 + 18 + 54 + \dots + 4374$.

Sol. Given series is $2 + 6 + 18 + 54 + \dots + 4374$.

Clearly, the successive ratio of the terms is same. So, the above series forms a GP with first term, $a = 2$, common ratio, $r = \frac{6}{2} = 3 > 1$ and last term, $l = 4374$.

$$\therefore \text{ Required sum} = \frac{(lr - a)}{(r-1)}$$

$$= \frac{(4374 \times 3 - 2)}{(3-1)}$$

$$= \frac{13120}{2} = 6560$$

Hence, the sum of the given series is 6560.

EXAMPLE |3| Find the sum to n terms of the sequence given by $a_n = 2^n + 3n, n \in N$.

Sol. Let S_n denote the sum to terms of the given sequence.

Then, $S_n = a_1 + a_2 + a_3 + \dots + a_n$

$$\Rightarrow \quad S_n = (2^1 + 3 \times 1) + (2^2 + 3 \times 2) + (2^3 + 3 \times 3) + \dots + (2^n + 3 \times n)$$

$$\Rightarrow \quad S_n = (2^1 + 2^2 + 2^3 + \dots + 2^n) + (3 \times 1 + 3 \times 2 + 3 \times 3 + \dots + 3 \times n)$$

$$= (2^1 + 2^2 + 2^3 + \dots + 2^n) + 3(1 + 2 + 3 + \dots + n)$$

$$= 2\left(\frac{2^n - 1}{2 - 1}\right) + 3\left\{\frac{n}{2}(1 + n)\right\}$$

$$\left[\because 2 + 2^2 + 2^3 + 2^n \text{ is a GP and } \Sigma n = \frac{n(n+1)}{2}\right]$$

$$= 2(2^n - 1) + \frac{3n}{2}(n+1)$$

EXAMPLE |4| Find the sum of series $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots$ to $2n$ terms.

Sol. We have, $S = \frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots$ to $2n$ terms

$$= \left[\frac{3}{5} + \frac{3}{5^3} + \dots + \text{ to } n \text{ terms}\right] + \left[\frac{4}{5^2} + \frac{4}{5^4} + \dots + \text{to } n \text{ terms}\right]$$

$$= \frac{\frac{3}{5}\left[1 - \left(\frac{1}{5^2}\right)^n\right]}{\left[1 - \left(\frac{1}{5^2}\right)\right]} + \frac{\frac{4}{5^2}\left[1 - \left(\frac{1}{5^2}\right)^n\right]}{\left[1 - \left(\frac{1}{5^2}\right)\right]}$$

$$\left[\because \text{sum of GP} = \frac{a(1 - r^n)}{1 - r}, |r| < 1\right]$$

$$= \frac{\frac{3}{5}\left[1 - \frac{1}{5^{2n}}\right]}{1 - \frac{1}{25}} + \frac{\frac{4}{25}\left[1 - \frac{1}{5^{2n}}\right]}{1 - \frac{1}{25}}$$

$$= \frac{\frac{3}{5}\left[1 - \frac{1}{5^{2n}}\right]}{\frac{24}{25}} + \frac{\frac{4}{25}\left[1 - \frac{1}{5^{2n}}\right]}{\frac{24}{25}} = \frac{5}{8}\left[1 - \frac{1}{5^{2n}}\right] + \frac{1}{6}\left[1 - \frac{1}{5^{2n}}\right]$$

$$= \left[1 - \frac{1}{5^{2n}}\right]\left[\frac{5}{8} + \frac{1}{6}\right] = \left[1 - \frac{1}{5^{2n}}\right]\left[\frac{15 + 4}{24}\right]$$

$$= \frac{19}{24}\left[1 - \frac{1}{5^{2n}}\right]$$

EXAMPLE |5| A GP consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Sol. Let the GP is $a, ar, ar^2, ar^3, ar^4, \dots, ar^{2n-2}, ar^{2n-1}$, where $a, ar^2, ar^4, ar^6, \dots$ occupy odd places and $ar, ar^3, ar^5, ar^7 \dots$ occupy even places.

According to the question,

Sum of all terms

= 5 × Sum of terms occupying odd places

i.e. $a + ar + ar^2 + \dots + ar^{2n-1}$

$$= 5 \times (a + ar^2 + ar^4 + \dots + ar^{2n-2})$$

$$\Rightarrow \frac{a(r^{2n} - 1)}{r - 1} = \frac{5a[(r^2)^n - 1]}{r^2 - 1} \qquad \left[\because S_n = \frac{a(r^n - 1)}{r - 1}, r > 1\right]$$

$$\Rightarrow \frac{r^{2n} - 1}{r - 1} = \frac{5(r^{2n} - 1)}{(r - 1)(r + 1)} \Rightarrow 1 = \frac{5}{r + 1}$$

$$\Rightarrow r + 1 = 5 \Rightarrow r = 4$$

Note

Students should remember that if a GP has $2n$ (even terms), then there are n even and n odd terms.

EXAMPLE |6| Find the sum of the products of the corresponding terms of the sequence 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

Sol. Given, sequences are 2, 4, 8, 16, 32 ...(i)

and 128, 32, 8, 2, $\frac{1}{2}$ ...(ii)

Multiplying the corresponding terms of Eqs. (i) and (ii) we get a new sequence, 256, 128, 64, 32, 16

Let $S = 256 + 128 + 64 + 32 + 16$

Clearly, the successive ratio of the terms is same. So, the above obtained series forms a GP, with first term,

$a = 256$ and comon ratio, $r = \frac{128}{256} = \frac{1}{2}$.

$$\therefore \text{ Required sum, } S_n = \frac{256\left[1 - \left(\frac{1}{2}\right)^5\right]}{1 - \frac{1}{2}}$$

$$= 256 \times 2\left(1 - \frac{1}{2^5}\right) \qquad \left[\because S_n = \frac{a(1 - r^n)}{1 - r}, r < 1\right]$$

$$= 512 \times \left(1 - \frac{1}{32}\right)$$

$$= 512\left(\frac{32 - 1}{32}\right) = 16 \times 31 = 496$$

EXAMPLE |7| The sum of the first three terms of a GP is 16 and the sum of next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the GP. **[NCERT]**

Sol. Let the GP be $a, ar, ar^2, ar^3, \dots$

According to the given condition,

Sum of first three terms $= a + ar + ar^2 = 16$ ...(i)

and sum of next three terms $= ar^3 + ar^4 + ar^5 = 128$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{a + ar + ar^2}{ar^3 + ar^4 + ar^5} = \frac{16}{128}$$

$$\Rightarrow \frac{a(1 + r + r^2)}{ar^3(1 + r + r^2)} = \frac{1}{8}$$

$$\Rightarrow \left(\frac{1}{r}\right)^3 = \left(\frac{1}{2}\right)^3$$

On comparing the base of the power 3 from both sides,

we get $\frac{1}{r} = \frac{1}{2} \Rightarrow r = 2$

On putting $r = 2$ in Eq. (i), we get

$$a + 2a + 4a = 16 \Rightarrow 7a = 16 \Rightarrow a = \frac{16}{7}$$

Now, sum of n terms, $S_n = \frac{a(r^n - 1)}{r - 1}$ $[\because r = 2 > 1]$

$$= \frac{\frac{16}{7}(2^n - 1)}{2 - 1} = \frac{16}{7}(2^n - 1)$$

Hence, $a = \frac{16}{7}$, $r = 2$ and $S_n = \frac{16}{7}(2^n - 1)$

EXAMPLE |8| If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a GP, prove that

$$S_1(S_3 - S_2) = (S_2 - S_1)^2.$$

Sol. Let a be the first term and r be the common ratio of the given GP. Then,

$$S_1(S_3 - S_2) = \frac{a(1 - r^n)}{(1 - r)} \cdot \left\{\frac{a(1 - r^{3n})}{(1 - r)} - \frac{a(1 - r^{2n})}{(1 - r)}\right\}$$

$$= \frac{a(1 - r^n)}{(1 - r)} \cdot \frac{(a - ar^{3n} - a + ar^{2n})}{(1 - r)}$$

$$= \frac{a(1 - r^n)}{(1 - r)} \cdot \frac{ar^{2n}(1 - r^n)}{(1 - r)}$$

$$= \frac{a^2 r^{2n}(1 - r^n)^2}{(1 - r)^2} \quad \text{...(i)}$$

And, $(S_2 - S_1)^2 = \left\{\frac{a(1 - r^{2n})}{(1 - r)} - \frac{a(1 - r^n)}{(1 - r)}\right\}^2$

$$= \frac{(a - ar^{2n} - a + ar^n)^2}{(1 - r)^2}$$

$$= \frac{\{(ar^n(1 - r^n)\}^2}{(1 - r)^2} = \frac{a^2 r^{2n}(1 - r^n)^2}{(1 - r)^2} \quad \text{...(ii)}$$

Hence, from Eqs. (i) and (ii), we get

$$S_1(S_3 - S_2) = (S_2 - S_1)^2$$

EXAMPLE |9| Let S be the sum, P be the product and R be the sum of reciprocals of n terms in a GP. Prove that $P^2 R^n = S^n$.

Sol. Let the GP is $a, ar, ar^2, ar^3 \dots ar^{n-1}$.

Given, $S =$ Sum of n terms

$$= a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

$$= \frac{a(r^n - 1)}{r - 1} \quad [\text{let } r > 1] \quad \text{...(i)}$$

and $R =$ Sum of the reciprocals of n terms

$$= \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots + \frac{1}{ar^{n-1}} \quad \left(\frac{1}{r} < 1\right)$$

$$= \frac{\frac{1}{a}\left[1 - \left(\frac{1}{r}\right)^n\right]}{1 - \frac{1}{r}} = \frac{1}{a}\left[\frac{1}{1} - \frac{1}{r^n}\right] \times \frac{1}{\frac{r - 1}{r}}$$

$$= \frac{1}{a}\left[\frac{r^n - 1}{r^n}\right] \times \frac{r}{r - 1} \Rightarrow R = \frac{(r^n - 1)r}{ar^n(r - 1)} \quad \text{...(ii)}$$

and $P =$ Product of n terms

$$= a \times ar \times ar^2 \times ar^3 \times \dots \times ar^{n-1}$$

$$= a^{1 + 1 + 1 + \dots + n \text{ terms}}\, r^{1 + 2 + 3 + \dots + (n - 1) \text{terms}}$$

$$= a^n r^{\frac{n(n - 1)}{2}} \quad \left[\because \Sigma n = \frac{n(n + 1)}{2}\right]$$

$$\Rightarrow \quad P^2 = a^{2n} r^{n(n - 1)} \quad \text{...(iii)}$$

Now, we have to prove $P^2 R^n = S^n$

or $P^2 = \frac{S^n}{R^n}$ or $P^2 = \left(\frac{S}{R}\right)^n$

$$\text{RHS} = \left(\frac{S}{R}\right)^n = \left[\frac{a(r^n - 1)}{r - 1} \times \frac{ar^n(r - 1)}{(r^n - 1)r}\right]^n$$

[using Eqs. (i) and (ii)]

$$= [a^2 r^n r^{-1}]^n = (a^2 r^{n-1})^n = [a^{2n} r^{n(n - 1)}]$$

$$= P^2 = \text{LHS} \quad [\text{from Eq. (iii)}]$$

Hence proved.

Note

Students should remembered that if you are taking $r < 1$ in S , then you have to take $r > 1$ in R.

TYPE II

PROBLEMS BASED ON FINDING THE SUM OF TERMS OF AN INFINITE GP

In this type of problems, an infinite GP is given and we have to find the sum of the infinite GP.

EXAMPLE |10| Find the sum of an infinite GP

$$1, \frac{1}{3}, \frac{1}{9}, \dots \infty.$$

Sol. Given infinite GP is $1, \frac{1}{3}, \frac{1}{9}, \dots \infty$

Here, $a = 1$ and $r = \frac{1}{3} \div 1 = \frac{1}{3}$

We know that, $S_\infty = \frac{a}{1 - r}$

$$\therefore \quad S_\infty = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

EXAMPLE |11| If $b = a + a^2 + a^3 + \ldots \infty$, then prove that $a = \frac{b}{1+b}$.

Sol. We have, $b = a + a^2 + a^3 + \ldots \infty$

Clearly, RHS is a geometric series with first term 'a' and common ratio 'a'

$$\therefore \quad b = \frac{a}{1-a} \Rightarrow b - ab = a \Rightarrow a = \frac{b}{1+b}$$

EXAMPLE |12| Find the sum of the series

$$9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty$$

Sol. Let $E = 9^{\frac{1}{3}} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\infty}$ ...(i)

Again let $S_\infty = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \infty$

Clearly, the successive ratio of the terms is same. So, the abvoe obtained series forms a GP, with first term, $a = \frac{1}{3}$

and common ratio, $\frac{1}{9} \div \frac{1}{3} = \frac{1}{3}$.

Then, $$S_\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2} \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$$

Then, from Eq. (i), we get $E = 9^{\frac{1}{2}} = 3$

EXAMPLE |13| If $|x| < 1$ and $|y| < 1$, find the sum of infinity of the following series:

$$(x+y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots$$

Sol. We have,

$$(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots \infty$$
$$= \frac{1}{x-y}\{(x^2-y^2) + (x^3-y^3) + (x^4-y^4) + \ldots \text{ to } \infty\}$$
$$\left[\because \frac{x^n - a^n}{x-a} = x^{n-1} \cdot a^0 + x^{n-2} \cdot a + x^{n-3}a^2 + \ldots + a^{n-1}, n \in N\right]$$
$$= \frac{1}{x-y}\{(x^2 + x^3 + x^4 + \ldots \text{ to } \infty) - (y^2 + y^3 + y^4 + \ldots \text{ to } \infty)\}$$
$$= \frac{1}{x-y}\left\{\frac{x^2}{1-x} - \frac{y^2}{1-y}\right\}$$
$$= \frac{1}{x-y} \frac{\{x^2 - x^2y - y^2 + y^2x\}}{(1-x)(1-y)}$$
$$= \frac{1}{(x-y)} \frac{\{(x^2-y^2) - xy(x-y)\}}{(1-x)(1-y)} = \frac{x+y-xy}{(1-x)(1-y)}$$

EXAMPLE |14| If $x = \sum_{n=0}^{\infty} \cos^{2n}\theta$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$, where $0 < \theta, \phi < \pi/2$, then prove that $xz + yz - z = xy$.

Sol. We have, $x = \sum_{n=0}^{\infty} \cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \ldots \infty$

$$\Rightarrow \quad x = \frac{1}{1-\cos^2\theta} \Rightarrow \sin^2\theta = \frac{1}{x}$$
$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty$$
$$\Rightarrow \quad y = \frac{1}{1-\sin^2\phi} \Rightarrow \cos^2\phi = \frac{1}{y}$$

and $$z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\phi$$
$$= 1 + \cos^2\theta\sin^2\phi + \cos^4\theta\sin^4\phi + \ldots \infty$$
$$\Rightarrow \quad z = \frac{1}{1-\cos^2\theta\sin^2\phi}$$
$$\Rightarrow \quad z = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$$
$$\Rightarrow \quad z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \Rightarrow z = \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}}$$
$$\Rightarrow \quad z = \frac{xy}{x+y-1} \Rightarrow xz + yz - z = xy$$

TYPE III

PROBLEMS BASED ON FINDING THE VALUE OF UNKNOWN/GP WHEN SUM OF *N* TERMS OF A FINITE GP IS GIVEN

In this type of problems, a GP and sum is given and we have to find r and n. Sometimes GP is not given and we have to find a, r and then GP with the help of given information related to sum.

EXAMPLE |15| How many terms of GP $3, \frac{3}{2}, \frac{3}{4}, \ldots$ are needed to give the sum $\frac{3069}{512}$? **[NCERT]**

Sol. Given, GP is $3, \frac{3}{2}, \frac{3}{4}, \ldots$

Here, $$a = 3, r = \frac{3}{2} \div 3 = \frac{1}{2}$$

Let n be the number of terms needed.

Then, $S_n = \frac{3069}{512} \Rightarrow \frac{a(1-r^n)}{1-r} = \frac{3069}{512}$ $[\because r<1]$

$$\Rightarrow \frac{3\left\{1-\frac{1}{2^n}\right\}}{1-\frac{1}{2}} = \frac{3069}{512}$$

$$\Rightarrow 6\left(1-\frac{1}{2^n}\right) = \frac{3069}{512} \Rightarrow 1-\frac{1}{2^n} = \frac{3069}{3072}$$

$$\Rightarrow \frac{1}{2^n} = 1 - \frac{3069}{3072} = \frac{3072-3069}{3072}$$

$$\Rightarrow \frac{1}{2^n} = \frac{3}{3072} = \frac{1}{1024} \Rightarrow 2^v = 1024 \Rightarrow 2^n = 2^{10}$$

On comparing the powers from both sides, we get

$$n = 10$$

Hence, 10 terms are needed to give the sum $\frac{3069}{512}$.

EXAMPLE |16| Find the least value of n for which the sum $1+3+3^2+\ldots$ to n terms is greater than 7000.

Sol. We have, $S_n = 1+3+3^2+\ldots$ to n terms

Clearly, the successive ratio of the terms is same. So, the above series forms a GP, with first term, $a=1$ and common ratio, $\frac{3}{1} = 3 > 1$.

$$\therefore \quad S_n = 1\times\left(\frac{3^n-1}{3-1}\right) = \frac{3^n-1}{2} \quad \left[\because S_n = a\left(\frac{r^n-1}{r-1}\right)\right]$$

Now, $S_n > 7000$

$$\Rightarrow \frac{3^n-1}{2} > 7000 \Rightarrow 3^n - 1 > 14000$$

$$\Rightarrow 3^n > 14001 \Rightarrow n\log 3 > \log 14001$$

$$\Rightarrow n > \frac{\log 14001}{\log 3} \Rightarrow n > \frac{4.1461}{0.4771} = 8.69$$

Hence, the least value of n is 9.

TYPE IV

PROBLEMS BASED ON FINDING THE VALUE OF UNKNOWN/GP WHEN SUM OF AN INFINITE GP IS GIVEN

In this type of problems, sum of an infinite GP is given and we have to find the values of unknown a, r and sometimes we have to find GP.

EXAMPLE |17| The sum of an infinite GP is 57 and the sum of their cubes is 9747. Find the GP.

Sol. Let a be the first term and r be the common ratio of an infinite GP.

Given, sum $= 57 \Rightarrow \frac{a}{1-r} = 57$ ...(i)

and sum of the cubes $= 9747$

$$\Rightarrow a^3 + a^3r^3 + a^3r^6 + \ldots = 9747 \Rightarrow \frac{a^3}{1-r^3} = 9747 \quad \text{...(ii)}$$

On dividing the cube of Eq. (i) by Eq. (ii), we get

$$\frac{a^3}{(1-r)^3}\cdot\frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747}$$

$$\Rightarrow \frac{1-r^3}{(1-r)^3} = 19 \Rightarrow \frac{1+r+r^2}{(1-r)^2} = 19$$

$[\because a^3 - b^3 = (a-b)(a^2+b^2+ab)]$

$$\Rightarrow 1+r+r^2 = 19(1+r^2-2r)$$

$[\because (a-b)^2 = a^2+b^2-2ab]$

$$\Rightarrow 18r^2 - 39r + 18 = 0$$

$$\Rightarrow (3r-2)(6r-9) = 0$$

$$\Rightarrow r = 2/3 \text{ or } r = 3/2$$

$$\Rightarrow r = 2/3$$

$[\because r \ne 3/2$, because $-1 < r < 1$ for an infinite GP$]$

On putting $r = 2/3$ in Eq. (i), we get

$$\frac{a}{1-(2/3)} = 57 \Rightarrow 3a = 57 \Rightarrow a = 19$$

Hence, the required GP is $19, 19\times\frac{2}{3}, 19\times\left(\frac{2}{3}\right)^2, \ldots$

i.e. 19, 38/3, 76/9, ...

TYPE V

PROBLEMS BASED ON FINDING A RATIONAL NUMBER WHOSE DECIMAL EXPANSION IS GIVEN

Sometimes a number in non-terminating repeating decimal expansion form i.e. of the form $0.a\overline{bc}$ is given and we have to write it as rational number. For finding the corresponding rational number,

we use the following steps

***Step* I** First, write the given decimal expansion in the form of series.

i.e. write $0.a\overline{bc}$ as

$0.a + 0.0bc + 0.000bc + 0.00000bc + \ldots \infty$.

***Step* II** Remove the decimal point from each term by multiply and divide by suitable power of 10.

i.e. $0.a\overline{bc} = \frac{a}{10} + \frac{bc}{10^3} + \frac{bc}{10^5} + \ldots \infty$.

***Step* III** Now, find the sum of infinite GP obtained in step II, by suitable formula, to get required rational number.

EXAMPLE |18| Write the rational number corresponding to the decimal expansion $0.3\overline{56}$.

Sol. Given decimal expansion is $0.3\overline{56}$.

In series form, it can be written as

$$0.3\overline{56} = 0.3 + 0.056 + 0.00056 + 0.0000056 + \dots \infty$$

$$\Rightarrow \quad 0.3\overline{56} = 0.3 \times \frac{10}{10} + 0.056 \times \frac{10^3}{10^3} + 0.00056 \times \frac{10^5}{10^5} + \dots \infty$$

$$= \frac{3}{10} + \frac{56}{10^3} + \frac{56}{10^5} + \dots \infty$$

$$\Rightarrow \quad 0.3\overline{56} = \frac{3}{10} + \left[\frac{56}{10^3} + \frac{56}{10^5} + \dots \infty\right]$$

Here, first term $a = \frac{56}{10^3}$

and common ratio $= \frac{56}{10^5} \div \frac{56}{10^3} = \frac{1}{10^2} < 1$

$$\therefore \quad 0.3\overline{56} = \frac{3}{10} + \frac{\frac{56}{10^3}}{1 - \frac{1}{10^2}} \qquad \left[\because S_\infty = \frac{a}{1-r}, \text{ if } r < 1\right]$$

$$= \frac{3}{10} + \frac{56}{10^3} \times \frac{10^2}{99} = \frac{3}{10} + \frac{56}{990}$$

$$\therefore \quad 0.3\overline{56} = \frac{353}{990}$$

TYPE VI

PROBLEMS BASED ON FINDING THE SUM OF SPECIAL SERIES BY CONVERTING IT INTO GP

In this type of problem a special series is given to find sum then we convert it into GP by suitable method and then find its sum.

Let the special series of the type

$S = a + aa + aaa + \dots n$ term be given.

The sum of this type of series can be find by using the following steps

***Step* I** First, take common a from each term.

i.e. write given series as $a\,(1 + 11 + 111 + \dots n$ th term).

***Step* II** Multiply and divide each term by 9.

i.e. $\frac{a}{9}(9 + 99 + 999 + \dots n\text{th term})$

***Step* III** Write each term as difference of multiple of 10 and 1.

i.e. $\frac{a}{9}[(10-1) + (100-1) + (1000-1) + \dots n\text{th term}]$

***Step* IV** Separate the terms i.e.

$\frac{a}{9}[(10 + 10^2 + 10^3 + \dots n\text{th term})] - (1 + 1 + \dots n\text{th term})]$

***Step* V** Find the sum of first series by using formula of GP

i.e. $\frac{a}{9}\left[\frac{10(10^n - 1)}{10-1}\right] - n = \frac{a}{9}\left[\frac{10(10^n - 1)}{9}\right] - n$

EXAMPLE |19| Find the sum of the series $4 + 44 + 444 + \dots n$ terms.

Sol. Given series is $4 + 44 + 444 + \dots n$ terms.

On taking common 4, we get

$$S_n = 4(1 + 11 + 111 + \dots n \text{ terms})$$

On multiplying and during each term of RHS by 9, we get

$$S_n = \frac{4}{9}(9 + 99 + 999 + \dots)$$

$$\Rightarrow S_n = \frac{4}{9}[(10-1) + (100-1) + (1000-1) + \dots n \text{ terms}]$$

$$\Rightarrow S_n = \frac{4}{9}[(10 + 100 + 1000 + \dots n \text{ terms}) - (1 + 1 + 1 + \dots n \text{ terms}] \quad \dots \text{(i)}$$

For first series, $10 + 100 + 1000 + \dots n$ terms

$$r = \frac{100}{10} = 10 > 1$$

$$\therefore S_n = 10(1 + 10 + 10^2 + \dots n \text{ terms})$$

$$= 10\left(\frac{10^n - 1}{10 - 1}\right) \qquad \left[\because \text{sum} = \frac{a(r^n - 1)}{r - 1}\right]$$

$$= \frac{10}{9}(10^n - 1)$$

Now, from Eq. (i), we get

$$S_n = \frac{4}{9}\left[\frac{10}{9}(10^n - 1) - n\right] = \frac{40}{81}(10^n - 1) - \frac{4}{9}n$$

Applications of GP

Sometimes we can solve the some real life problems by using the concept of GP. Some of them are given below.

EXAMPLE |20| At the end of each year the value of a certain machine has depreciated by 20% of its value at the beginning of that year. If its initial value was ₹ 1250, then find the value at the end of 5 yr. **[NCERT Exemplar]**

Sol. Given, depreciation in value of machine = 20%

After each year the value of the machine is 80% (100 – 20)% of its value of the previous year, so at the end of 5 yr, the machine will depreciate as many times as 5.

Hence, we have to find the 6th terms of the G P whose first term a_1 is 1250 and common ratio r is 8.

Hence, value at the end 5 yrs

$$= t_6 = a_1 r^5$$
$$= 1250\,(0.8)^5 = 409.6$$

Note In depreciation, the cost value decrease every year.

EXAMPLE |21| The lengths of three unequal edges of a rectangular solid block are in GP. If the volume of the block is 216 cm^3 and the total surface area is 252 cm^2, then find the length of its edges. **[NCERT Exemplar]**

Sol. Let the length, breadth and height of rectangular solid block is $\frac{a}{r}$, a and ar, respectively.

$$\therefore \quad \text{Volume} = \frac{a}{r} \times a \times ar = 216 \text{ cm}^3$$
$$\Rightarrow \quad a^3 = 216 \Rightarrow a^3 = 6^3$$
$$\therefore \quad a = 6$$
$$\text{Surface area} = 2\left(\frac{a^2}{r} + a^2 r + a^2\right) = 252$$
$$\Rightarrow \quad 2a^2\left(\frac{1}{r} + r + 1\right) = 252$$
$$\Rightarrow \quad 2 \times 36\left(\frac{1 + r^2 + r}{r}\right) = 252$$
$$\Rightarrow \quad \frac{1 + r^2 + r}{r} = \frac{252}{2 \times 36}$$
$$\Rightarrow \quad 1 + r^2 + r = \frac{126}{36} r$$
$$\Rightarrow \quad 1 + r^2 + r = \frac{21}{6} r$$
$$\Rightarrow \quad 6 + 6r^2 + 6r = 21r$$
$$\Rightarrow \quad 6r^2 - 15r + 6 = 0$$
$$\Rightarrow \quad 2r^2 - 5r + 2 = 0$$
$$\Rightarrow \quad (2r - 1)(r - 2) = 0$$
$$\Rightarrow \quad r = \frac{1}{2} \text{ or } 2$$

For $r = \frac{1}{2}$, Length $= \frac{a}{r} = \frac{6 \times 2}{1} = 12$

Breadth $= a = 6$

Height $= ar = 6 \times \frac{1}{2} = 3$

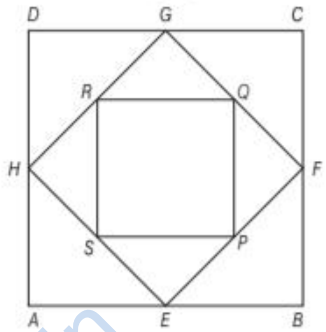
For $r = 2$, Length $= \frac{a}{r} = \frac{6}{2} = 3$

Breadth $= a = 6$

Height $= ar = 6 \times 2 = 12$

EXAMPLE |22| The side of a given square is 10 cm. The mid-points of its sides are joined to form a new square. Again, the mid-point of the sides of this new square are joined to form another square. This process is continued indefinitely. Find the sum of the area and the sum of the perimeteres of the squares.

Sol. Let $ABCD$ be the given square with each side equal to 10 cm. Let E, F, G, H be the mid-points of the sides AB, BC, CD and DA respectively. Let P, Q, R, S be the mid-point of the sides EF, FG, GH and HE respectively.



$$\therefore \quad BE = BF = 5 \text{ cm}$$
$$\Rightarrow \quad EF = \sqrt{BE^2 + BF^2}$$
$$= \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2} \text{ cm}$$
$$FQ = FP = \frac{1}{2} EF = \frac{5\sqrt{2}}{2} = \frac{5}{\sqrt{2}} \text{ cm}$$
$$\Rightarrow \quad PQ = \sqrt{FP^2 + FQ^2}$$
$$= \sqrt{\frac{25}{2} + \frac{25}{2}} = \sqrt{25} = 5 \text{ cm}$$

Thus, the sides of the squares are 10 cm, $5\sqrt{2}$ cm, 5cm, ...

(i) Sum of the areas of squares formed

$$= \{(10)^2 + (5\sqrt{2})^2 + 5^2 + \ldots \infty\}$$
$$= (100 + 50 + 25 + \ldots \infty) \quad \left[\because \text{ it is infinite GP with } a = 100 \text{ and } r = \frac{1}{2}\right]$$
$$= \frac{100}{\left(1 - \frac{1}{2}\right)} = 200 \text{ cm}^2$$

(ii) Sum of perimeters of the given squares

$$= (40 + 20\sqrt{2} + 20 + \ldots)$$
$$= \frac{40}{\left(1 - \frac{1}{\sqrt{2}}\right)} = \frac{40\sqrt{2}}{(\sqrt{2} - 1)} \times \frac{(\sqrt{2} + 1)}{(\sqrt{2} + 1)}$$
$$= (80 + 40\sqrt{2}) \text{ cm}$$

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 Let $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$ be a GP, then

(a) $S_n = na, r \neq 1$ (b) $S_n = \frac{a(r^n - 1)}{r - 1}, r \neq 1$

(c) $S_n = \frac{r - 1}{a(r^n - 1)}, r \neq 1$ (d) $S_n = \frac{a(r^n + 1)}{r - 1}, r \neq 1$

2 The value of $\sum_{k=1}^{10}(1+2^k)$ is

(a) 2085 (b) 2805 (c) 2056 (d) 2508

3 A person has 2 parents, 4 grandparents, 8 great grandparents and so on. Then, the number of ancestors during the ten generations preceding his own is

(a) 1084 (b) 2046 (c) 2250 (d) 1024

4 In a GP of even number of terms, then the sum of all terms is 5 times the sum of the odd terms. The common ratio of the GP is **[NCERT Exemplar]**

(a) $-\frac{4}{5}$ (b) $\frac{1}{5}$

(c) 4 (d) None of these

5 Sum of infinite terms of the sequence $a, ar, ar^2, \ldots$ is equal to

(a) $\frac{a}{1-r}, r \neq 1$ (b) $\frac{a}{1+r}, r \neq -1$

(c) $\frac{1}{1+r}, r \neq -1$ (d) None of these

VERY SHORT ANSWER Type Questions

6 Find the sum of following geometric progressions. (Each part carries 1 mark)

(i) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots n$ terms

(ii) $1, -a, a^2, -a^3, \ldots n$ terms (if $a \neq -1$)

(iii) 2, 6, 18, ... upto 7 terms

(iv) $1, \sqrt{3}, 3, 3\sqrt{3} \ldots$ upto 10 terms

(v) 0.15, 0.015, 0.0015, ... 20 terms

7 Evaluate $\sum_{k=1}^{11}(2 + 3^k)$.

8 The sum of some terms of GP is 315, whose first term and common ratio are 5 and 2, respectively. Find the last term and number of terms. **[NCERT]**

9 Find the sum to infinity of the following GP. (Each part carries 1 mark)

(i) 6, 1.2, 0.24, ... ∞

(ii) 10, –9, 8.1, ... ∞

10 The common ratio of a GP is $-\frac{4}{5}$ and the sum to infinity is $\frac{80}{9}$. Find the first term.

SHORT ANSWER Type I Questions

11 How many terms of the series $1 + 3 + 3^2 + 3^3 + \ldots$ must be taken to make 3280?

12 The first term of a GP is 27 and its 8th term is $\frac{1}{81}$. Find the sum of its first 10 terms.

13 The 2nd and 5th terms of a GP are $\frac{-1}{2}$ and $\frac{1}{16}$, respectively. Find the sum of the GP upto 8 terms.

14 Given a GP with $a = 729$ and 7th term 64, determine S_7. **[NCERT]**

15 Show that the ratio of the sum of the first n terms of a GP to the sum of terms from $(n+1)$th to $(2n)$th term is $\frac{1}{r^n}$. **[NCERT]**

16 The sum of the some terms of a GP is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms. **[NCERT]**

17 The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the series.

SHORT ANSWER Type II Questions

18 Find the sum of the following series. (Each part carries 4 marks)

(i) $5 + 55 + 555 + \ldots$ upto n terms.

(ii) $7 + 77 \times 777 + \ldots$ upto n terms. **[NCERT]**

19 Find the sum of the following series. (Each part carries 4 marks)

(i) $0.6 + 0.66 + 0.666 + \ldots$

(ii) $0.3 + 0.33 + 0.333 + \ldots$ **[NCERT]**

20 Represent the following as rational numbers $0.1\overline{5}$.

21 One side of an equilateral triangle is 18 cm. The mid-points of its sides are joined to form another triangle whose mid-points, in term, are joined to form further another triangle and so on up to infinity. Find the sum of
(i) Perimeters of all the triangles.
(ii) Areas of all the triangles.

22 Find the three numbers in GP, whose sum is 52 and sum of whose product in pairs is 624.

23 In a GP $[a_n]$, if $T_1 = 3$, $T_n = 96$ and $S_n = 189$, then find n.

24 If $x = a + \frac{a}{r} + \frac{a}{r^2} + \dots \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - \dots \infty$ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + \dots \infty$, then prove that $\frac{xy}{z} = \frac{ab}{c}$.

25 Prove that $2^{\frac{1}{2}} \cdot 4^{\frac{1}{8}} \cdot 8^{\frac{1}{24}} \cdot 16^{\frac{1}{64}} \dots \infty = 2$.

26 The inventor of the chessboard suggested a reward of one grain of wheat for the first square; 2 grains for the second; 4 grains for the third and so on, doubling the number of grains for subsequent squares. How many grains would have to be given to the inventor? (Note that there are 64 squares in the chessboard.)

27 A man writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they move the chain similarly. Assuming that the chain is not broken and it costs ₹ 2 to mail one letter, find the amount spent on postage when 8th set of letters is mailed.

28 Sanjeev deposited ₹ 10000 in a bank at the rate of 5% simple interest to annually. Find the amount in 15th yr, since he deposited the amount and also calculate the amount after 20 yr.

29 Rajeev buys a scooter for ₹ 22000. He pays ₹ 4000 cash and agrees to pay the balance in annual instalment of ₹ 1000 plus 10% interest on the unpaid amount. How much will scooter cost him?

30 A farmer buys a used tractor of ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalment of ₹ 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

LONG ANSWER Type Questions

31 If $S_1, S_2, S_3, \dots, S_p$ denote the sum of infinite GP whose first terms are 1, 2, 3, ..., p respectively and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{p+1}$, respectively. Show that $S_1 + S_2 + S_3 + \dots + S_p = \frac{p(p+3)}{2}$.

32 If S_p denotes the sum of the series $1 + r^p + r^{2p} + \dots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - \dots$ to ∞, then prove that $S_p + s_p = 2S_{2p}$.

33 If S denotes the sum of an infinite GP and S_1 denotes the sum of the squares of its terms, then prove that the first term and common ratio are respectively

$$\frac{2SS_1}{S^2 + S_1} \text{ and } \frac{S^2 - S_1}{S^2 + S_1}.$$

34 Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies $f(x+y) = f(x) \cdot f(y)$ for all natural numbers x, y and further $f(1) = 2$.

35 If f is a function satisfying $f(x+y) = f(x) \cdot (y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

HINTS & ANSWERS

1. (b) When $r \neq 1$, then $S_n = \frac{a(r^n - 1)}{r - 1}$ where, a = first term of GP and r = common difference of a GP.

2. (c) $\sum_{k=1}^{10} (1 + 2^k) = 1 \times 10 + (2^1 + 2^2 + 2^3 + 2^4 + 2^5 + \dots + 2^{10})$

$$= 10 + \frac{2(2^{10} - 1)}{2 - 1} = 2056$$

3. (b) Here $a = 2$, $r = 2$ and $n = 10$

$\therefore$ We have, $S_{10} = \frac{2(2^{10} - 1)}{2 - 1} = 2046$

4. (c) Let the given GP be $a, ar, ar^2 \ldots$ with $2n$ terms

Then, we have $\frac{a(r^{2n}-1)}{r-1} = \frac{5a((r^2)^n-1)}{r^2-1}$

$\Rightarrow \quad a(r+1) = 5a \Rightarrow r = 4$

5. (a) Sum to infinity is given by

$$S_\infty = \frac{a}{1-r}, r \neq 1$$

6. (i) Here, $a = \sqrt{7}, r = \frac{\sqrt{21}}{\sqrt{7}} = \frac{\sqrt{7\times 3}}{\sqrt{7}} = \sqrt{3} > 1$

$$\Rightarrow S_n = \frac{\sqrt{7}[(\sqrt{3})^n - 1]}{\sqrt{3}-1} = \frac{\sqrt{7}}{2}(\sqrt{3}+1)(3^{n/2}-1)$$

(ii) Here, $a = 1, r = -\frac{a}{1} = -a < 1$

$$S_n = \frac{1\{1-(-a)^n\}}{1-(-a)} = \frac{1-(-a)^n}{1+a}$$

(iii) Solve as part (i). **Ans.** 2186

(iv) Solve as part (i). **Ans.** $121(\sqrt{3}+1)$

(v) Solve as part (i). **Ans.** $\frac{1}{6}[1-(0.1)^{20}]$

7. $\sum_{k=1}^{11} 2 + \sum_{k=1}^{11} 3^k = 2\times 11 + (3^1 + 3^2 + 3^3 + \ldots + 3^{11})$

$$= 22 + \frac{3(3^{11}-1)}{3-1} = 22 + \frac{3(3^{11}-1)}{2}$$

8. Here, $a = 5$ and $r = 2$

$\because \quad \frac{a(r^n-1)}{r-1} = 315 \Rightarrow \frac{5(2^n-1)}{2-1} = 315$

Ans. Number of terms = 6 and last term = 160.

9. (i) Here, $a = 6$ and $r = \frac{T_2}{T_1} = \frac{1.2}{6} = 0.2$

Since, $|0.2| = 0.2 < 1$ **Ans.** $S_\infty = 7.5$

(ii) Here, $a = 10$ and $r = \frac{T_2}{T_1} = \frac{-9}{10} = -0.9$

Since, $|-0.9| = 0.9 < 1$ **Ans.** $S_\infty = \frac{100}{19}$

10. We have, $r = -\frac{4}{5}$ and $S = \frac{80}{9}$.

$\therefore \quad \frac{80}{9} = \frac{a}{1-\left(\frac{-4}{5}\right)}$ **Ans.** $a = 16$

11. Here, $a = 1, r = \frac{3}{1} = 3 \therefore \frac{1\cdot(3^n-1)}{3-1} = 3280$ **Ans.** $n = 8$

12. $\frac{1}{81} = 27(r)^{8-1} \Rightarrow r = \frac{1}{3}$ **Ans.** $\frac{81}{2}\left(1-\frac{1}{3^{10}}\right)$

13. $\frac{85}{128}$

14. $ar^{7-1} = 64 \Rightarrow r^6 = \left(\frac{2}{3}\right)^6 \Rightarrow r = \frac{2}{3}$

$$\therefore \quad S_7 = \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-\frac{2}{3}} = 2059$$

15. Let the GP is $\underbrace{ar, ar^2, ar^3, ar^4, ar^5, \ldots, ar^{n-1}}_{n \text{ terms}}$, $\underbrace{ar^n, ar^{n+1}, \ldots, ar^{2n-1}}_{n \text{ terms}}$

Now, required ratio $= \dfrac{\frac{a(r^n-1)}{r-1}}{\frac{ar^n(r^n-1)}{r-1}}$

16. Given, $a = 5$, $r = 2$ and $S_n = 315$

Therefore, $315 = \frac{5(2^n-1)}{2-1} \Rightarrow n = 6$

Ans. $T_6 = 160$

17. Sum = 15 $\Rightarrow \frac{a}{1-r} = 15$

Sum of the squares = 45 $\Rightarrow (a^2 + a^2r^2 + a^2r^2 + \ldots \infty) = 45$

$\Rightarrow \frac{a^2}{1-r^2} = 45 \; \frac{a^2}{(1-r)^2} \times \frac{1-r^2}{a^2} = \frac{(15)^2}{45}$

Ans. $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \ldots \infty$.

18. (i) $5 + 55 + 555 + \ldots n$ terms

$= \frac{5}{9}(9 + 99 + 999 + \ldots\ n \text{ terms})$

$= \frac{5}{9}[(10-1) + (10^2-1) + (10^3-1) + \ldots + (10^n-1)]$

$= \frac{5}{9}[(10 + 10^2 + 10^3 + \ldots + 10^n) - (1 + 1 + \ldots + 1)\ n \text{ terms}]$

$= \frac{5}{9}\left[10\left(\frac{10^n-1}{10-1}\right) - n\right]$

Ans. $\frac{50}{81}(10^n-1) - \frac{5n}{9}$

(ii) Solve as part (i). **Ans.** $\frac{70}{81}(10^n-1) - \frac{7n}{9}$

19. (i) $6\times 0.1 + 6\times 0.11 + 6\times 0.111 + \ldots n$ terms

$= \frac{6}{9}[0.9 + 0.99 + 0.999 + \ldots n \text{ terms}]$

$= \frac{2}{3}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots n \text{ terms}\right]$

$= \frac{2}{3}\left[\left(1-\frac{1}{10}\right) + \left(1-\frac{1}{100}\right) + \left(1-\frac{1}{1000}\right) + \ldots n \text{ terms}\right]$

$$= \frac{2}{3}\left[(1+1+1+\ldots n \text{ terms}) - \left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots n \text{ terms}\right)\right]$$

Ans. $\frac{2}{3}n - \frac{2}{27}(1-10^{-n})$

(ii) $\frac{1}{3}n - \frac{1}{27}(1-10^{-n})$

20. $0.1\bar{5} = 0.15555\ldots = 0.1 + 0.05 + 0.005 + 0.0005 + \ldots \infty]$

$$= 0.1 + \left[\frac{5}{100}+\frac{5}{1000}+\frac{5}{10000}+\ldots\infty\right] = 0.1 + \frac{\left(\frac{5}{100}\right)}{1-\frac{1}{10}} = \frac{7}{45}$$

21. (i) Sum of the perimeters of all the triangles is given by

$$S_p = 54 + 27 + 13.5 + \ldots\infty = \frac{54}{1-\frac{1}{2}} = 54 \times 2 = 108 \text{ cm}$$

(ii) We know that the area of the triangle formed by joining the mid-points of the sides one-forth of the given triangle.

$\therefore$ Area of equilateral $\Delta ABC = \frac{\sqrt{3}}{4} \times (18)^2 = 81\sqrt{3}$ cm

Area of $\Delta DEF = \frac{1}{4} \times$ area of ΔABC

$$= \frac{1}{4} \times 81\sqrt{3} = \frac{81}{4}\sqrt{3} \text{ cm}$$

and area of $\Delta GHI = \frac{1}{4} \times$ area of ΔABC

$$= \frac{1}{4} \times \frac{81}{4}\sqrt{3} = \frac{81}{16}\sqrt{3} \text{ cm}$$

$\therefore$ Sum of the areas of all the triangles is

$$S_a = 81\sqrt{3} + \frac{81}{4}\sqrt{3} + \frac{81}{16}\sqrt{3} + \ldots\infty$$

Clearly, it is geometric series whose $a = 81\sqrt{3}$ and $r = \frac{1}{4}$

$$\therefore \quad S_a = \frac{a}{1-r} = \frac{81\sqrt{3}}{1-\frac{1}{4}} = 81\sqrt{3} \times \frac{4}{3} = 108\sqrt{3} \text{ cm}^2$$

22. $\frac{a}{r} + a + ar = 52 \Rightarrow a\left(\frac{1}{r}+1+r\right) = 52$ ...(i)

and $\frac{a}{r}\cdot a + a\cdot ar + \frac{a}{r}\cdot ar = 624 \Rightarrow a^2\left(\frac{1}{r}+r+1\right) = 624$ ...(ii)

On dividing Eq. (ii) by Eq. (i), we get $a = 12$

On putting $a = 12$ in Eq. (i), we get

$$12\left(\frac{1}{r}+r+1\right) = 52$$

$$\Rightarrow \quad r = \frac{1}{3} \text{ or } r = 3$$

Ans. When $r = \frac{1}{3}$, then numbers are 36, 12, 4.

When $r = 3$, then numbers are 4, 12, 36.

23. Here, $T_1 = a = 3$, $T_n = 96$ and $S_n = 189$

$$S_n = \frac{a(r^n-1)}{r-1} = \frac{ar^n - a}{r-1} \Rightarrow S_n = \frac{(ar^{n-1})r - a}{r-1}$$

$$\Rightarrow 189 = \frac{96r - a}{r-1} \Rightarrow r = 2$$

Now, $\quad T_n = ar^{n-1}$

$\Rightarrow \quad 96 = 3 \times 2^{n-1} \Rightarrow \quad 2^5 = 2^{n-1}$

On equating exponential power from both sides, we get

$n = 6$

24. Clearly, x, y and z are the sums of infinite geometric progressions.

$$\therefore \quad x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1},\ y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$$

and $$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

Now, $$xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$

$$\Rightarrow \quad \frac{xy}{z} = \left\{\left(\frac{abr^2}{r^2-1}\right) \div \frac{cr^2}{(r^2-1)}\right\} = \frac{ab}{c}$$

25. $2^{1/2}\cdot 4^{1/8}\cdot 8^{1/24}\cdot 16^{1/64}\ldots\infty$

$$= 2^{\frac{1}{2}}\cdot 2^{\frac{1}{4}}\cdot 2^{\frac{3}{24}}\cdot 2^{\frac{1}{16}}\ldots = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\infty}$$

26. Required number of grains $= 1 + 2 + 2^2 + \ldots$ to 64 terms

$= 1 + (2 + 2^2 + 2^3 + \ldots + 2^{63})$

$$= \left\{1 + \frac{2(2^{63}-1)}{(2-1)}\right\} = (2^{64}-1)$$

27. Successive number of letters are 4, 16, 64, ...

This is a GP with $a = 4$ and $r = 4$.

Number of letters in the 8th set $= 4 \times 4^{(8-1)} = 4^8 = 65536$

Ans Cost of postage = ₹ (65526×2) = ₹ 131072

28. $10000\left(\frac{21}{20}\right)^{15}$ **29.** ₹ 39100 **30.** ₹ 16680

31. $S_1 = \frac{1}{1-\frac{1}{2}} = 2,\ S_2 = \frac{2}{1-\frac{1}{3}} = 3,\ S_3 = \frac{3}{1-\frac{1}{4}} = 4$

....

....

and $$S_p = \frac{p}{1-\frac{1}{p+1}} = p+1$$

34. Given, $\quad f(x+y) = f(x)\cdot f(y)$ and $f(1) = 2$

Therefore, $f(2) = f(1+1) = f(1)\cdot f(1) = 2^2$

$f(3) = f(1+2) = f(1)\cdot f(2) = 2^3$

$f(4) = f(1+3) = f(1)\cdot f(3) = 2^4$

and so on. continuing the process, we obtain

$f(k) = 2^k$ and $f(a) = 2^a$

Hence, $\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)\cdot f(k) = f(a)\sum_{k=1}^{n} f(k)$

$$= 2^a(2^1 + 2^2 + 2^3 + \ldots + 2^n)$$

$$= 2^a\left\{\frac{2\cdot(2^n-1)}{2-1}\right\} = 2^{a+1}(2^n-1) \quad \ldots\text{(i)}$$

But we are given, $\sum_{k=1}^{n} f(a+k) = 16(2^n-1)$ ...(ii)

From Eqs. (i) and (ii), we get

$$= 2^{a+1}(2^n-1) = 16(2^n-1)$$

$$\Rightarrow \quad 2^{a+1} = 2^4$$

$$\Rightarrow \quad a+1 = 4 \Rightarrow a = 3$$

35. Given, $f(x+y) = f(x)\cdot f(y)$ ...(i)

Putting $x = y = 1$ in Eq. (i), we get

$f(1+1) = f(1)f(1) \Rightarrow f(2) = 3\times 3 = 9$ $[\because f(1) = 3]$

Putting $x = 2,\ y = 1$ in Eq. (i), we get

$f(2+1) = f(2)f(1)$

$\Rightarrow f(3) = 9\times 3 = 27$ $[\because f(2) = 9,\ f(1) = 3]$

Putting $x = 3,\ y = 1$ in Eq. (i), we get

$f(3+1) = f(3)f(1)$

$\Rightarrow f(4) = 27\times 3 = 81$ $[\because f(3) = 27]$

Now, $\sum_{x=1}^{n} f(x) = 120$

$\Rightarrow f(1) + f(2) + f(3) + \ldots + f(n) = 120$

$\Rightarrow 3 + 9 + 27 + \ldots n$ terms $= 120$

$$= \frac{3(3^n-1)}{3-1} = 120 \quad \left[\text{here, } a = 3,\ r = \frac{9}{3} = 3\right]$$

$\Rightarrow 3(3^n-1) = 120\times 2 \Rightarrow 3(3^n-1) = 240$

$\Rightarrow 3^n - 1 = \frac{240}{3} \Rightarrow 3^n - 1 = 80 \Rightarrow 3^n = 80 + 1$

On comparing the power of 3 from both sides, we get

$$3^n = 81 \Rightarrow 3^n = 3^4 \Rightarrow n = 4$$

|TOPIC 6|

Geometric Mean and Its Relation with Arithmetic Mean

GEOMETRIC MEAN (GM)

Let a and b be two positive numbers. If we insert a single number G between them, so that a, G, b is a GP, then G is called **geometric mean**. Thus, a, G, b are in GP.

$$\therefore \quad \frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \sqrt{ab}$$

Inserting *n* GM's between Two Positive Numbers

Let a and b be two positive numbers and $G_1, G_2, G_3, \ldots, G_n$ be the n GM's between a and b.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP with common ratio r and number of terms are $n+2$.

Here, b is $(n+2)$th term.

i.e. $$b = T_{n+2} = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Thus, the n geometric means are given below

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

....

....

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

EXAMPLE |1| Find a geometric mean of 4 and 16.

Sol. Geometric mean between 4 and 16

$$= \sqrt{4\times 16} = \sqrt{64} = 8$$

EXAMPLE |2| Insert two numbers between 3 and 81, so that resulting sequence is GP. **[NCERT]**

Sol. Let the two numbers are a and b, then $3, a, b, 81$ are in GP.

$\because$ nth term, $T_n = Ar^{n-1}$

$\therefore T_4 = 81 = 3r^{4-1} \Rightarrow r^3 = \frac{81}{3} \Rightarrow r^3 = 27 \Rightarrow r^3 = 3^3$

On comparing the base of power 3 from both sides, we get $r = 3$,

Hence, $a = Ar = 3\times 3 = 9,\ b = Ar^2 = 3\times 3^2 = 27.$

EXAMPLE |3| Insert 4 GM's between 3 and 96.

Sol. Let G_1, G_2, G_3 and G_4 be the required GM's. Then, $3, G_1, G_2, G_3, G_4, 96$ are in GP.

Let r be the common ratio. Here, 96 is the 6th term.

$\therefore \quad 96 = ar^{6-1} = 3r^5$

$\Rightarrow \quad 32 = r^5 \Rightarrow (2)^5 = r^5 \Rightarrow r = 2$

$\therefore \quad G_1 = ar = 3 \cdot 2 = 6$

$G_2 = ar^2 = 3 \cdot 2^2 = 12$

$G_3 = ar^3 = 3 \cdot 2^3 = 24$

and $\quad G_4 = ar^4 = 3 \cdot 2^4 = 48$

EXAMPLE |4| If the 4th, 10th and 16th terms of a GP are x, y and z, respectively, then prove that x, y, z are in GP.

Sol. Given, $T_4 = x \Rightarrow ar^{4-1} = x \Rightarrow ar^3 = x$ ...(i)

$T_{10} = y \Rightarrow ar^{10-1} = y \Rightarrow ar^9 = y$ ...(ii)

$T_{16} = z \Rightarrow ar^{16-1} = z \Rightarrow ar^{15} = z$ ...(iii)

Now, on multiplying Eq. (i) by Eq. (iii), we get

$ar^3 \times ar^{15} = x \times z \Rightarrow a^2r^{3+15} = xz$

$\Rightarrow \quad a^2r^{18} = xz \quad \Rightarrow \quad (ar^9)^2 = xz$

$\Rightarrow \quad y^2 = xz$ [from Eq. (ii)]

Therefore, x, y, z are in GP.

EXAMPLE |5| If a, b, c, d are in GP, then prove that $a+b, b+c, c+d$ are also in GP.

Sol. Let r be the common ratio of the GP a, b, c, d. Then,

$b = ar, c = ar^2$ and $d = ar^3$

$\therefore \quad a + b = a + ar = a(1+r), b + c = ar + ar^2$

$= ar(1+r)$ and $c + d = ar^2 + ar^3 = ar^2(1+r)$

Now, $(b+c)^2 = \{ar(1+r)\}^2 = a^2r^2(1+r)^2$

$= \{a(1+r)\}\{ar^2(1+r)\} = (a+b)(c+d)$

$[\because a + b = a(1+r)$ and $c + d = ar^2(1+r)]$

Here, $a+b, b+c, c+d$ are in GP.

EXAMPLE |6| If $a^2 + b^2, ab + bc$ and $b^2 + c^2$ are in GP, then prove that a, b, c are also in GP.

Sol. Given that $a^2 + b^2, ab + bc, b^2 + c^2$ are in GP.

$\therefore \quad (ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$

$\Rightarrow a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$

$\Rightarrow \quad b^4 + a^2c^2 - 2ab^2c = 0$

$\Rightarrow (b^2 - ac)^2 = 0 \Rightarrow b^2 = ac \Rightarrow a, b, c$ are in GP.

EXAMPLE |7| If a, b, c, d are in GP, then prove that $a^n + b^n, b^n + c^n, c^n + d^n$ are in GP. **[NCERT]**

Sol. Given a, b, c, d are in GP.

$\Rightarrow \quad b = ar, c = ar^2, d = ar^3$ ...(i)

Now, we have to prove $a^n + b^n, b^n + c^n, c^n + d^n$ are in GP.

$\Rightarrow \quad (b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$

Now, $\quad \text{RHS} = (a^n + b^n)(c^n + d^n)$

$= [a^n + a^nr^n] \times [a^nr^{2n} + a^nr^{3n}]$ [from Eq. (i)]

$= a^n(1 + r^n)a^nr^{2n}(1 + r^n)$

$= a^{2n}r^{2n}(1 + r^n)^2 = [a^nr^n(1 + r^n)]^2$

$= [a^nr^n + a^nr^{2n}]^2$

$= [(ar)^n + (ar^2)^n]^2 = [b^n + c^n]^2$

[from Eq. (i)] $=$ LHS

$\therefore \quad \text{LHS} = \text{RHS}$ **Hence proved.**

EXAMPLE |8| The sum of two numbers is 6 times their geometric mean, show that the numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$. **[NCERT]**

After taking the given condition, we will use componendo and dividendo,

i.e. $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Sol. Let the numbers are a and b.

According to the question $a + b = 6\sqrt{ab}$

$$\Rightarrow \quad \frac{a+b}{2\sqrt{ab}} = \frac{3}{1}$$

Now, applying componendo and dividendo, we get

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3+1}{3-1}$$

$$\Rightarrow \quad \frac{(\sqrt{a})^2 + (\sqrt{b})^2 + 2\sqrt{ab}}{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{ab}} = \frac{4}{2}$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{2}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{2}}{1}$$

Again, applying componendo and dividendo, we get

$$\frac{\sqrt{a}+\sqrt{b}+\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow \quad \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

Now, on squaring both sides, we get

$$\frac{a}{b} = \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)^2}$$

$\Rightarrow \quad \frac{a}{b} = \frac{2+1+2\sqrt{2}}{2+1-2\sqrt{2}}$ $\left[\because (a+b)^2 = a^2+b^2+2ab \text{ and } (a-b)^2 = a^2+b^2-2ab\right]$

$\Rightarrow \quad \frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$

$\therefore \quad a:b = (3+2\sqrt{2}):(3-2\sqrt{2})$

EXAMPLE |9| Prove that the product of n GMs between any two positive numbers is equal to nth power of the GM between them.

Sol. Let $G_1, G_2, \ldots, G_n$ be the n GMs between positive numbers a and b. Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ are in GP.

Let r be the common ratio of this GP and b is $(n+2)$ th term.

Now, $\quad b = T_{n+2} = ar^{n+1}$

$\Rightarrow \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Product of n GMs between a and b

$= G_1, G_2, \ldots, G_n = ar \cdot ar^2, \ldots, ar^n$

$= a^n r^{1+2+\ldots+n} = a^n \cdot r^{\frac{n}{2}[2a+(n-1)d]}$

$= a^n \cdot r^{\frac{n(n+1)}{2}} = a^n \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}}$ $\left[\because r = \left(\frac{b}{a}\right)^{1/n+1}\right]$

$= a^n\left(\frac{b}{a}\right)^{\frac{n}{2}} = a^{n-\frac{n}{2}} \cdot b^{\frac{n}{2}} = a^{\frac{n}{2}} \cdot b^{\frac{n}{2}} = (\sqrt{ab})^n$

Hence, product of n GMs between a and b is equal to the nth power of the GM between a and b.

Relation between AM and GM

Let A and G be the AM and GM of two positive real numbers a and b, respectively.

Then, $\quad A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Now, $\quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$\Rightarrow \quad A - G = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0 \Rightarrow A \geq G$

METHOD OF FINDING TWO NUMBERS, WHEN AM AND GM ARE GIVEN

Let A and G be the AM and GM between two positive numbers, then the two positive numbers will be

$$a = A + \sqrt{A^2 - G^2} \text{ and } b = A - \sqrt{A^2 - G^2}$$

e.g. Suppose arithmetic mean is $A = 5$ and geometric mean is $G = 3$. Then, the two numbers will be

$a = A + \sqrt{A^2 - G^2} = 5 + \sqrt{(5)^2 - (3)^2}$

$= 5 + \sqrt{25-9} = 5 + \sqrt{16} = 5 + 4 = 9$

and $\quad b = A - \sqrt{A^2 - G^2} = 5 - \sqrt{(5)^2 - (3)^2}$

$= 5 - \sqrt{25-9} = 5 - \sqrt{16} = 5 - 4 = 1$

EXAMPLE |10| If arithmetic mean and geometric mean between two numbers is 5 and 4 respectively, then find the two numbers.

Sol. Given, arithmetic mean, $A = 5$

and geometric mean, $G = 4$

Let the two numbers be a and b.

Then, $\quad a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$

$\therefore \quad a = 5 + \sqrt{5^2 - 4^2}$ and $b = 5 - \sqrt{5^2 - 4^2}$

$\Rightarrow \quad a = 5 + \sqrt{25-16}$ and $b = 5 - \sqrt{25-16}$

$\Rightarrow \quad a = 5 + \sqrt{9}$ and $b = 5 - \sqrt{9}$

$\Rightarrow \quad a = 5 + 3$ and $b = 5 - 3 \Rightarrow a = 8$ and $b = 2$

Hence, the required numbers are 2 and 8.

EXAMPLE |11| Find two positive numbers whose difference is 12 and whose AM exceeds the GM by 2.

Sol. Let the two numbers be a and b such that $a > b$.

Given $\quad a - b = 12 \quad$ ...(i)

and $\quad$ AM − GM = 2

$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} = 2$ $\left[\because \text{AM} = \frac{a+b}{2} \text{ and GM} = \sqrt{ab}\right]$

$\Rightarrow \quad a + b - 2\sqrt{ab} = 4$

$\Rightarrow \quad (\sqrt{a} - \sqrt{b})^2 = 4$

$\Rightarrow \quad \sqrt{a} - \sqrt{b} = 2 \quad$ ...(ii)

Now, $\quad a - b = 12$

$\Rightarrow \quad (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = 12$

$[\because x^2 - y^2 = (x-y)(x+y)]$

$\Rightarrow \quad (\sqrt{a} + \sqrt{b}) \times (2) = 12 \quad$ [using Eq. (ii)]

$\Rightarrow \quad \sqrt{a} + \sqrt{b} = 6 \quad$ ...(iii)

On adding Eqs. (ii) and (iii), we get

$2\sqrt{a} = 8 \Rightarrow \sqrt{a} = 4$

$\Rightarrow \quad a = (4)^2 \Rightarrow a = 16$

Then, from Eq. (i), we get

$16 - b = 12$

$\Rightarrow \quad b = 16 - 12 = 4$

Hence, required numbers are 16 and 4.

EXAMPLE |12| Find the minimum value of $4^x + 4^{1-x}$, $x \in R$. **[NCERT Exemplar]**

Sol. We know that, AM $\geq$ GM

$$\therefore \quad \frac{4^x + \frac{4}{4^x}}{2} \geq \sqrt{4^x \times \frac{4}{4^x}}$$

$$\Rightarrow \quad 4^x + \frac{4}{4^x} \geq 2\sqrt{4} \Rightarrow 4^x + \frac{4}{4^x} \geq 4$$

Hence, the minimum value of given expression is 4.

EXAMPLE |13| If a, b and c be positive numbers, then prove that $a^2 + b^2 + c^2$ is greater than $ab + bc + ca$.

Sol. We know that, AM > GM

$$\therefore \quad \frac{a^2+b^2}{2} > \sqrt{a^2b^2} \Rightarrow \frac{a^2+b^2}{2} > ab \quad \text{...(i)}$$

Similarly, $$\frac{b^2+c^2}{2} > \sqrt{b^2c^2} \Rightarrow \frac{b^2+c^2}{2} > bc \quad \text{...(ii)}$$

and $$\frac{c^2+a^2}{2} > \sqrt{c^2a^2} \Rightarrow \frac{c^2+a^2}{2} > ca \quad \text{...(iii)}$$

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{a^2+b^2}{2} + \frac{b^2+c^2}{2} + \frac{c^2+a^2}{2} > ab + bc + ca$$

$$\Rightarrow \quad a^2 + b^2 + c^2 > ab + bc + ca$$ **Hence proved.**

EXAMPLE |14|

(i) If a, b, c, d are four distinct positive quantities in AP, then show that $bc > ad$.

(ii) If a, b, c, d are four distinct positive quantities in GP then show that $a + d > b + c$. **[NCERT Exemplar]**

Sol. (i) Given a, b, c, d are in AP and we know that AM > GM, then for the first three terms.

$$b > \sqrt{ac} \quad \left[\text{here, } \frac{a+c}{2} = b\right]$$

On squaring both sides, we get

$$b^2 > ac \quad \text{...(i)}$$

Similarly, for the last three terms

$$c > \sqrt{bd} \quad \left[\text{here, } \frac{b+d}{2} = c\right]$$

$$\Rightarrow \quad c^2 > bd \quad \text{...(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$b^2c^2 > (ac)(bd) \Rightarrow bc > ad$$

(ii) Given, a, b, c, d are in GP.

and we know that AM > GM then for the first three terms

$$\frac{a+c}{2} > b \quad [\because \sqrt{ac} = b]$$

$$\Rightarrow \quad a + c > 2b \quad \text{...(iii)}$$

Similarly, for the last three terms

$$\frac{b+d}{2} > c \quad [\because \sqrt{bd} = c]$$

$$\Rightarrow \quad b + d > 2c \quad \text{...(iv)}$$

On adding Eqs. (iii) and (iv), we get

$$(a+c) + (b+d) > 2b + 2c \Rightarrow a + d > b + c$$

EXAMPLE |15| If A and G be AM and GM respectively, between two positive numbers, then prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

We know that, if roots of a quadratic equation are given, then quadratic equation is
$x^2 -$ (Sum of roots)x + Product of roots = 0.

Sol. Let the numbers are α and β.

Given, sum of the roots, $\frac{\alpha+\beta}{2} = A$ [arithmetic mean]

$$\Rightarrow \quad \alpha + \beta = 2A$$

and product of the roots,

$$\sqrt{\alpha\beta} = G \quad \text{[geometric mean]}$$

$$\Rightarrow \quad \alpha\beta = G^2$$

Now, quadratic equation having roots α and β is

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

$$\Rightarrow \quad x^2 - 2Ax + G^2 = 0$$

$$\Rightarrow x = \frac{2A \pm \sqrt{4A^2 - 4 \times 1 \times G^2}}{2 \times 1} \quad \left[\because x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\right]$$

$$= \frac{2A \pm 2\sqrt{A^2 - G^2}}{2}$$

$$= A \pm \sqrt{(A+G)(A-G)} \quad [\because a^2 - b^2 = (a+b)(a-b)]$$

Hence proved.

EXAMPLE |16| The ratio of the AM and GM of two positive numbers a and b is $m : n$. Show that $a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2})$. **[NCERT]**

Sol. Let the AM of the number a and b is A and GM of a and b is G, then $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Given, $A : G = m : n$

i.e. $$\frac{A}{G} = \frac{m}{n} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$

On applying componendo and dividendo rule, we get

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{m+n}{m-n}$$

$$\Rightarrow \frac{(\sqrt{a})^2 + (\sqrt{b})^2 + 2\sqrt{ab}}{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{ab}} = \frac{m+n}{m-n}$$

$$\Rightarrow \quad \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{m+n}{m-n} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{m+n}}{\sqrt{m-n}}$$

Again, applying componendo and dividendo rule, we get

$$\frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})}=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

$$\Rightarrow \quad \frac{2\sqrt{a}}{2\sqrt{b}}=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

$$\Rightarrow \quad \frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

Now, squaring both sides, we get

$$\frac{a}{b}=\frac{(\sqrt{m+n}+\sqrt{m-n})^2}{(\sqrt{m+n}-\sqrt{m-n})^2} \quad \left[\begin{array}{l}\because (a+b)^2=a^2+b^2+2ab\\ \text{and } (a-b)^2=a^2+b^2-2ab\end{array}\right]$$

$$\Rightarrow \frac{a}{b}=\frac{m+n+m-n+2\sqrt{m+n}\sqrt{m-n}}{m+n+m-n-2\sqrt{m+n}\sqrt{m-n}}$$

$$\Rightarrow \frac{a}{b}=\frac{2m+2\sqrt{m^2-n^2}}{2m-2\sqrt{m^2-n^2}} \quad [\because (a+b)(a-b)=a^2-b^2]$$

$$\Rightarrow \frac{a}{b}=\frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}$$

or $a:b=(m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$ **Hence proved.**

Problems Based on AP and GP

Sometimes, in given problem we have some terms in AP or in GP and we have to show that other given terms will be in GP or in AP.

To solve these types of problems we use the following conditions and then simplify

(i) a, b, c are in AP iff $b=\dfrac{a+c}{2}$

(ii) a, b, c are in GP iff $b^2=ac$

EXAMPLE |17| Three numbers whose sum is 15 are in A.P. If 1, 4, 19 be added to them respectively, then they are in GP. Find the numbers.

Sol. Let the three numbers $a-d$, a, $a+d$. Then,

Sum $=15 \Rightarrow (a-d)+a+(a+d)=15 \Rightarrow a=5.$

So, the numbers are $5-d$, 5, $5+d$. Adding 1, 4, 19, respectively to these numbers, we get $6-d$, 9, $24+d$. These numbers are in GP.

$$\therefore \quad 9^2=(6-d)(24+d) \Rightarrow 81=144+6d-24d-d^2$$

$$\Rightarrow \quad d^2+18d-63=0 \Rightarrow (d+21)(d-3)=0$$

$$\Rightarrow \quad d=-21 \text{ or } d=3.$$

When $d=-21$ then numbers are $5-(21)$, 5, $5+(-21)$ i.e., 26, 5, -16 and when $d=3$, then numbers are $5-3$, 5, $5+3$ i.e., 2, 5, 8.

Hence, the numbers are 26, 5, -16 or 2, 5, 8.

EXAMPLE |18| If reciprocals of $\dfrac{x+y}{2}$, y, $\dfrac{y+z}{2}$ are in AP. Show that x, y, z in GP.

Sol. Given, $\dfrac{2}{x+y}, \dfrac{1}{y}, \dfrac{2}{y+z}$ are in AP.

$$\therefore \quad \frac{2}{y}=\frac{2}{x+y}+\frac{2}{y+z} \quad \left[\begin{array}{l}\because \text{ if } a, b, c \text{ are in AP, then}\\ b=\dfrac{a+c}{2} \text{ or } 2b=a+c\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{y}=\frac{y+z+x+y}{(x+y)(y+z)} \Rightarrow \frac{1}{y}=\frac{x+2y+z}{xy+xz+y^2+yz}$$

$$\Rightarrow \quad xy+xz+y^2+yz=xy+2y^2+yz$$

$$\Rightarrow \quad y^2+xz=2y^2 \Rightarrow y^2=xz$$

Hence, x, y, z are in GP.

EXAMPLE |19| If pth, qth, rth terms of an AP and GP are both a, b and c, respectively, then show that $a^{b-c}\cdot b^{c-a}\cdot c^{a-b}=1$ **[NCERT Exemplar]**

Sol. Let A and d be the first term and common difference of AP and x, R be the first term and common ratio of GP, respectively.

According to the given condition,

$$A+(p-1)d=a \quad \text{...(i)}$$

$$A+(q-1)d=b \quad \text{...(ii)}$$

$$A+(r-1)d=c \quad \text{...(iii)}$$

and

$$a=xR^{p-1} \quad \text{...(iv)}$$

$$b=xR^{q-1} \quad \text{...(v)}$$

$$c=xR^{r-1} \quad \text{...(vi)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$d(p-1-q+1)=a-b$$

$$\Rightarrow \quad a-b=d(p-q) \quad \text{...(vii)}$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$d(q-1-r+1)=b-c$$

$$\Rightarrow \quad b-c=d(q-r) \quad \text{...(viii)}$$

On subtracting Eq. (i) from Eq. (iii), we get

$$d(r-1-p+1)=c-a \Rightarrow c-a=d(r-p) \quad \text{...(ix)}$$

Now, we have to prove $a^{b-c}\,b^{c-a}\,c^{a-b}=1$

LHS $=a^{b-c}\,b^{c-a}\,c^{a-b}$

Using Eqs. (iv), (v), (vi) and (vii), (viii), (ix), we get

$$\text{LHS}=(xR^{p-1})^{d(q-r)}\,(xR^{q-1})^{d(r-p)}\,(xR^{r-1})^{d(p-q)}$$

$$=x^{d(q-r)+d(r-p)+d(p-q)}\;R^{(p-1)d(q-r)+(q-1)d(r-p)+(r-1)d(p-q)}$$

$$=x^{d(q-r+r-p+p-q)}\;R^{d(pq-pr-q+r+qr-pq-r+p+rp-rq-p+q)}$$

$$=x^0\,R^0=1=\text{RHS}$$ **Hence proved.**

TOPIC PRACTICE 6

OBJECTIVE TYPE QUESTIONS

1 The geometric mean of two positive numbers a and b is

(a) $\frac{a+b}{2}$ (b) $\frac{a-b}{2}$ (c) $\sqrt{ab}$ (d) ab

2 The geometric mean of 2 and 8 is

(a) 4 (b) 6 (c) 7 (d) 5

3 The numbers which can be inserted between two positive numbers a and b to make the resulting sequence in a G.P., are

(a) three (b) four
(c) many (d) infinitely many

4 If A and G are AM and GM of two given positive real numbers a and b respectively, then A and G are related as

(a) $A \geq G$ (b) $G \geq A$ (c) $A = G$ (d) $A = -G$

5 If A and G be the AM and GM between two positive numbers, then two positive numbers a and b are respectively.

(a) $A+\sqrt{A^2-G^2}$ and $A-\sqrt{A^2-G^2}$
(b) $A-\sqrt{A^2-G^2}$ and $A+\sqrt{A^2-G^2}$
(c) $A+\sqrt{A^2+G^2}$ and $A+\sqrt{A^2+G^2}$
(d) None of the above

VERY SHORT ANSWER Type Question

6 Find the GM between the following numbers. (Each part carries 1 mark)

(i) 1 and $\frac{9}{16}$ (ii) a^3b and ab^3.

SHORT ANSWER Type I Questions

7 Insert three GMs between 1 and 256. **[NCERT]**

8 Insert two numbers between 9 and 243, so that the resulting sequence is an GP.

9 Insert three numbers between $\frac{1}{3}$ and 432, so that the resulting sequence is a GP.

10 Insert four numbers between 6 and 192, so that the resulting sequence is an GP.

11 Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.

12 Find the minimum values of the expression $3^x + 3^{1-x}$, $x \in R$. **[NCERT Exemplar]**

13 If x, y, z are distinct positive integers, then prove that $(x+y)(y+z)(z+x) > 8xyz$. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

14 If a, b, c are in GP, then prove that

(i) $a(b^2+c^2) = c(a^2+b^2)$

(ii) $\frac{1}{a^2-b^2} + \frac{1}{b^2} = \frac{1}{b^2-c^2}$.

15 If a, b, c, d are in GP, then prove that

(i) $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$

(ii) $(b+c)(b+d) = (c+a)(c+d)$.

16 The sum of the three numbers in GP is 56. If we substract the 1, 7, 21, from these numbers in that order, we obtain an arithmetic progression (AP). Find the numbers. **[NCERT]**

17 If x, $2y$ and $3z$ are in AP, where the distinct numbers x, y, z are in GP, then find the common ratio of the GP. **[NCERT Exemplar]**

18 If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then prove that

$$\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = 2A.$$

19 Find all the sequences which are simultaneously arithmetic and geometric progressions.

20 If a, b, c are in AP and b, c, d are in GP and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP, then prove that a, c, e are in GP. **[NCERT]**

LONG ANSWER Type Questions

21 If a, b, c are in GP, then prove that

$$\frac{a^2+ab+b^2}{bc+ca+ab} = \frac{b+a}{c+a}$$

22 Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b. **[NCERT]**

HINTS & ANSWERS

1. (c) The geometric mean of two positive numbers a and b is $\sqrt{ab}$.

2. (a) The geometric mean of 2 and 8 is $\sqrt{16}$ i.e. 4.

3. (d) Infinitely many numbers can be inserted between two positive numbers 'a' and 'b' to make the resulting sequence in a GP.

4. (a) If A and G are AM and GM of two given positive real numbers a and b respectively.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Now, $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$

$\Rightarrow A - G \geq 0 \Rightarrow A \geq G$

5. (a) If A and G be the AM and GM between two positive numbers a and b, then

$a = A + \sqrt{A^2 - G^2}$

and $b = A - \sqrt{A^2 - G^2}$

6. Use the formula, $G^2 = ab$

Ans. (i) $\frac{3}{4}$ (ii) a^2b^2

7. Let 1, G_1, G_2, G_3, 256 are in GP.

$\therefore T_5 = 256$

$\Rightarrow 1(r)^{5-1} = 256$

$\Rightarrow r = \pm 4$

If $r = 4$, then $G_1 = 1 \times (4)^1 = 4$;

$G_2 = 1(4)^2 = 16$

$G_2 = 1(4)^3 = 64$

and if $r = -4$,

then $G_1 = 1(-4) = -4$;

$G_2 = 1(-4)^2 = 16$;

$G_3 = -(-4)^3 = -64$

Ans. Hence, 4, 16, 64 or -4, 16, -64 are the three GMs.

8. 27, 81

9. (2, 12, 72) or (-2, 12, -72)

10. 12, 24, 48, 96

11. Let the two numbers be a and b such that $a > b$.

Then, $\frac{a+b}{2} = 34$ and $\sqrt{ab} = 16$

Now, $(a-b)^2 = (a+b)^2 - 4ab$

$\Rightarrow a - b = 60$

Ans. 64 and 4.

12. We know that,

$AM \geq GM \Rightarrow \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3^x \cdot 3^{1-x}}$

Ans. Minimum $= 2\sqrt{3}$

13. Since, AM > GM

$\frac{x+y}{2} > \sqrt{xy}$, $\frac{y+z}{2} > \sqrt{yz}$ and $\frac{z+x}{2} > \sqrt{zx}$

On multiplying the above inequalities, we get

$\frac{x+y}{2} \cdot \frac{y+z}{2} \cdot \frac{y+z}{2} > \sqrt{(xy)(yz)(zx)}$

or $(x+y)(y+z)(z+x) > 8xyz$

16. Given, $a + ar + ar^2 = 56$ ...(i)

Again, $a - 1$, $ar - 7$, $ar^2 - 21$ are in AP.

$\Rightarrow 2(ar - 7) = (a - 1) + (ar^2 - 21)$

$\Rightarrow a + ar^2 - 2ar = 8$ ...(ii)

On dividing Eq. (i) by Eq. (ii), we get

$\frac{a + ar + ar^2}{a + ar^2 - 2ar} = \frac{56}{8}$

$\Rightarrow 6r^2 - 15r + 6 = 0$

$\Rightarrow r = 2, \frac{1}{2}$

If $r = 2$, then from Eq. (i),

$a + 2a + 4a = 56$

$\Rightarrow a = 8$

If $r = \frac{1}{2}$, then from Eq. (i),

$\frac{a}{1} + \frac{a}{2} + \frac{a}{4} = 56$

$\Rightarrow a = 32$

Ans. Required numbers are 8, 16, 32 or 32, 16, 8.

17. Since, x, $2y$ and $3z$ are in AP.

$\therefore 4y = x + 3z$

And x, y, z are in GP.

$\therefore y = rx$ and $z = xr^2$

On putting the value of y and z in Eq. (i), we get

$4xr = x + 3xr^2$

$\Rightarrow 3r^2 - 4r + 1 = 0$

Ans. $r = \frac{1}{3}$ [$\because r = 1$ is not possible]

18. $A = \frac{a+b}{2}$ and $G_1 = ar$, $G_2 = ar^2$

where, $r = \left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3}$ ...(i)

$\therefore \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{a^2r^2}{ar^2} + \frac{a^2r^4}{ar}$

19. Let $T_1, T_2, T_3, \ldots$ be a sequence which is AP as well as GP.

Let $T_n = a + (n-1)d, \forall\, n \in N$

So, the sequence is $a, a+d, a+2d, \ldots$

This is also a GP.

$$\therefore \quad \frac{T_{n+1}}{T_n} = \frac{T_{n+2}}{T_{n+1}}, \forall\, n \in N$$

$$\Rightarrow \quad \frac{a+nd}{a+(n-1)d} = \frac{a+(n+1)d}{a+nd}$$

Ans. $a, a, a \ldots$

20. b, c, d are in GP.

$$\Rightarrow \quad c^2 = bd \qquad \ldots\text{(ii)}$$

Similarly, $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP.

$$\Rightarrow \quad \frac{2}{d} = \frac{1}{c} + \frac{1}{e} \Rightarrow d = \frac{2ce}{c+e} \qquad \ldots\text{(iii)}$$

On putting the values of b and d from Eq. (i) and (iii), in Eq. (ii), we get

$$c^2 = \left(\frac{a+c}{2}\right) \times \left(\frac{2ce}{c+e}\right) \Rightarrow c^2 = ae$$

Therefore, a, c, e are in GP.

22. Given, $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n} = \dfrac{a^{\frac{1}{2}} b^{\frac{1}{2}}}{1}$

$$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{n+\frac{1}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{n+\frac{1}{2}}$$

$$\Rightarrow \quad (a^{n+1} - a^{n+\frac{1}{2}} b^{\frac{1}{2}}) + (b^{n+1} - a^{\frac{1}{2}} b^{n+\frac{1}{2}}) = 0$$

$$\Rightarrow \quad (a^{n+\frac{1}{2}} - b^{n+\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) = 0$$

$$\Rightarrow \quad a^{n+\frac{1}{2}} - b^{n+\frac{1}{2}} = 0 \qquad [\because a^{\frac{1}{2}} - b^{\frac{1}{2}} \neq 0]$$

$$\therefore \quad a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = 1 = \left(\frac{a}{b}\right)^0$$

$$\Rightarrow \quad n + \frac{1}{2} = 0$$

$$\Rightarrow \quad n = -\frac{1}{2}$$

SUMMARY

- A **sequence** is a succession of numbers or terms formed according to some rule. A sequence is finite or infinite, according as number of terms in the corresponding sequence is finite or infinite.
- Let a sequence is $a_1, a_2, a_3, \ldots, a_n$. Then, the expression $a_1 + a_2 + a_3 + \ldots + a_n$ is called the **series** associated with the given sequence. A series is finite or infinite, according as number of terms in the corresponding sequence is finite or infinite.
- A sequence whose terms follow a certain pattern or rule, is called a **progression**.
- A sequence whose terms increases or decreases by a fixed number, is an **arithmetic progression**.
- **General Term of an AP** is T_n or $a_n = a + (n-1)d$. Also, $l = a + (n-1)d$

  where, a = first term, d = common difference and l = last term.
- ***m*th Term of an AP from the End** is $a_{n-m+1} = a + (n-m)d, n > m$ Also, $a_{n-m+1} = a_n - (m-1)d$ or $l - (m-1)d$
- If a, A and b are in AP, then arithmetic mean is $A = \frac{a+b}{2}$
- Let $A_1, A_2, \ldots, A_n$ be n numbers between positive numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ is an AP. Then, n arithmetic means between a and b are

$$A_1 = a + \frac{b-a}{n+1}, A_2 = a + \frac{2(b-a)}{n+1}, \ldots, A_n = a + \frac{n(b-a)}{n+1}.$$

- A sequence of non-zero numbers is said to be Geometric Progression (GP), if the ratio of each term except the first one, by its preceding term is always the same.
- **General Term of a GP** is $T_n = ar^{n-1}$ or $l = ar^{n-1}$, where a = first term, r = common ratio and l = last term.
- ***m*th Term of a Finite GP from the End** is ar^{n-m}, $n > m$ or $l\left(\frac{1}{r}\right)^{n-1}$.
- **Sum of finite *n* Terms of a GP** is $S_n = \frac{a(1-r^n)}{1-r}$, where $r < 1$ and $S_n = \frac{a(r^n - 1)}{r-1}$, where $r > 1 (r \neq 1)$
- **Sum of an infinite GP** is $S_\infty = \frac{a}{1-r}$.
- Relation between AM and GM is AM $\geq$ GM.
- Let a, G, b, are in GP, then **geometric mean**, $G = \sqrt{ab}$.
- Let $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, G_3, \ldots, G_n, b$ is a GP. Then, n geometric means

$$G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+2}}, \ldots, Gn = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. A sequence is called ...A... sequence, if it is not a finite sequence. Here, A refers to
(a) bounded (b) limited
(c) infinite (d) None of these

2. The sequence 3,3.3, 3.33, 3.333, 3.3333, ... is a/an ...A... sequence, since it ...B.. ends. Here, A and B refer to
(a) finite, never (b) infinite, never
(c) finite, always (d) infinite, always

3. The nth term a_n of a sequence can also be denoted as
(a) $n(a)$ (b) $a(n)$
(c) $(n)(a)$ (d) None of these

4. The series $a_1 + a_2 + a_3 + \ldots a_n$ is abbreviated as ...
(a) $\sum_{k=1}^{n} a_k$ (b) $\prod_{k=1}^{n} a_k$
(c) $\sum_{k=1}^{n} a_n$ (d) $\prod_{k=1}^{n} a_n$

5. The general term of a GP, whose first non-zero term is a and common ratio is r, can be written as
(a) $a_n = ar^n$ (b) $a_{n+1} = ar^{n+1}$
(c) $a_n = ar^{n-1}$ (d) $a_n = ar^{n+1}$

6. If the numbers $\frac{-2}{7}$, x, $\frac{-7}{2}$ are in GP, then the values of x are
(a) ± 1 (b) 1, 2 (c) -1, 2 (d) ± 2

7. If the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots, ar^{n-1}$ and $A, AR, AR^2, \ldots, AR^{n-1}$ form a GP, then the common ratio is ...Y... . Here, Y refers to
(a) r/R (b) rR (c) R (d) r

8. If a, b, c are in arithmetic progression, then the value of $(a+2b-c)(2b+c-a)(a+2b+c)$ is
(a) $16abc$ (b) $4abc$ (c) $8abc$ (d) $3abc$

SHORT ANSWER Type I Questions

9. Find the sum of the series
$\frac{3}{\sqrt{5}} + \frac{4}{\sqrt{5}} + \sqrt{5} + \ldots + 25$ terms.

10. If 5th and 8th terms of a GP be 48 and 384 respectively. Find the GP, if terms of GP are real numbers ?

11. If the pth term of an AP is x and qth term is y, show that the sum of $(p+q)$ terms is $\frac{p+q}{2}\left[x + y + \left(\frac{x-y}{p-q}\right)\right]$.

12. If the number of terms of an AP be $2n+1$, then find the ratio of sum of the odd terms to sum of even terms.

13. If a, b, c be the 1st, 3rd, and nth terms respectively of an AP, prove that the sum to n terms is $\frac{c+a}{2} + \frac{c^2-a^2}{b-a}$.

14. Find an inifinite GP whose first term is 1 and each term is the sum of all the terms which follow it.

15. The sum of first two terms of an infinite GP is 5 and each term is three times the sum of the succeeding terms. Find the GP. **[NCERT]**

16. If a, b, c are in GP, then prove that $\log a, \log b, \log c$ are in AP.

17. If $a^x = b^y = c^z$ and x, y, z are in GP, show that $\log_b a = \log_c b$.

18. Evaluate $\sum_{r=1}^{n}(3^r - 2^r)$.

19. Find the sum of n terms of the series
$$(a+b) + (a^2+2b) + (a^3+3b) + \ldots$$

20. The fourth term of a GP in the square of its second term and the first term is -3. Determine its 7th term.

SHORT ANSWER Type II Questions

21. If $ab + bc + ca \neq 0$ and a, b, c are is AP. Prove that $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in AP.

22. 150 workers were engaged to finish a job in a certain number of days, 4 workers dropped out on second day, 4 more workers dropped out an third day and so on. It took 8 more days of finish the work. Find the numbers of days in which the work was completed. **[NCERT]**

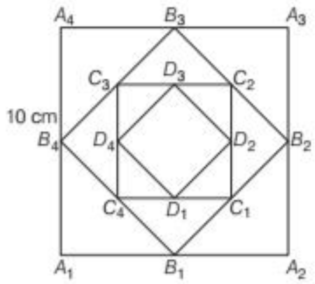
23. A trophy is to be made out of waste material in the form of equilateral triangle as base, on this base another equailateral triangle is kept so that its vertices are mid-point of sides of the base, again another equilateral triangle is kept on the second equilateral triangle obtained in the same manner and process continues. Now sides of each equilateral triangle are decorated with green ribbon to give natural environmental look. If side of equilateral tirangle at the base is 30 cm. Find the total length of ribbon required to decorate the trophy.

24. One side of an equilateral triangle is 24 cm. The mid-point of its sides are joined to form another triangle whose mid-points, in turn, are joined to form still another triangle. The process continues idenfinitely. Find the sum of the perimeters fo all the triangles.

25. An equilateral triangle is drawn by joining the mid-points of the sides of an given equilateral triangle. A third equilateral triangle is drawn inside the second triangle in the same manner. This process is repeated indefinitely. If each side of the first equilateral triangle is 6 cm, find the sum of the areas of all the triangles.

CASE BASED Questions

26. A student of class XI draw a square of side 10 cm. Another student join the mid-point of this square to form new square. Again, the mid-points of the sides of this new square are joined to form another square by another student. This process is continued indefinitely.



Based on above information answer the following question

(i) The side (in cm) of fourth square is

(a) 5 (b) $\frac{\sqrt{5}}{2}$ (c) $\sqrt{5}$ (d) None of these

(ii) The area (in sq cm) of the fifth square is

(a) $\frac{25}{2}$ (b) 50 (c) 25 (d) $\frac{25}{4}$

(iii) The perimeter (in cm) of the 7th square is

(a) 10 (b) 20 (c) 5 (d) $\frac{5}{2}$

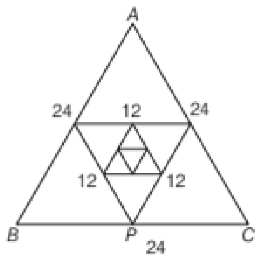
(iv) The sum of areas (in sq cm) of all the square formed is

(a) 150 (b) 200 (c) 250 (d) None of these

(v) The sum of the perimeter (in cm) of all the squae formed is

(a) $80 + 40\sqrt{2}$ (b) $40 + 40\sqrt{2}$ (c) 40 (d) None of these

27. Each side of an equilateral triangle is 24 cm. The mid-point of its sides are joined to form another triangle whose mid-point, in terms, are joined to form still another triangle. This process continuous infinitely.



Based on above information answer the following questions.

(i) The side (in cm) of the 5th triangle is

(a) 3 (b) 6 (c) 1.5 (d) 0.75

(ii) The sum of perimeter (in cm) of first 6 triangle is

(a) $\frac{569}{4}$ (b) $\frac{567}{4}$ (c) 120 (d) 144

(iii) The area (in sq cm) of all the triangle is
(a) 576 (b) $192\sqrt{3}$ (c) $144\sqrt{3}$ (d) $169\sqrt{3}$

(iv) The sum (in cm) of perimeter of all triangle is
(a) 144 (b) 169 (c) 400 (d) 625

(v) The perimeter (in cm) of 7th triangle is
(a) $\frac{7}{8}$ (b) $\frac{9}{8}$ (c) $\frac{5}{8}$ (d) $\frac{3}{4}$

HINTS & ANSWERS

1. (c) A sequence is called infinite sequence, if it is not a finite sequence.

2. (b) The sequence 3, 3.3, 3.33, ... is an infinite sequence, since it never ends.

3. (b) The nth term a_n of a sequence can also be denoted as $a(n)$.

4. (a) The series $a_1 + a_2 + a_3 + ... + a_n$ is abbreviated as $\sum_{k=1}^{n} a_k$.

5. (c) The general term of a GP whose first non-zero term is 'a' and common ratio is 'r' can be written as $a_n = ar^{n-1}$.

6. (a) Since, $\frac{-2}{7}, x, -\frac{7}{2}$ are in GP, there fore

$$x^2 = \left(-\frac{2}{7}\right) \times \left(-\frac{7}{2}\right)$$

$$\Rightarrow \quad x^2 = \frac{2}{7} \times \frac{7}{2} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

7. (b) Given sequences are $a, ar, ar^2, ..., ar^{n-1}$ ...(i)

and $A, AR, AR^2, ..., AR^{n-1}$ ...(ii)

Now, multiplying the corresponding terms of (i) and (ii) to obtain a new sequence.

$aA, ar\ AR, ar^2\ AR^2, ..., ar^{n-1}\ AR^{n-1}$

Clearly, common ratio = rR

8. (a) Since, $2b = a + c$

Now, $(a + 2b - c)(2b + c - a)(a + 2b + c)$

$= (a + a + c - c)(a + c + c - a)(2b + 2b)$

$= 2a \cdot 2c \cdot 4b = 16abc$

9. Let $S_n = \frac{3}{\sqrt{5}} + \frac{4}{\sqrt{5}} + \sqrt{5} + ...$ 25th terms

$\Rightarrow S_n = \frac{3}{\sqrt{5}} + \frac{4}{\sqrt{5}} + \frac{5}{\sqrt{5}} + ...$ 25th terms

Clearly, the successive difference of the terms is same. So, RHS of the above series forms an AP, with first term,

$a = \frac{3}{\sqrt{5}}$ and common difference, $d = \frac{4}{\sqrt{5}} - \frac{3}{\sqrt{5}} = \frac{1}{\sqrt{5}}$.

$$\therefore \quad S_{25} = \frac{25}{2}\left[2 \times \frac{3}{\sqrt{5}} + (25-1)\frac{1}{\sqrt{5}}\right]$$

$$= 25\left[\frac{3}{\sqrt{5}} + \frac{12}{\sqrt{5}}\right]$$

$$= 25 \times \frac{15}{5} \times \sqrt{5} = 75\sqrt{5}$$

10. Let a be the first term and r be the common ratio of the given GP.

According to the question,

$$T_5 = 48 \Rightarrow ar^4 = 48 \quad ...(i)$$

and $$T_8 = 384 \Rightarrow ar^7 = 384 \quad ...(ii)$$

On dividing Eq. (ii) by Eq. (i), we get

$$\Rightarrow \quad \frac{ar^7}{ar^4} = \frac{384}{48} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

On putting $r = 2$ in Eq^n (i), we get $a = 3$

$\therefore$ 3, 6, 12, ... are in GP.

11. Let A be the first term and D be the common difference of the given AP.

Then, $T_p = x \Rightarrow A + (p-1)D = x$ ...(i)

and $T_q = y \Rightarrow A + (q-1)D = y$ ...(ii)

On subtracting Eq. (ii) from Eq. (i) we get

$$(p-q)D = x - y \Rightarrow D = \frac{x-y}{p-q} \quad ...(iii)$$

On adding Eqs. (i) and (ii), we get

$$2A + (p + q - 2)D = x + y$$

$$\Rightarrow \quad 2A + pD + qD - 2D = x + y$$

$$\Rightarrow \quad 2A + pD + qD - D = x + y$$

$$\Rightarrow \quad 2A + (p + q - 1)D = x + y$$

$$2A + (p+q-1)D = x + y\left(\frac{x-y}{p-q}\right) \quad \text{[from Eq. (iii)]...(iv)}$$

Now, $S_{p+q} = \frac{p+q}{2}[2A + (p+q-1)D]$

$$= \frac{p+q}{2}\left[x + y + \left(\frac{x-y}{p-q}\right)\right] \quad \text{[from Eq. (iv)]}$$

Hence proved.

12. $\frac{n+1}{n}$

13. Let A be the first term and D be the common difference of the given AP.

According to the question,

$$T_1 = A = a \quad ...(i)$$

$$T_3 = A + 2D = b \quad ...(ii)$$

and $$T_n = A + (n-1)D = c \quad ...(iii)$$

On solving Eqs. (i) and (ii), we get

$$D = \frac{b-a}{2}$$

On putting $A = a$ and $D = \left(\frac{b-a}{2}\right)$ in Eq. (iii), we get

$$a + (n-1)\left(\frac{b-a}{2}\right) = c$$

$$\Rightarrow \quad (n-1) = \frac{2(c-a)}{b-a} \Rightarrow n = \frac{2c-3a+b}{b-a}$$

Now, $S_n = \frac{n}{2}(A + T_n) = \frac{(2c-3a+b)}{2(b-a)}(a+c)$

$$= \frac{1}{2(b-a)}[2ac - 3a^2 + ab + 2c^2 - 3ac + bc]$$
$$= \frac{1}{2(b-a)}[bc - ac + ab - a^2 + 2c^2 - 2a^2]$$
$$= \frac{1}{2(b-a)}[c(b-a) + a(b-a) + 2(c^2 - a^2)]$$
$$= \frac{1}{2(b-a)}[(b-a)(c+a) + 2(c^2 - a^2)]$$

14. Let a be the first term and $r(|r|<1)$, the common ratio of the GP.

$\therefore$ The GP is $a, ar, ar^2, \ldots$

According to the question, $a = 1$

and $\quad T_n = T_{n+1} + T_{n+2} + T_{n+3} + \ldots$

$\Rightarrow \quad ar^{n-1} = ar^n + ar^{n+1} + ar^{n+2} + \ldots$

$\Rightarrow \quad \frac{ar^n}{r} = ar^n \; [1 + r + r^2 + \ldots]$

$\Rightarrow \quad 1 = r\left(\frac{1}{1-r}\right) \Rightarrow 1 - r = r$

$\Rightarrow \quad r = \frac{1}{2}$

$\therefore 1, \frac{1}{2}, \frac{1}{4}, \ldots$ is the required GP.

15. Let a be the first term and $r(|r|<1)$ be the common ratio of the GP.

$\therefore$ The GP is $a, ar, ar^2, \ldots$

According to the question,

$T_1 + T_2 = 5 \Rightarrow a + ar = 5 \Rightarrow a(1+r) = 5 \quad \ldots$(i)

and $\quad T_n = 3(T_{n+1} + T_{n+2} + T_{n+3} + \ldots)$

$\Rightarrow \quad ar^{n-1} = 3(ar^n + ar^{n+1} + ar^{n+2} + \ldots)$

$\Rightarrow \quad ar^{n-1} = 3ar^n(1 + r + r^2 + \ldots)$

$\Rightarrow \quad 1 = 3r\left(\frac{1}{1-r}\right)$

$\Rightarrow \quad 1 - r = 3r$

$\Rightarrow \quad r = \frac{1}{4}$

From Eq. (i), $a = 4$

$\therefore \; 4, 1, \frac{1}{4}, \ldots$ is the required GP.

17. Let $a^x = b^y = c^z = k$

$\therefore x = \log_a k, \; y = \log_b k, \; z = \log_c k$

Since, x, y, z are in GP.

$\therefore \frac{\log_b k}{\log_a k} = \frac{\log_c k}{\log_b k} \Rightarrow \frac{\log_k a}{\log_k b} = \frac{\log_k b}{\log_k c}$

$\Rightarrow \log_b a = \log_c b$ **Hence proved.**

18. Let $S_n = \sum_{r=1}^{n}(3^r - 2^r)$

$\Rightarrow \quad S_n = (3^1 + 3^2 + 3^3 + \ldots$ upto n terms$)$
$+ (2^1 + 2^2 + 2^3 + \ldots$ upto n terms$)$

$\Rightarrow \quad S_n = \frac{3(3^n - 1)}{3 - 1} + \frac{2(2^n - 1)}{2 - 1} \qquad \left[\because S_n = \frac{a(r^n - 1)}{r - 1}, r > 1\right]$

$$= \frac{3^{n+1} - 3}{2} + \frac{2^{n+1} - 2}{1}$$
$$= \frac{3^{n+1} - 3 + 2^{n+2} - 4}{2}$$
$$= \frac{1}{2}(3^{n+1} + 2^{n+2} - 7)$$

19. Let $S_n = (a+b) + (a^2 + 2b) + (a^3 + 3b) + \ldots$ upto n terms

$= (a + a^2 + a^3 + \ldots$ upto n terms$)$
$+ (b + 2b + 3b + \ldots$ upto n terms$)$

$$= a\left(\frac{a^n - 1}{a - 1}\right) + \frac{n}{2}[2b + (n-1)b]$$

$\left[\because S_{n(GP)} = \frac{A(R^n - 1)}{R - 1}, R > 1 \text{ and } S_{n(AP)} = \frac{n}{2}\{2A + (n-1)D\}\right]$

Ans. $\frac{a(1 - a^n)}{1 - a} + \frac{bn(n+1)}{2}$

20. Let a be the first term and r be the common ratio of the given GP.

According to the question,

$T_4 = (T_2)^2$ and $a = -3$

$\because \quad T_4 = (T_2)^2 \therefore ar^3 = (ar)^2$

$\Rightarrow \quad -3r^3 = (-3)^2 r^2 \qquad [\because a = -3]$

$\Rightarrow \quad r = -3$

Now, $T_7 = ar^6 = -3(-3)^6 = -3 \times 729 = -2187$

21. $a^2(b+c), b^2(c+a), c^2(a+b)$ are in AP.

$\Rightarrow a^2b + a^2c, b^2c + b^2a, c^2a + c^2b$ are in AP.

$\Rightarrow a^2b + a^2c + abc, \; b^2c + b^2a + abc, c^2a + c^2b + abc$ are in AP. [adding abc to each term]

$\Rightarrow a(ab + ac + bc), b(bc + ba + ac), c(ca + cb + ab)$ are in AP.

$\Rightarrow a, b, c$ are in AP. [dividing each term by $(ab + bc + ca)$]

22. Let the number or days in which the work is completed, is n.

Now, according to the question, 4 workers dropped on everyday i.e. the devils of number of workers is 150, 146, 142, 138, ...

Clearly, the work done in both conditions is same.

[had the workers not dropped, then the work would have finished in $(n-8)$ days with 150 workers working on each day. Hence, the total number of workers who would have worked for all the n days is $150(n-8)$]

$\Rightarrow \quad 150(n-8) = \frac{n}{2}[2 \times 150 + (n-1)(-4)]$

$\left[\because \; S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$

$\Rightarrow 150n - 1200 = \frac{n}{2} \times 2(150 - 2n + 2)$

$\Rightarrow 150n - 1200 = \frac{n}{2} \times 2(152 - 2n)$

Dividing each term by 2, we get

$$75n - 600 = n(76 - n)$$

$$\Rightarrow \quad 75n - 600 = 76n - n^2$$

$$\Rightarrow n^2 - 76n + 75n - 600 = 0 \Rightarrow n^2 - n - 600 = 0$$

Now factorising it by splitting the middle term,

$$\Rightarrow \quad n^2 - (25n - 24n) - 600 = 0$$

$$\Rightarrow \quad n^2 - 25n + 24n - 600 = 0$$

$$\Rightarrow \quad n(n - 25) + 24(n - 25) = 0$$

$$\Rightarrow \quad (n - 25)(n + 24) = 0$$

$$\Rightarrow \quad n = 25$$

and $n \neq -24$, because it is not possible.

Hence, the work will be completed in 25 days.

23. 180 cm

24. 144 cm

25. $12\sqrt{3}$ cm^2

26. Let A_1, A_2, A_3, A_4 be the vertices of the first square with each side equal to 10 cm. Let B_1, B_2, B_3, B_4 be the mid-point of its side.

Then, $\quad B_1B_2 = \sqrt{A_2B_1^2 + A_2B_2^2} = \sqrt{5^2 + 5^2} = 5\sqrt{2}$

$\therefore \quad C_1B_2 = B_2C_2 = \dfrac{5\sqrt{2}}{2}$

Similarly, $C_1C_2 = \sqrt{B_2C_2^2 + B_2C_1^2}$

$$= \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2} = 5 \text{ cm}$$

Similarly, the side of fourth square is $\dfrac{5}{\sqrt{2}}$ cm.

$\therefore$ Side are $10, 5\sqrt{2}, 5, \dfrac{5}{\sqrt{2}}, \dfrac{5}{2}, \dfrac{5}{2\sqrt{2}}, \dfrac{5}{4}$, ... respectively

which form a GP with $a = 10$ and $r = \dfrac{1}{\sqrt{2}}$.

(i) (d) Side of fourth square $= ar^3 = 10\left(\dfrac{1}{\sqrt{2}}\right)^3 = \dfrac{10}{2\sqrt{2}} = \dfrac{5}{\sqrt{2}}$ cm

(ii) (d) Side of fifth square $= ar^4 = 10\left(\dfrac{1}{\sqrt{2}}\right)^4 = \dfrac{10}{4} = \dfrac{5}{2}$

$\therefore$ Area of fifth square $= \left(\dfrac{5}{2}\right)^2 = \dfrac{25}{4}$ cm^2

(iii) (c) The side of 7th square $= ar^6 = 10\left(\dfrac{1}{\sqrt{2}}\right)^6 = \dfrac{10}{8} = \dfrac{5}{4}$

$\therefore$ Perimeter of 7th square $= \dfrac{5}{4} \times 4 = 5$ cm.

(iv) (b) Sum of area of all square is

$$10^2 + (5\sqrt{2})^2 + (5)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 \ldots.$$

$$= 100 + 50 + 25 + \frac{25}{2} + \ldots\ldots$$

$$= \frac{100}{1 - \frac{1}{2}} = 200 \text{ cm}^2 \qquad \left[\because a = 100, r = \frac{50}{100} = \frac{1}{2}\right]$$

(v) (a) Sum of perimeter of all square is

$$4\left(10 + 5\sqrt{2} + 5 + \frac{5}{\sqrt{2}} \ldots\right)$$

$$= 4 \times \frac{10}{1 - \frac{1}{\sqrt{2}}} = \frac{40\sqrt{2}}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$

$$= \frac{40\sqrt{2}(\sqrt{2} + 1)}{(2 - 1)} = 80 + 40\sqrt{2}$$

27. (i) (c) Side of first triangle is 24.

Side of second triangle is $\dfrac{24}{2} = 12$

$\therefore \quad a = 24, r = \dfrac{1}{2}$

$\therefore$ Side of the fifth triangle $= a_5 = ar^4 = 24 \times \left(\dfrac{1}{2}\right)^4$

$$= \frac{24}{16} = \frac{3}{2} = 1.5 \text{ cm}$$

(ii) (b) Perimeter of first triangle $= 24 \times 3 = 72$

Perimeter of second triangle $= \dfrac{72}{2} = 36$

$\therefore \quad a = 72, r = \dfrac{1}{2}$

$\therefore$ Sum of perimeter of first 6 triangle $= S_6 = \dfrac{a(1 - r^6)}{1 - r}$

$$= \frac{72\left(1 - \left(\frac{1}{2}\right)^6\right)}{1 - \frac{1}{2}} = \frac{72 \times 63 \times 2}{2^6} = \frac{567}{4} \text{ cm}$$

(iii) (b) Area of first triangle is $\dfrac{\sqrt{3}}{4}(24)^2$

Area of second triangle $= \dfrac{\sqrt{3}}{4}\left(\dfrac{24}{2}\right)^2 = \dfrac{\sqrt{3}}{4}(24)^2 \times \dfrac{1}{4}$

$\therefore \; a = \dfrac{\sqrt{3}}{4}(24)^2, \; r = \dfrac{1}{4}$

$\therefore$ Sum of area of all triangles $= \dfrac{\frac{\sqrt{3}}{4}(24)^2}{1 - \frac{1}{4}}$

$$= \frac{\sqrt{3} \times (24)^2}{3}$$

$$= 192\sqrt{3} \text{ cm}^2$$

(iv) (a) The sum of perimeter of all triangle

$3(24 + 12 + 6 + \ldots)$

$$3 \times 24\left(\frac{1}{1 - \frac{1}{2}}\right) = 144 \text{ cm}$$

(v) (b) Here, $a = 72, r = \dfrac{1}{2}$

$$a_7 = (72)\left(\frac{1}{2}\right)^6 = \frac{72}{64} = \frac{9}{8} \text{ cm}$$