01

In previous class, we initiated the study of relations and functions, where we studied about domain, codomain and range alongwith different types of specific real-valued functions and their graphs. All these concepts are the basics of relations and functions. In this chapter, we will study about different types of relations and functions.

RELATIONS AND FUNCTIONS

|TOPIC 1|
Relations

ORDERED PAIR

A pair of elements listed in a specific order separated by comma and enclosing within the parentheses, is called an ordered pair. e.g. (a, b) is an ordered pair with a as the first element and b as the second element.

CHAPTER CHECKLIST
- Relations
- Functions and Their Types

CARTESIAN PRODUCT

The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, is called the cartesian product or cross product of sets A and B; and it is denoted by $A \times B$. Similarly, the set of all ordered pairs (b, a) such that $b \in B$ and $a \in A$, is called the cartesian product or cross product of sets B and A; and it is denoted by $B \times A$.

Thus, $A \times B = \{(a, b) : a \in A, b \in B\}$ and $B \times A = \{(b, a) : a \in A, b \in B\}$

e.g. If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then

$A \times B$ is $\{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

and $B \times A$ is $\{(4, 1), (4, 2) (4, 3), (5, 1), (5, 2), (5, 3)\}$.

RELATION

In Mathematics, the concept of the term 'Relation' has been drawn from the meaning of relation in English language, according to which two objects or quantities are related, if there is a recognisable connection or link between two objects or quantities.

Relation on Sets A and B

Let A and B be two non-empty sets, then a relation R from set A to set B is a subset of $A \times B$, i.e. $R \subseteq A \times B$.

Here, the subset R is derived by describing a relationship between first and second elements of the ordered pairs in $A \times B$. The second element is called image of first element.

e.g. Let A = Set of students of class XII of a school and B = Set of students of class XI of the same school. Then, $R = \{(a, b) \in A \times B$: age of a is greater than age of $b\}$ is a subset of $A \times B$ and thus R is a relation from set A to set B.

Note If $(a, b) \in R$, then we write it as aRb and we say that a is related to b under the relation R. If $(a,b) \notin R$, then we write it as $a \not{R} b$ and we say that a is not related to b under the relation R.

Relation on a Set

Let A be a non-empty set, then a relation from A to itself, i.e. a subset of $A \times A$, is called a relation on set A (or a relation in set A). **e.g.** Let $A = \{1, 2, 3, 4\}$, then

$R = \{(a, b) \in A \times A : a - b = 3\} = \{4, 1\}$ is a relation on set A.

Domain, Range and Codomain of Relation

Let us consider a relation R from set A to set B i.e. $R \subseteq A \times B$. Then, the set of all first elements of the ordered pairs in R is called the **domain** of relation and the set of all second elements of the ordered pairs in R is called **range of the relation, i.e.** domain $(R) = \{a : (a, b) \in R\}$ and range $(R) = \{b : (a, b) \in R\}$. The set B is called the **codomain** of relation R. Thus, the domain of a relation R from set A to set B is a subset of set A and range is a subset of set B, i.e. range is a subset of codomain.

EXAMPLE |1| If $R = \{(x, y) : x + 2y = 8\}$ is a relation on a set of natural numbers (N), then write the domain, range and codomain of R. **[All India 2014]**

Sol. Given, $R = \{(x, y) : x + 2y = 8\}$ on a set of natural numbers.

Consider, $x + 2y = 8$, which can be rewritten as $y = \frac{8 - x}{2}$.

Now, as $x, y \in N$, therefore substitute values of x from natural numbers such that $y \in N$.

On putting $x = 2$, we get $y = \frac{8-2}{2} = 3$

On putting $x = 4$, we get $y = \frac{8-4}{2} = 2$

On putting $x = 6$, we get $y = \frac{8-6}{2} = 1$

Thus, $R = \{(2, 3), (4, 2), (6, 1)\}$

[$\because$ there is no other value of x, for which $y \in N$]

$\therefore$ Domain of $R = \{2, 4, 6\}$, codomain of $R = N$ and range of $R = \{3, 2, 1\}$.

TYPES OF RELATIONS

There are various types of relations on a set A, which are given below

EMPTY OR VOID RELATION

Relation R in a set A is called an empty relation, if no element of A is related to any element of A, i.e. $R = \phi \subset A \times A$.

e.g. Let set $A = \{1, 2, 3, 8, 10, 11\}$ and R be a relation in A, given by $R = \{(a, b) : a - b = 4\}$. Then, R is an empty relation. Since, no element $(a, b) \in A \times A$ satisfy the property $a - b = 4$.

UNIVERSAL RELATION

Relation R in a set A is called an universal relation, if each element of A is related to every element of A, i.e. $R = A \times A$.

e.g. Let set $A = \{1, 2, 3, 4\}$ and relation R is given by $R = \{(a, b) : |a - b| \geq 0\}$. Then, R is a universal relation. Since, all ordered pairs $(a, b) \in A \times A$ satisfy the property $|a - b| \geq 0$.

IDENTITY RELATION

Relation R in a set A is called an identity relation, if each element of A is related to itself only and it is denoted by I_A, i.e. $I_A = R = \{(a, a) : a \in A\}$.

e.g. Let set $A = \{1, 2\}$, then the identity relation R on A is given by $R = \{(1, 1), (2, 2)\}$. But the relation $R_1 = \{(1, 1), (2, 2), (1, 2)\}$ is not an identity relation on A, because element 1 is related to elements 1 and 2.

REFLEXIVE RELATION

Relation R in a set A is called reflexive relation, if $(a, a) \in R$, for every $a \in A$, i.e. $a R a, \forall a \in A$.

e.g.

(i) Let $A = \{11, 12, 15\}$ and relation on it is defined as $R = \{(11, 11), (12, 12), (11, 12), (15, 15), (11, 15), (12, 15)\}$. Then, R is a reflexive relation. Since, for every element 11, 12 and 15 of A, (11, 11), (12, 12) and $(15, 15) \in R$.

(ii) Let A be the set of real numbers and relation on it is defined as $R = \{(x, y) : x + y \in A\}$. Then, R is a reflexive relation.

Since, sum of two real numbers is also a real number, therefore $R = A \times A$.

Hence, R is reflexive, as $(a, a) \in R$, for every $a \in A$.

SYMMETRIC RELATION

Relation R in a set A is called symmetric relation, if $(a, b) \in R \Rightarrow (b, a) \in R$, for every $a, b \in A$

i.e. $aRb \Rightarrow bRa$, $\forall a, b \in A$.

e.g.

(i) Let $A = \{11, 12, 13\}$ and relation on it is defined as $R = \{(11, 11), (11, 12), (12, 12), (12, 11)\}$.

Then, R is a symmetric relation.

Since, $(11, 12) \in R \Rightarrow (12, 11) \in R$

(ii) Let A be the set of even natural numbers and relation on it is defined as

$$R = \{(x, y) : x + y \text{ is divisible by } 2\}.$$

Then, it is a symmetric relation.

Since, sum of two even natural numbers is always even and hence divisible by 2, therefore $R = A \times A$. Thus, R is symmetric as $(a, b) \in R \Rightarrow (b, a) \in R$, for every $a, b \in A$.

TRANSITIVE RELATION

Relation R in a set A is called transitive relation, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, $\forall a, b, c \in A$

i.e. aRb and $bRc \Rightarrow aRc$, $\forall a, b, c \in A$.

e.g. (i) Let $A = \{11, 12, 14\}$ and relation on it is defined as $R = \{(11, 11), (11, 12), (12, 14), (12, 12), (11, 14)\}$.

Then, it is a transitive relation.

Since, $(11, 11) \in R$ and $(11, 12) \in R \Rightarrow (11, 12) \in R$,

$(11, 11) \in R$ and $(11, 14) \in R \Rightarrow (11, 14) \in R$,

$(11, 12) \in R$ and $(12, 14) \in R \Rightarrow (11, 14) \in R$,

$(11, 12) \in R$ and $(12, 12) \in R \Rightarrow (11, 12) \in R$

and $(12, 12) \in R$ and $(12, 14) \in R \Rightarrow (12, 14) \in R$.

(ii) Let A be the set of positive integers and relation on it is defined as $R = \{(x, y) : x < y, \forall x, y \in A\}$. Then, it is a transitive relation.

Since, $x < y$ and $y < z$

$\Rightarrow x < z$, for every $x, y, z \in A$.

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$,

for every $x, y, z \in A$.

Note

(i) A relation R on set A is not reflexive, if there exists an element $a \in A$ such that $(a, a) \notin R$.

e.g. Let $A = \{4, 5, 7\}$ and $R = \{(4, 4), (4, 5), (7, 7)\}$, then R is not reflexive. Since, $5 \in A$ but $(5, 5) \notin R$.

(ii) If R does not have any element of the type (a, b), where $a \neq b$, then R is symmetric. Also, if R does not have elements of the type (a, b) and (b, c), then R is transitive.

e.g. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2)\}$, then R is symmetric and transitive.

(iii) Identity relation is always reflexive, symmetric and transitive.

Method to Solve the Problems Based on Types of Relations

In these types of problems, a set and a relation defined on that set is given to us and we have to check or show that given relation is reflexive or symmetric or transitive. For this, first we denote the given set as A and the given relation as R. Then, check the given relation is reflexive or symmetric or transitive as follows

FOR REFLEXIVE

We have to show that for all $a \in A$, $(a, a) \in R$. For this, we take an arbitrary element of set A in form of a variable (say x) and then check whether (x, x) satisfy the given condition of R or not i.e. $(x, x) \in R$ or not. If it satisfy the given condition, then R is reflexive otherwise not.

FOR SYMMETRIC

We have to show that for $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$. For this, we take two arbitrary elements of set A in form of two variables (say x and y) such that $(x, y) \in R$ and then check (y, x) satisfy the given condition of R or not i.e. $(y, x) \in R$ or not. If they satisfy the given condition, then R is symmetric otherwise not.

FOR TRANSITIVE

We have to show that for $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$. For this, we take three arbitrary elements of set A in the form of three variables (say x, y and z) such that $(x, y) \in R$ and $(y, z) \in R$ and then check whether (x, z) satisfy the given condition of R or not i.e. $(x, z) \in R$ or not. If they satisfy the given condition, then R is transitive otherwise not.

Note Generally, whenever we have to prove or show that given relation is reflexive, symmetric or transitive, then we take variables and when we have to show that given relation is not reflexive or symmetric or transitive, then we take elements of set to justify.

EXAMPLE |2| Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive but neither symmetric nor transitive.

Sol. Let the given set be $A = \{1, 2, 3\}$

and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$

Reflexive Here, $1, 2, 3 \in A$ and $(1, 1), (2, 2), (3, 3) \in R$

i.e. for all $a \in A$, $(a, a) \in R$.

So, R is reflexive.

Symmetric Here, $(1, 2) \in R$ but $(2, 1) \notin R$.

So, R is not symmetric.

Transitive Here, $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

So, R is not transitive.

EXAMPLE |3| Check whether the relation R defined in the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(x, y) : y$ is divisible by $x\}$ is reflexive, symmetric and transitive. **[NCERT]**

For reflexive, show for all $x \in A$, $(x, x) \in R$. For symmetric, show $(x, y) \in R \Rightarrow (y, x) \in R$, $x, y \in A$. For transitive, show $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$, $x, y, z \in A$.

Sol. Given, $R = \{(x, y) : y$ is divisible by $x\}$

and $A = \{1, 2, 3, 4, 5, 6\}$

Reflexive Let $x \in A$ be any arbitrary element.

We know that x is divisible by x.

[$\because$ every real number except zero is divisible by itself]

$\Rightarrow \quad (x, x) \in R$

Since, $x \in A$ was arbitrary element, therefore $(x, x) \in R, \forall x \in A$. So, R is reflexive.

Symmetric Clearly, $2, 4 \in A$ and 4 is divisible by 2, but 2 is not divisible by 4.

$\therefore \quad (2, 4) \in R$ but $(4, 2) \notin R$

So, R is not symmetric.

Transitive Let $x, y, z \in A$ such that $(x, y) \in R$ and $(y, z) \in R$.

Now, as $(x, y) \in R$, therefore y is divisible by x.

i.e. $\quad \frac{y}{x} = k_1 \text{(say)} \quad$ ...(i)

where, k_1 is a natural number

and as $(y, z) \in R$, therefore z is divisible by y.

i.e. $\quad \frac{z}{y} = k_2 \text{ (say)} \quad$ ...(ii)

where, k_2 is a natural number.

On multiplying Eqs. (i) and (ii), we get

$$\frac{y}{x} \times \frac{z}{y} = k_1 k_2 \Rightarrow \frac{z}{x} = k_1 k_2$$

where, $k_1 k_2$ is a natural number.

$\therefore \quad z$ is divisible by x.

Thus, $(x, z) \in R$, for $(x, y), (y, z) \in R$,

i.e. $\quad (x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$

Hence, R is transitive.

EXAMPLE |4| Check whether the relation R defined in set $A = \{1, 2, 3, \ldots, 13, 14\}$ as $R = \{(x, y) : 3x - y = 0\}$ is reflexive, symmetric and transitive.

Sol. Given, $R = \{(x, y) : 3x - y = 0\}$

and $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

Reflexive Here, $2 \in A$

If $(2, 2) \in R$, then $3(2) - 2 = 0$

$\Rightarrow \quad 4 = 0 \quad$ [not true]

$\therefore \quad (2, 2) \notin R$

So, R is not reflexive.

Symmetric Here, $1, 3 \in A$

Clearly, $(1, 3) \in R$, as $3(1) - 3 = 0$

But if $(3, 1) \in R$, then $3(3) - 1 = 0$

$\Rightarrow \quad 8 = 0 \quad$ [not true]

Thus, $(1, 3) \in R \not\Rightarrow (3, 1) \in R$

So, R is not symmetric.

Transitive Here, $1, 3, 9 \in A$

Clearly, $(1, 3) \in R$, as $3(1) - 3 = 0$

and $(3, 9) \in R$, as $3(3) - 9 = 0$

But if $(1, 9) \in R$, then $3(1) - 9 = 0 \Rightarrow -6 = 0 \Rightarrow (1, 9) \notin R$

Thus, $(1, 3) \in R$ and $(3, 9) \in R \not\Rightarrow (1, 9) \in R$

So, R is not transitive.

EXAMPLE |5| Give an example of a relation, which is

(i) symmetric but neither reflexive nor transitive.

(ii) transitive but neither reflexive nor symmetric.

(iii) reflexive and symmetric but not transitive.

(iv) reflexive and transitive but not symmetric.

(v) symmetric and transitive but not reflexive. **[NCERT]**

Sol. (i) Let $A = \{1, 2, 3\}$ and defined a relation R on A as $R = \{(1, 2), (2, 1)\}$.

Then, R is symmetric, as $(1, 2) \in R \Rightarrow (2, 1) \in R$.

R is not reflexive, as $1 \in A$ but $(1, 1) \notin R$.

R is not transitive, as $(1, 2), (2, 1) \in R$ but $(1, 1) \notin R$.

(ii) Let $A = \{1, 2, 3\}$ and defined a relation R on A as $R = \{(1, 2), (2, 2)\}$.

Then, R is transitive, as $(1, 2), (2, 2) \in R \Rightarrow (1, 2) \in R$

R is not reflexive, as $1 \in A$ but $(1, 1) \notin R$.

R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$.

(iii) Let $A = \{1, 2, 3\}$ and defined a relation R on A, as $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$

Then, R is reflexive, as for each $a \in A$, $(a, a) \in R$.

R is symmetric, as $(1, 2) \in R \Rightarrow (2, 1) \in R$

and $(2, 3) \in R \Rightarrow (3, 2) \in R$.

R is not transitive, as $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$.

(iv) Let $A = \{1, 2, 3\}$ and defined a relation R on A as $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$

Then, R is reflexive, as for each $a \in A$, $(a, a) \in R$.

R is transitive, as $(1, 1) \in R$, $(1, 2) \in R \Rightarrow (1, 2) \in R$

and $(1, 2) \in R$, $(2, 2) \in R \Rightarrow (1, 2) \in R$.

R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$.

(v) Let $A = \{1, 2, 3\}$ and defined a relation R on A as $R = \{(2, 2), (3, 3)\}$.

Then, R is symmetric, as R does not have any element of the type (a, b), where $a \neq b$.

R is transitive, as R does not have elements of the type (a, b) and (b, c).

R is not reflexive, as $1 \in A$ but $(1, 1) \notin R$.

EXAMPLE |6| Let a relation R on the set N of natural number be define as $R = \{(x, y) : 3x^2 - 7xy + 4y^2 = 0;\ x, y \in N\}$. Then, show that relation R is reflexive but neither symmetric nor transitive.

Sol. Given, $R = \{(x, y): 3x^2 - 7xy + 4y^2 = 0;\ x, y \in N\}$

$= \{(x, y): 3x^2 - 3xy - 4xy + 4y^2 = 0;\ x, y \in N\}$

$= \{(x, y): 3x(x - y) - 4y(x - y) = 0;\ x, y \in N\}$

$= \{(x, y): (x - y)(3x - 4y) = 0;\ x, y \in N\}$

Here, we get those ordered pairs (x, y) which satisfy the equation defined in R.

Reflexive Since, $xRx \Rightarrow (x - x)(3x - 4x) = 0$

$\Rightarrow 0 = 0$, which is true, so R is reflexive.

Symmetric Now, $xRy \Rightarrow (x - y)(3x - 4y) = 0$

and $yRx \Rightarrow (y - x)(3y - 4x) = 0$

Clearly, $xRy \not\Rightarrow yRx$

So, R is not symmetric.

Transitive Now, $xRy \Rightarrow (x - y)(3x - 4y) = 0$

$yRz \Rightarrow (y - z)(3y - 4z) = 0$

and $xRz \Rightarrow (x - z)(3x - 4z) = 0$

Here, $(x - y)(3x - 4y) = 0$ and $(y - z)(3y - 4z) = 0$

$\not\Rightarrow \quad (x - z)(3x - 4z) = 0$

So, R is not transitive.

EQUIVALENCE RELATION

A relation R on a set A is said to be an equivalence relation on A iff it is

(i) **reflexive** i.e. aRa or $(a, a) \in R,\ \forall\ a \in A$.

(ii) **symmetric** i.e. $aRb \Rightarrow bRa$ or $(a, b) \in R$
$\Rightarrow \quad (b, a) \in R$, where $a, b \in A$.

(iii) **transitive** i.e. if aRb and bRc, then aRc or $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, where $a, b, c \in A$.

EXAMPLE |7| Let T be the set of all triangles in a plane with R is a relation in T given by $R = \{(T_1, T_2): T_1$ is congruent to T_2 and $T_1, T_2 \in T\}$. Show that R is an equivalence relation.

Sol. Given, T = Set of all triangles in a plane

and $R = \{(T_1, T_2): T_1$ is congruent to T_2 and $T_1, T_2 \in T\}$

We know that two triangles are said to be congruent, if they have same shape and same size.

Reflexive Let $T_1 \in T$ be any arbitrary element. We know that every triangle is congruent to itself. So, T_1RT_1, i.e. $(T_1, T_1) \in R$. Now, as T_1 was arbitrary element of T, therefore R is reflexive.

Symmetric Let $T_1, T_2 \in T$, such that

$(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2.

$\Rightarrow \quad T_2$ is congruent to T_1.

[$\because$ if a triangle is congruent to another triangle, then they are congruent to each other]

$\Rightarrow \quad (T_2, T_1) \in R$

$\therefore$ R is symmetric.

Transitive Let $T_1, T_2, T_3 \in T$, such that

$(T_1, T_2) \in R$ and $(T_2, T_3) \in R$.

$\Rightarrow \quad T_1$ is congruent to T_2 and T_2 is congruent to T_3.

$\Rightarrow \quad T_1$ is congruent to T_3

$\Rightarrow \quad (T_1, T_3) \in R$

$\therefore$ R is transitive.

Thus, the relation R is reflexive, symmetric and transitive. So, R is an equivalence relation.

EXAMPLE |8| Given a non-empty set X. Consider $P(X)$, which is the set of all subset of X. Defined the relation R in $P(X)$ as follows

For subsets A and B in $P(X)$, ARB if and only if $A \subseteq B$

Is R an equivalence relation on $P(X)$? Justify your answer.

Sol. Given, $R = \{(A, B): A \subseteq B$ and $A, B \in P(X)\}$.

Reflexive We know that every set is a subset of itself, then $ARA,\ \forall\ A \in P(X)$.

So, R is reflexive.

Symmetric Let $A, B \in P(X)$ such that $ARB \Rightarrow A \subseteq B$

This may not be implied that $B \subseteq A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then A is related to B, but it cannot be implied that B is related to A.

So, R is not symmetric.

Hence, R is not an equivalence relation on $P(X)$, since it is not symmetric.

EQUIVALENCE CLASSES

Consider an equivalence relation R on a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called the equivalence class of R determined by a; and it is denoted by $[a]$. Thus, $[a] = \{b \in A : (a, b) \in R\}$.

Now, by finding the classes determined by the elements of A, we can write A as union of disjoint classes. We call these classes as equivalence classes of R. Thus, R divides A into mutually disjoint equivalence classes.

In other words, **if R is an equivalence relation on a set X, then R divides X into mutually disjoint subsets A_i called partitions or sub-divisions of X, satisfying**

(i) for each i; all elements of A_i are related to each other.

(ii) no element of A_i is related to any element of A_j, $i \neq j$.

(iii) $\cup A_i = X$ and $A_i \cap A_j = \phi, i \neq j$.

Then, subsets A_i are called equivalence classes of R.

e.g. Consider a equivalence relation R on the set Z of all integers defined as $R = \{(a, b): a, b \in Z,\ a - b$ is divisible by 2$\}$, then $Z = [0] \cup [1]$, i.e. R divides Z into mutually disjoint equivalence classes, [0] and [1].

Justification We know that

(i) if both a and b are even or odd integer, then $a - b$ is divisible by 2 and hence $(a, b) \in R$.

(ii) if one of a and b is an odd integer and other is an even integer, then $a-b$ is not divisible by 2 and hence $(a, b) \notin R$.

Therefore, all even integers including 0 are related to 0, which implies as $(0, 0)$, $(0, \pm 2)$, $(0, \pm 4)$, etc., lie in R. Also, all odd integers including 1 are related to 1which implies as $(1, \pm 1)$, $(1, \pm 3)$, etc., lie in R.

Thus, all even integers belongs to the class determined by 0 (i.e. denoted by $[0]$) and all odd integers belongs to the class determined by 1 (i.e. denoted by $[1]$). Also, we know that the set Z is the union of set of even integers and set of odd integers. Hence, $Z = [0] \cup [1]$.

EXAMPLE |9| Let R be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2 \text{ divides } (a-b)\}$. Then, write equivalence class $[0]$. **[Delhi 2014C]**

Sol. Given, $A = \{0, 1, 2, 3, 4, 5\}$ and $R = \{(a, b) : 2 \text{ divides } (a-b)\}$.

Clearly, $[0] = \{b \in A : (0, b) \in R\} = \{b \in A : 2 \text{ divides } (0-b)\}$
$= \{b \in A : 2 \text{ divides } (-b)\}$.

So, b can be 0, 2, 4.

Hence, equivalence class $[0] = \{0, 2, 4\}$.

EXAMPLE |10| Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b) R (c, d)$, if $a+d = b+c$ for (a, b), (c, d) in $A \times A$. Prove that R is an equivalence relation and also obtain the equivalence class $[(2, 5)]$. **[Delhi 2014; NCERT Exemplar]**

Sol. Given a relation R in $A \times A$, where $A = \{1, 2, 3, \ldots, 9\}$, defined as

$$R = \{((a, b), (c, d)) : a+d = b+c\}$$

or $\quad (a, b)\, R\, (c, d)$, if $a+d = b+c$

Reflexive Let (a, b) be any arbitrary element of $A \times A$.

i.e. $\quad (a, b) \in A \times A$, where $a, b \in A$

Now, as $\quad a+b = b+a \quad$ [$\because$ addition is commutative]

$\therefore \quad (a, b) R (a, b)$

So, R is reflexive.

Symmetric Let $(a, b), (c, d) \in A \times A$, such that $(a, b) R (c, d)$, then $a+d = b+c$

$\Rightarrow \quad b+c = a+d \Rightarrow c+b = d+a \Rightarrow (c, d) R (a, b)$

So, R is symmetric.

Transitive Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$.

Then, $a+d = b+c$ and $c+f = d+e$

On adding LHS to LHS terms and RHS to RHS terms, we get $\quad a+d+c+f = b+c+d+e$

$\Rightarrow \quad a+f = b+e \Rightarrow (a, b) R (e, f)$

So, R is transitive. Thus, R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

Now, equivalence class containing an element x of A is given by $\{y : x R y\}$. Here, $(a, b) R (c, d)$

$\Rightarrow \quad a+d = b+c$

So, for $(2, 5)$, we will find (c, d) such that $2+d = 5+c$.

Clearly, $\quad (2, 5) R (1, 4)$ as $2+4 = 5+1$
$(2, 5) R (2, 5)$ as $2+5 = 5+2$
$(2, 5) R (3, 6)$ as $2+6 = 5+3$
$(2, 5) R (4, 7)$ as $2+7 = 5+4$
$(2, 5) R (5, 8)$ as $2+8 = 5+5$
and $\quad (2, 5) R (6, 9)$ as $2+9 = 5+6$

where, $1, 2, 3, \ldots, 9 \in A$.

Hence, equivalence class $[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

EXAMPLE |11| Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a-b| \text{ is divisible by } 2\}$ is an equivalence relation. Write all the equivalence classes of R. **[All India 2015C]**

Sol. We have a relation R in set $A = \{1, 2, 3, 4, 5\}$ defined as

$$R = \{(a, b) : |a-b| \text{ is divisible by } 2\}$$

Clearly, $R = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)\}$

Reflexive For any $a \in A$, we have $|a-a| = 0$, which is divisible by 2.

$\Rightarrow (a, a) \in R, \forall\ a \in A$

Thus, R is reflexive.

Symmetric Let $a, b \in A$, such that $(a, b) \in R$

$\Rightarrow \quad |a-b|$ is divisible by 2

$\Rightarrow \quad |a-b| = 2\lambda$ for some $\lambda \in N$

$\Rightarrow \quad |b-a| = 2\lambda$ for some $\lambda \in N \quad [\because |a-b| = |b-a|]$

$\Rightarrow \quad (b, a) \in R$

Thus, R is symmetric.

Transitive Let $a, b, c \in A$, such that $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a-b|$ is divisible by 2 and $|b-c|$ is divisible by 2.

$\Rightarrow \quad |a-b| = 2\lambda$ and $|b-c| = 2\mu$, for some $\lambda, \mu \in N$

$\Rightarrow \quad (a-b) = \pm 2\lambda$ and $(b-c) = \pm 2\mu$

Now, $|a-c| = |(a-b)+(b-c)| = |\pm 2\lambda + (\pm 2\mu)|$
$= |\pm 2\lambda \pm 2\mu| = 2|\pm \lambda \pm \mu|$
$= 2 \quad$ [some positive number]

$\Rightarrow |a-c|$ is divisible by 2 $\Rightarrow (a, c) \in R$

Thus, R is transitive.

Hence, R is an equivalence relation.

Now, $[a] = \{x \in A : (a, x) \in R\}$

$\therefore$ Equivalence class $[1] = \{1, 3, 5\}$, $[2] = \{2, 4\}$, $[3] = \{1, 3, 5\}$,
$[4] = \{2, 4\}$ and $[5] = \{1, 3, 5\}$

Hence, $[1] = [3] = [5] = \{1, 3, 5\}$ and $[2] = [4] = \{2, 4\}$

EXAMPLE |12| Let L be the set of all lines in XY-plane and R be the relation in L defined as

$$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}.$$

Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$. **[NCERT]**

Sol. Given, $R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$.

Reflexive R is reflexive as any line L is parallel to itself, i.e. $(L, L) \in R$.

Symmetric Now, let $(L_1, L_2) \in R$

$\Rightarrow$ L_1 is parallel to $L_2 \Rightarrow L_2$ is parallel to L_1

$\Rightarrow$ $(L_2, L_1) \in R$

So, R is symmetric.

Transitive Now, let $(L_1, L_2), (L_2, L_3) \in R$

Then, L_1 is parallel to L_2 and L_2 is parallel to L_3.

$\Rightarrow$ L_1 is parallel to L_3.

$\Rightarrow$ $(L_1, L_3) \in R$.

So, R is transitive.

Hence, R is an equivalence relation.

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of line, $y = 2x + 4$ is $m = 2$.

It is known that parallel lines have the same slope.

So, the line parallel to the given line will be of the form $y = 2x + c$, where $c \in R$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in R$.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 A relation f from C to R is defined by $x \ f \ y \Leftrightarrow |x| = y$. Then, the correct option is

(a) $(2 + i) \ f \ 3$ (b) $3 \ f \ (-3)$

(c) $i \ f \ 1$ (d) $(2 + 3i) \ f \ 13$

2 If a relation R on the set $\{1, 2, 3\}$ be defined by $R = \{(1, 2)\}$, then R is **[NCERT Exemplar]**

(a) reflexive (b) transitive

(c) symmetric (d) None of these

3 The relation R in the set of natural numbers N defined as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$ is

(a) reflexive (b) symmetric

(c) transitive (d) None of these

4 Let set $X = \{1, 2, 3\}$ and a relation R is defined in X as : $R = \{(1, 3), (2, 2), (3, 2)\}$, then minimum ordered pairs which should be added in relation R to make it reflexive and symmetric are **[CBSE 2021 (Term I)]**

(a) $\{(1, 1), (2, 3), (1, 2)\}$

(b) $\{(3, 3), (3, 1), (1, 2)\}$

(c) $\{(1, 1), (3, 3), (3, 1), (2, 3)\}$

(d) $\{(1, 1), (3, 3), (3, 1), (1, 2)\}$

5 If $A = \{x \in Z : 0 \le x \le 12\}$ and R is the relation in A given by $R = \{(a, b) : a = b\}$. Then, the set of all elements related to 1 is

(a) $\{1, 2\}$ (b) $\{2, 3\}$

(c) $\{1\}$ (d) $\{2\}$

VERY SHORT ANSWER Type Questions

6 If R is a relation 'is divisor of' from the set $A = \{1, 2, 3\}$ to $B = \{4, 10, 15\}$, then write down the set of ordered pairs corresponding to R.

7 Let $R = \{(a, a^3) : a$ is a prime number less than 5$\}$ be a relation. Find the range of R. **[Foreign 2014]**

8 State the reason for the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is not to be transitive. **[Foreign 2011]**

SHORT ANSWER Type I Questions

9 Let $A = \{0, 1, 2, 3\}$ and define a relation R on A as $R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$. Is R reflexive, symmetric and transitive? **[NCERT Exemplar]**

10 For real numbers x and y, define $x R y$ if and only if $x - y + \sqrt{2}$ is an irrational number. Is R transitive? Explain your answer.

11 Let $A = \{a, b, c\}$ and the relation R be defined on A as $R = \{(a,a), (b,c), (a,b)\}$. Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

SHORT ANSWER Type II Questions

12 Let $A = \{x \in Z : 0 \le x \le 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to 1. Also, write the equivalence class $[2]$. **[CBSE 2018]**

13 Show that the relation R in the set A of real numbers defined as $R = \{(a, b) : a \le b\}$ is reflexive and transitive but not symmetric. **[NCERT]**

14 Show that the relation S in the set R of real numbers defined as $S = \{(a, b) : a, b \in R \text{ and } a \le b^3\}$ is neither reflexive nor symmetric nor transitive. **[Delhi 2010]**

15 Let a relation R on the set A of real numbers be defined as $(a, b) \in R \Rightarrow 1 + ab > 0, \forall a, b \in A$. Show that R is reflexive and symmetric but not transitive.

16 Let N be the set of all natural numbers and let R be a relation in N, defined by

$$R = \{(a, b) : a \text{ is a multiple of } b\}.$$

Show that R is reflexive and transitive but not symmetric.

17 Let A be the set of all points in a plane and R be a relation on A defined as $R = \{(P, Q)$: distance between P and Q is less than 2 units$\}$. Show that R is reflexive and symmetric but not transitive.

18 Let R be a relation defined on the set of natural numbers N as $R = \{(x, y): x \in N, y \in N$ and $2x + y = 24\}$. Then, find the domain and range of the relation R. Also, find whether R is an equivalence relation or not. **[Delhi 2014C]**

19 Let $A = \{1, 2, 3\}$. Then, find the number of equivalence relations containing $(1, 2)$.

20 Prove that the relation R on Z, defined by $R = \{(x, y) : (x - y)$ is divisible by $5\}$ is an equivalence relation. **[Delhi 2020]**

21 Let R be a relation on the set A of ordered pairs of positive integers defined by $(x, y)\ R\ (u, v)$ if and only if $xv = yu$. Show that R is an equivalence relation. **[NCERT]**

22 Show that the relation R defined by $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ on the set $N \times N$ is an equivalence relation. **[All India 2010]**

LONG ANSWER Type Questions

23 Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b|$ is even$\}$ is an equivalence relation. Also, show that all elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other, but no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$. **[NCERT]**

24 Show that the relation R in the set A of points in a plane given by $R = \{(P, Q)$: distance of the point P from the origin is same as the distance of the point Q from the origin$\}$ is an equivalence relation. Further, show that the set of all points related to a point $P \neq (0,0)$ is the circle passing through P with origin as centre. **[NCERT]**

25 Show that the relation 'is similar to' on the set of all triangles in a plane is an equivalence relation.

26 If R_1 and R_2 are equivalence relation in a set A, then show that $R_1 \cap R_2$ is also an equivalence relation. Also, give an example to show that the union of two equivalence relations on a set A need not be an equivalence relation on a set A.

27 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let R_1 be a relation on X given by $R_1 = \{(x, y) : x - y$ is divisible by $3\}$ and R_2 be another relation on X given by $R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$. **[NCERT]**

28 In the set of natural numbers N, define a relation R as follows $\forall\ n, m \in N, nRm$, if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of numbers 0, 1, 2, 3 and 4. Show that R is an equivalence relation. Also, obtain the pairwise disjoint subset determined by R. **[NCERT Exemplar]**

29 If N denotes the set of all natural numbers and R is the relation on $N \times N$ defined by $(a, b)\ R\ (c, d)$, if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation. **[Delhi 2015]**

HINTS & SOLUTIONS

1. (c) **Hint** $|a + ib| = \sqrt{a^2 + b^2}$
2. (b) **Hint** R does not have elements of the type (a,b) and (b, c).
3. (c) **Hint** $R = \{(1, 6), (2, 7), (3, 8)\}$
4. (c) **Hint** For R to be reflexive and symmetric, we should add $\{(1, 1), (3, 3), (3, 1), (2, 3)\}$.
5. (c) **Hint** The set of all elements related to 1 is $\{a \in A : a = 1\}$.
6. Here, $R = \{(a, b): b/a\ ; a \in A, b \in B\}$
$= \{(1, 4), (1, 10), (1, 15), (2, 4), (2, 10), (3, 15)\}$
7. Given, $R = \{(a, a^3): a$ is a prime number less than $5\}$
$\Rightarrow \quad R = \{(2, 8), (3, 27)\}$
Hence, range of $R = \{8, 27\}$

8. Here, $(1, 2) \in R$ and $(2, 1) \in R$, but $(1, 1) \notin R$.
Hence, R is not transitive.

9. **Hint** (i) R is reflexive, as $(a, a) \in R, \forall a \in A$.
(ii) R is symmetric, as $(0, 1) \in R \Rightarrow (1, 0) \in R$ and $(0, 3) \in R \Rightarrow (3, 0) \in R$.
(iii) R is not transitive, as $(3, 0), (0, 1) \in R \not\Rightarrow (3, 1) \in R$.
[**Ans.** Reflexive, symmetric and not transitive]

10. **Hint** Consider $(x, y) = (1, \sqrt{3})$ and $(y, z) = (\sqrt{3}, \sqrt{2})$
Ans. No

11. For R to be reflexive, (b, b) and (c, c) should belong to R and for R to be transitive (a, c) should belong to R, as $(a, b) \in R$ and $(b, c) \in R$. Hence, minimum number of ordered pairs to be added in R is 3.

12. Given, a relation R on $A = \{0, 1, 2, 3, ..., 12\}$ defined as

$R = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$

To show R is an equivalence relation

(i) **Reflexivity** Let $a \in A$ be an arbitrary element.
Then, $|a - a| = 0$, which is divisible by 4.
$\therefore \quad (a, a) \in R$
But $a \in A$ was arbitrary.
$\therefore \quad (a, a) \in R, \forall a \in A$
Thus, R is reflexive.

(ii) **Symmetricity** Let $a, b \in A$ such that $(a, b) \in R$
$\because \quad (a, b) \in R$
$\therefore \quad |a - b|$ is divisible by 4.
$\Rightarrow \quad |-(b - a)|$ is divisible by 4.
$\Rightarrow \quad |b - a|$ is divisible by 4.
$\Rightarrow \quad (b, a) \in R$
Thus, $(a, b) \in R \Rightarrow (b, a) \in R$
Hence, R is symmetric.

(iii) **Transitivity** Let $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$
$\because$ (a, b) and (b, c) belongs to R.
$\therefore$ Both $|a - b|$ and $|b - c|$ are divisible by 4.
Let $|a - b| = 4\lambda$ and $|b - c| = 4\mu$ for some real numbers λ and μ.
$\therefore \quad a - b = \pm 4\lambda$ and $b - c = \pm 4\mu$

Now, consider $a - c = (a - b) + (b - c)$
$= \pm 4\lambda \pm 4\mu = 4(\pm\lambda \pm \mu)$,
which is divisible by 4.
$\therefore |a - c|$ is divisible by 4 $\Rightarrow (a, c) \in R$
Thus $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$
Hence, R is transitive.

From above, we can conclude that R is an equivalence relation.

Clearly, the set of all elements related to 1
= equivalence class of 1
$= \{a \in A : (1, a) \in R\}$
$= \{a \in A : |1 - a| \text{ is divisible by } 4\}$
$= \{1, 5, 9\}$

Similarly, $[2] = \{a \in A : |2 - a| \text{ is divisible by } 4\}$
$= \{2, 6, 10\}$

13. Given, A = Set of real numbers and $R = \{(a, b) : a \le b\}$.
Reflexive Let $a \in A$ be any arbitrary real number.
We know that every real number is equal to itself, i.e. $a = a$.
So, we can write $a \le a \Rightarrow (a, a) \in R$
So, R is reflexive.
Symmetric Let $(a, b) \in R$, then $a < b$ or $a = b$
If $a = b$, then $b = a$
But if we consider $a < b$, then $b \nless a \Rightarrow (b, a) \notin R$.
e.g. $\quad 4 < 5$ but $5 \nless 4$
So, R is not symmetric.
Transitive Let $(a, b), (b, c) \in R$, then

$$(a, b) \in R \Rightarrow a \le b \quad \text{...(i)}$$
$$(b, c) \in R \Rightarrow b \le c \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $a \le c \Rightarrow (a, c) \in R$
So, R is transitive.

14. Solve as Question 13.

15. Given, A = Set of real numbers and $R = \{(a, b) : 1 + ab > 0\}$.
Reflexive Let a be any real number.
Then, $\quad 1 + aa = 1 + a^2 > 0 \quad [\because a^2 \ge 0, \forall a \in A]$
$\Rightarrow \quad (a, a) \in R, \forall a \in A$
So, R is reflexive.
Symmetric Let $(a, b) \in R$, then
$(1 + ab) > 0 \Rightarrow 1 + ba > 0 \quad [\because ab = ba, \forall a, b \in A]$
$\Rightarrow \quad (b, a) \in R$
Thus, $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$
So, R is symmetric.
Transitive We observe that $\left(1, \frac{1}{2}\right) \in R$ as

$$1 + 1 \cdot \frac{1}{2} = 1 + \frac{1}{2} > 0$$

and $\left(\frac{1}{2}, -1\right) \in R$ as $1 + (-1)\left(\frac{1}{2}\right) = 1 - \frac{1}{2} > 0$

But $(1, -1) \notin R$, because $1 + 1 \times (-1) = 0 \not> 0$
So, R is not transitive.

16. Similar as Example 3.

17. Given, A = Set of all points in a plane.
$R = \{(P, Q)$: Distance between P and Q is less than 2 units$\}$
Reflexive Let $P \in A$ be any arbitrary point. Then, distance between P and P is 0, which is less than 2 units.
$\Rightarrow \quad (P, P) \in R$
Thus, $(P, P) \in R$ for all $P \in A$
So, R is reflexive.
Symmetric Let $P, Q \in A$ such that $(P, Q) \in R$, i.e. distance between P and Q is less than 2 units.
$\Rightarrow$ Distance between Q and P is less than 2 units.
$\Rightarrow (Q, P) \in R$
So, R is symmetric.

Transitive Consider the points P, Q and S having coordinates (0, 0), (1, 0) and (2, 0), respectively. We can observe that the distance between P and Q is 1 unit, which is less than 2 units and also the distance between Q and S is 1 unit.

But the distance between P and S is 2 units, which is not less than 2 units. Thus, $(P, Q) \in R, (Q, S) \in R$ but $(P, S) \notin R$.

So, R is not transitive.

18. Given, $R = \{(x, y): x \in N, y \in N \text{ and } 2x + y = 24\}$

Here if $x = 1$, then $y = 22$

[$\because$ put $x = 1$ in $2x + y = 24$, we get $y = 22$]

If $x = 2$, then $y = 20$; If $x = 3$, then $y = 18$;
If $x = 4$, then $y = 16$; If $x = 5$, then $y = 14$;
If $x = 6$, then $y = 12$; If $x = 7$, then $y = 10$;
If $x = 8$, then $y = 8$; If $x = 9$, then $y = 6$;
If $x = 10$, then $y = 4$; If $x = 11$, then $y = 2$;

$\therefore \quad R = \{(1, 22), (2, 20), (3,18), (4,16), (5,14), (6,12), (7,10), (8, 8), (9, 6), (10, 4), (11, 2)\}$

Hence, domain $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

and range $= \{22, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2\}$.

Reflexive Clearly, $1 \in N$ but $(1, 1) \notin R$, so R is not reflexive.

Hence, R is not an equivalence relation.

19. It is given that $A = \{1, 2, 3\}$. An equivalence relation is reflexive, symmetric and transitive.

The smallest equivalence relation containing (1, 2) is given by

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

Now, we are left with only four pairs, (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one pair [say (2, 3)] to R_1, then for symmetry, we must add (3, 2).

Also, for transitivity, we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

20. Given, $R = \{(x, y): 5 \text{ divides } (x - y)\}$

and $\quad Z =$ Set of integers

Reflexive Let $x \in Z$ be any arbitrary element. Now, if $(x, x) \in R$, then 5 divides $x - x$, which is true.

So, R is reflexive.

Symmetric Let $x, y \in Z$, such that

$(x, y) \in R \Rightarrow 5$ divides $(x - y)$

$\Rightarrow$ 5 divides $[-(x - y)]$

$\Rightarrow$ 5 divides $(y - x) \Rightarrow (y, x) \in R$

So, R is symmetric.

Transitive Let $x, y, z \in Z$, such that $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow x - y$ and $y - z$ both are divisible by 5.

$\Rightarrow x - y + y - z$ is divisible by 5.

$\Rightarrow (x - z)$ is divisible by 5 $\Rightarrow (x, z) \in R$

So, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

21. Given, $A = N \times N$ and a relation R on A, defined as $(x, y) R (u, v)$, iff $xv = yu$.

Reflexive Let $(x, y) \in A$ be any arbitrary element.

Now, we have to show that $(x, y) R (x, y)$.

Clearly, $xy = yx \Rightarrow (x, y) R (x, y)$

$\because \quad (x, y) \in A$ was arbitrary.

$\therefore \quad (x, y) R (x, y), \forall (x, y) \in A$

So, R is reflexive.

Symmetric Let (x, y) and $(u, v) \in A$ such that $(x, y) R (u, v)$.

Now, we have to show that $(u, v) R (x, y)$, i.e. $uy = vx$.

$\because \quad (x, y) R (u, v)$

$\Rightarrow \quad xv = yu \Rightarrow yu = xv$

$\Rightarrow \quad uy = vx \Rightarrow (u, v) R (x, y)$

So, R is symmetric.

Transitive Let (x, y), (u, v) and $(a, b) \in A$, such that $(x, y) R (u, v)$ and $(u, v) R (a, b)$

Now, we have to show that $(x, y) R (a, b)$, i.e. $xb = ya$.

$\because \quad (x, y) R (u, v) \Rightarrow xv = yu \quad$ ...(i)

and $\quad (u, v) R (a, b) \Rightarrow ub = va \quad$ ...(ii)

Now, multiplying both sides of Eq. (i) by $\frac{a}{u}$, we get

$$xv\left(\frac{a}{u}\right) = yu\left(\frac{a}{u}\right)$$

$$\Rightarrow \quad xv\left(\frac{b}{v}\right) = yu\left(\frac{a}{u}\right) \quad \text{[using Eq. (ii)]}$$

$$\Rightarrow \quad xb = ya$$

$\Rightarrow (x, y) R (a, b)$

So, R is transitive.

Hence, R is an equivalence relation.

22. Similar as Example 10.

23. Given, $A = \{1, 2, 3, 4, 5\}$ and $R = \{(a, b): |a - b| \text{ is even}\}$.

Reflexive For any $a \in A$, we have $|a - a| = 0$, which is even.

Thus, $(a, a) \in R, \forall a \in A$

So, R is reflexive.

Symmetric Let $a, b \in A$ such that $(a, b) \in R \Rightarrow |a - b|$ is even.

$\Rightarrow \quad |-(b - a)|$ is even.

$\Rightarrow \quad |b - a|$ is even.

$\Rightarrow \quad (b, a) \in R$.

So, R is symmetric.

Transitive Let $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a - b|$ is even and $|b - c|$ is even.

$\Rightarrow$ (a and b both are even or both are odd) and (b and c both are even or both are odd)

***Case* I** *When b is even.*

In this case, a is even and c is even

$\Rightarrow |a-c|$ is even $\Rightarrow (a, c) \in R.$

***Case* II** *When b is odd.*

In this case, a is odd and c is odd

$\Rightarrow |a-c|$ is even $\Rightarrow (a, c) \in R$

Thus, $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

So, R is transitive.

Hence, R is an equivalence relation.

Now, all elements of the set $\{1, 3, 5\}$ are related to each other as all the elements of this set are odd.

So, the modulus of the difference between any two elements will be even.

Similarly, all elements of the set $\{2, 4\}$ are related to each other as all the elements of this set are even. So, the modulus of the difference between any two elements will be even.

But no element of the set $\{1, 3, 5\}$ can be related to any element of $\{2, 4\}$ as all the elements of set $\{1, 3, 5\}$ are odd and all the elements of set $\{2, 4\}$ are even.

So, the modulus of the difference between the two elements each from these two subsets [as $3-2=1$ not even, $5-2=3$ not even, etc.] will not be even.

24. Given, $A = \{$all points in a plane$\}$ and

$R = \{(P, Q):$ distance of the point P from the origin is the same as the distance of the point Q from the origin$\}$

Reflexive Let $P \in A$ be any arbitrary point. Then, distance of a point P from the origin is always same.

$\therefore \quad (P, P) \in R$

So, R is reflexive.

Symmetric Let $(P, Q) \in R$, where $P, Q \in A$.

$\Rightarrow$ The distance of point P from the origin is same as the distance of point Q from the origin.

$\Rightarrow$ The distance of point Q from the origin is same as the distance of point P from the origin.

$\Rightarrow \quad (Q, P) \in R$

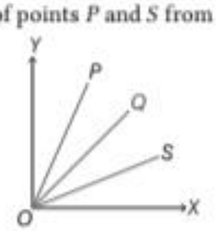
Thus, $(P, Q) \in R \Rightarrow (Q, P) \in R$, where $P, Q \in A$.

So, R is symmetric.

Transitive Let $(P, Q), (Q, S) \in R$, where $P, Q, S \in A$.

$$\left.\begin{array}{ll}\Rightarrow & OP = OQ \\ \text{and} & OQ = OS\end{array}\right\} \Rightarrow OP = OS$$

i.e. the distance of points P and S from the origin is same.



$\Rightarrow \quad (P, S) \in R$

So, R is transitive.

Hence, R is an equivalence relation on A.

Let $P \neq (0, 0)$ be a fixed point in the plane and $Q(x, y)$ be any point in the plane related to P, i.e. $(P, Q) \in R$ or distance of P from origin is same as distance of Q from origin.

Then, $OP = OQ = k$ (say)

$\because \quad OQ = \sqrt{(x-0)^2 + (y-0)^2}$

$\therefore \quad k = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = k^2$

This is an equation of the circle with centre $(0, 0)$ and radius k (distance of point P from origin). Thus, all the points related to P satisfy this equation. Hence, the set of all points related to a point $P \neq (0, 0)$ is the circle passing through P with origin as centre.

25. Similar as Example 7.

26. Given, R_1 and R_2 are two equivalence relations on set A.

Now, we have to show that $R_1 \cap R_2$ is also an equivalence relation on set A.

Reflexive Let $a \in A$ be any arbitrary element.

Then, $(a, a) \in R_1$ and $(a, a) \in R_2$

[$\because$ both R_1 and R_2 are reflexive]

$\Rightarrow \quad (a, a) \in R_1 \cap R_2$

$\because \quad a \in A$ was arbitrary.

So, $R_1 \cap R_2$ is reflexive.

Symmetric Let $a, b \in A$ such that $(a, b) \in R_1 \cap R_2$

$\Rightarrow \quad (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow \quad (b, a) \in R_1$ and $(b, a) \in R_2$

[$\because$ both R_1 and R_2 are symmetric]

$\Rightarrow \quad (b, a) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ is symmetric.

Transitive Let $a, b, c \in A$ such that $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$.

$\because \quad (a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

and $\quad (b, c) \in R_1 \cap R_2 \Rightarrow (b, c) \in R_1$ and $(b, c) \in R_2$

$\Rightarrow \quad (a, c) \in R_1$ and $(a, c) \in R_2$

[$\because$ both R_1 and R_2 are transitive]

$\Rightarrow \quad (a, c) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ is transitive.

Hence, $R_1 \cap R_2$ is an equivalence relation.

Now, define relations R_1 and R_2 on a set $A = \{1, 2, 3\}$

as $R_1 = \{(1, 1), (2, 2), (3, 3) (1, 2), (2, 1)\}$

$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$

Clearly, R_1 and R_2 are equivalence relations but

$R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$

is not an equivalence relation, as

$(2, 1) \in R_1 \cup R_2, (1, 3) \in R_1 \cup R_2$ but $(2, 3) \notin R_1 \cup R_2$.

27. Note that the characteristic of sets $\{1, 4, 7\}, \{2, 5, 8\}$ and $\{3, 6, 9\}$ is that difference between any two elements of these sets is a multiple of 3.

Therefore, $(x, y) \in R_1 \Rightarrow x - y$ is divisible by $3 \Rightarrow x - y$ is a multiple of $3 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence, $R_1 \subseteq R_2$.

Similarly, $(x, y) \in R_2 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$ is divisible by $3 \Rightarrow (x, y) \in R_1$.

This shows that $R_2 \subseteq R_1$.

Hence, $R_1 = R_2$.

28. R is reflexive, since for each $a \in N$, aRa.

R is symmetric, since if aRb, then bRa for $a, b \in N$.

Also, R is transitive, since for $a, b, c \in N$, if aRb and bRc, then aRc.

Hence, R is an equivalence relation in N which will partition the set N into the pairwise disjoint subsets.

The equivalence classes are as mentioned below

A_0 = {Set of numbers, which leaves remainder 0}
$= \{5, 10, 15, 20, \ldots\}$

A_1 = {Set of numbers, which leaves remainder 1}
$= \{1, 6, 11, 16, 21, \ldots\}$

A_2 = {Set of numbers, which leaves remainder 2}
$= \{2, 7, 12, 17, 22, \ldots\}$

A_3 = {Set of numbers, which leaves remainder 3}
$= \{3, 8, 13, 18, 23, \ldots\}$

A_4 = {Set of numbers, which leaves remainder 4}
$= \{4, 9, 14, 19, 24, \ldots\}$

It is evident that the above five sets are pairwise disjoint and $A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = N$

29. We have a relation R on $N \times N$ defined by $(a, b) R (c, d)$, if $ad(b + c) = bc(a + d)$

Reflexive Let $(a, b) \in N \times N$ be any arbitrary element.

We have to show that $(a, b) R (a, b)$, i.e.

$ab(b + a) = ba(a + b)$, which is always true, as natural numbers are commutative under usual multiplication and addition.

Since, $(a, b) \in N \times N$ was arbitrary, so R is reflexive.

Symmetric Let $(a, b), (c, d) \in N \times N$ such that $(a, b) R (c, d)$, i.e. $ad(b + c) = bc(a + d)$ ...(i)

We have to show that $(c, d) R (a, b)$,

i.e. $cb(d + a) = da(c + b)$

From Eq. (i), we have $ad(b + c) = bc(a + d)$

$\Rightarrow \quad da(c + b) = cb(d + a)$

[$\because$ natural numbers are commutative under usual addition and multiplication]

$\Rightarrow \quad cb(d + a) = da(c + b)$

$\Rightarrow \quad (c, d) R (a, b)$

So, R is symmetric.

Transitive Let $(a, b), (c, d)$ and $(e, f) \in N \times N$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$.

$\because \quad (a, b) R (c, d)$

$\Rightarrow \quad ad(b + c) = bc(a + d)$

$\Rightarrow \quad \frac{b + c}{bc} = \frac{a + d}{ad}$

$\Rightarrow \quad \frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$ ...(ii)

and $(c, d) R (e, f) \Rightarrow cf(d + e) = de(c + f)$

$\Rightarrow \quad \frac{d + e}{de} = \frac{c + f}{cf}$

$\Rightarrow \quad \frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$ ...(iii)

Now, adding Eqs. (ii) and (iii), we get

$$\left(\frac{1}{c} + \frac{1}{b}\right) + \left(\frac{1}{e} + \frac{1}{d}\right) = \left(\frac{1}{d} + \frac{1}{a}\right) + \left(\frac{1}{f} + \frac{1}{c}\right)$$

$\Rightarrow \quad \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f}$

$\Rightarrow \quad \frac{e + b}{be} = \frac{f + a}{af}$

$\Rightarrow \quad af(e + b) = be(f + a)$

$\Rightarrow \quad af(b + e) = be(a + f)$

$\Rightarrow \quad (a, b) R (e, f)$

So, R is transitive.

Hence, R is an equivalence relation.

|TOPIC 2|

Functions and Their Types

FUNCTION (MAPPING) AS A RULE

For any two non-empty sets A and B, a function f from A to B is a rule which associates each element of set A to a unique element of set B. A function f from A to B is represented by $f : A \to B$ or $A \xrightarrow{f} B$.

e.g. $f : N \to N$ define as $f(x) = x^2$ is a function, where N is the set of natural numbers.

Function (Mapping) as a Set of Ordered Pairs

For any two non-empty sets A and B, a function from set A to set B is a relation from A to B (denoted by f) satisfying the following conditions

(i) For each $a \in A$, there exists $b \in B$ such that $(a, b) \in f$.

EXAMPLE |1| Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8\}$. Consider the rule $f : A \to B$, defined as $f(x) = 2x, \forall x \in A$. Find the domain, codomain and range of f.

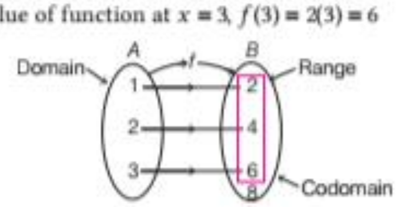
Find the image of 1, 2, 3 using definition of f.

Sol. Given, $f(x) = 2x, \forall x \in A$

Value of function at $x = 1$, $f(1) = 2(1) = 2$

Value of function at $x = 2$, $f(2) = 2(2) = 4$

Value of function at $x = 3$, $f(3) = 2(3) = 6$



We can write it as $f = \{(1, 2), (2, 4), (3, 6)\}$

$\therefore$ Domain of $f = \{1, 2, 3\}$, Codomain of $f = \{2, 4, 6, 8\}$

(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.

e.g. If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then

$f = \{(1, 4), (2, 5), (3, 4)\}$ is a function but

$f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$ is not a function as

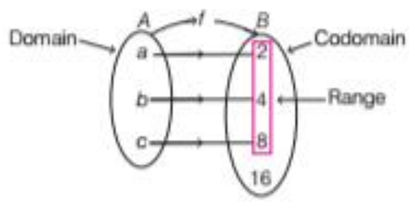
$(1, 4) \in f$, $(1, 5) \in f \not\Rightarrow 4 = 5$.

Note Every function is a relation but converse is not always true. e.g. If $A = \{1, 2, 3\}$ and $B = \{1, 4\}$, then $R = \{(1, 1), (1, 4), (2, 4)\}$ is a relation but not a function.

Domain, Codomain and Range of a Function

Consider a function $f : A \to B$, then elements of set A are called domain of f and elements of set B are called codomain of f, and the set of all f-images obtained in set B corresponding to each element belongs to set A is called range of f.

e.g. Let $f: A \to B$ be a function, where $A = \{a, b, c\}$ and $B = \{2, 4, 8, 16\}$, defined as



Then, domain $= \{a, b, c\}$, codomain $= \{2, 4, 8, 16\}$

and range $= \{2, 4, 8\}$.

and range of $f = \{2, 4, 6\}$

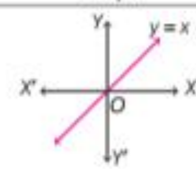
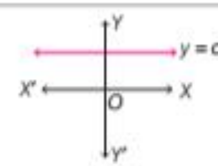
RELATION BETWEEN RANGE AND CODOMAIN

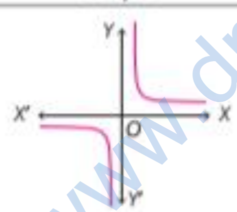
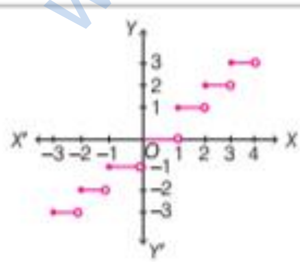
Range and codomain may be same or may be different. A function, whose codomain is a set (or a subset) of real numbers, is known as **real-valued function** and a function whose domain and codomain both are the sets (or subsets) of real numbers, is known as **real function**. Note that every real function is a real-valued function, but converse may not be true.

e.g. $f : C \to R$ is a real-valued function but not real function, where C is a set of complex numbers.

Some Special Real Functions

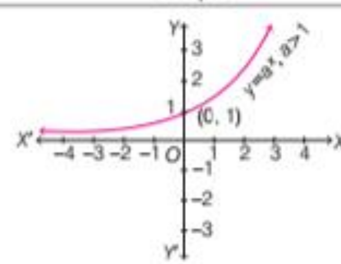
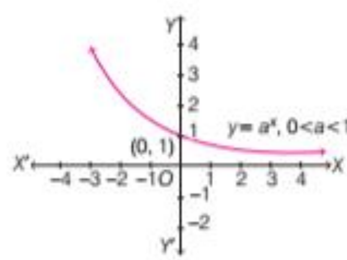
Some special real functions and their domain, range and graphs are given below

Function	Graph
Identity function $f(x) = x, \forall x \in R$ Domain – R Range – R	
Constant function $f(x) = c, \forall x \in R$ Domain – R Range – $\{c\}$	

Function	Graph
Reciprocal function $f(x) = \frac{1}{x}, \forall x \in R - \{0\}$ Domain – $R - \{0\}$ Range – $R - \{0\}$	
Greatest integer function $f(x) = [x], \forall x \in R$ Domain – R Range – I	

Properties of greatest integer function

If $n < x < n + 1$, where n is a greatest integer less than x and x is a

Function	Graph
Exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1, \forall x \in R$ **Case I** When $a > 1$, $f(x) = a^x \to \begin{cases} < 1, \text{ for } x < 0 \\ = 1, \text{ for } x = 0 \\ > 1, \text{ for } x > 0 \end{cases}$	
Case II When $0 < a < 1$, $f(x) = a^x \to \begin{cases} > 1, \text{ for } x < 0 \\ = 1, \text{ for } x = 0 \\ < 1, \text{ for } x > 0 \end{cases}$ Domain – R Range – $(0, \infty)$	

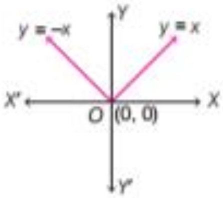
If $n < x < n + 1$, where n is a greatest integer less than x and x is a real number, then

- $[x] = n$
- $[-x] = -[x] - 1$
- $[x + m] = [x] + m$, for any integer m.

Modulus function

$f(x) = |x|, \forall x \in R$

$$= \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



Domain – R

Range – $[0, \infty)$

Properties of modulus function

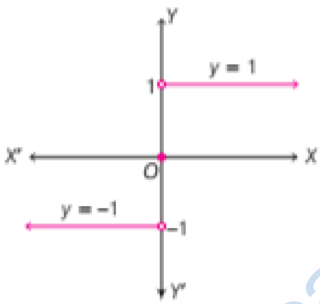
Let $x, y \in R$ (set of real numbers). Then,

- $\sqrt{x^2} = |x|$
- $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$
- $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$
- $|x \pm y| \leq |x| + |y|$
- $|x \pm y| \geq ||x| - |y||$

Signum function

$f(x) = \text{sgn}(x) = \dfrac{|x|}{x}$

or $f(x) = \begin{cases} -1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1, & \text{for } x > 0 \end{cases}$



Domain – R

Range – $\{-1, 0, 1\}$

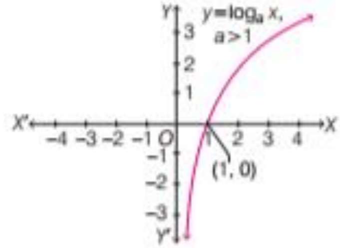
Logarithmic function

$f(x) = \log_a x, \forall x > 0$,

where $a > 0$ and $a \neq 1$

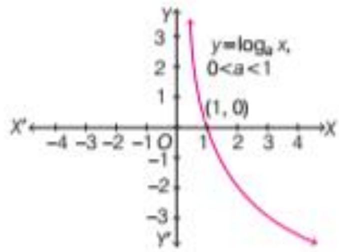
Case I When $a > 1$,

$$f(x) \to \begin{cases} < 0, & \text{for } 0 < x < 1 \\ = 0, & \text{for } x = 1 \\ > 0, & \text{for } x > 1 \end{cases}$$



Case II When $0 < a < 1$,

$$f(x) \to \begin{cases} > 0, & \text{for } 0 < x < 1 \\ = 0, & \text{for } x = 1 \\ < 0, & \text{for } x > 1 \end{cases}$$



Domain – R^+

Range – R

TYPES OF FUNCTIONS

As we know that if $f : A \to B$ is a function, then f associates all the elements of set A to the elements in set B such that an element of set A is associated to a unique element of set B. But there are some more possibilities, which may occur in a function, such as

(i) more than one elements of A may have same image in B.

(ii) each elements of B is image of some elements of A.

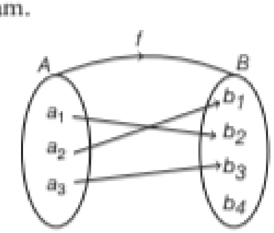
(iii) there may be some elements in B, which are not the images of any element of A.

Corresponding to these possibilities, we define the following types of functions.

One-One (Injective) Function

A function $f : A \to B$ is called a one-one or injective function, if distinct elements of A have distinct images in B, i.e. for every $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$ and if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

e.g. Let $f : A \to B$ be a function represented by the following diagram.



Here, f is a one-one function, because each element have distinct image.

Many-One Function

EXAMPLE |2| Determine whether the function $f : A \to B$ defined by $f(x) = 4x + 7$, $x \in A$ is one-one.

Show that no two elements in domain have same image in codomain.

Sol. Given, $f : A \to B$ defined by $f(x) = 4x + 7$, $x \in A$

Let $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + 7 = 4x_2 + 7 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$

So, f is one-one function.

EXAMPLE |3| Show that the signum function $f : R \to R$, given by $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$ is many-one.

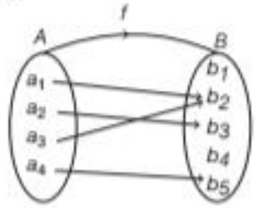
Show that the images of any two or more elements of domain are same in codomain.

Sol. Given, $f : R \to R$ defined by $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$

Clearly, $f(1) = f(2) = f(3) = 1$, where $1, 2, 3 \in R$

A function $f: A \rightarrow B$ is called a many-one function, if there exist atleast two distinct elements in A, whose images are same in B, i.e. if there exist $a_1, a_2 \in A$, such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$, then f is many-one. In other words, a function $f: A \rightarrow B$ is called a many-one function, if it is not one-one.

e.g. Let $f: A \rightarrow B$ be a function represented by the following diagram.



Here, f is many-one function, because a_1 and a_3 have same image b_2.

METHOD TO CHECK WHETHER A FUNCTION IS ONE-ONE OR MANY-ONE

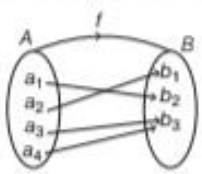
Sometimes, a function is given to us and we have to check whether this function is one-one or many-one. For this, we use the following steps

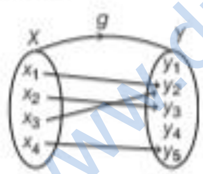
I. First, consider any two arbitrary elements, say $a_1, a_2 \in$ (domain of function).

II. Put $f(a_1) = f(a_2)$ and simplify the equation.

III. If we get $a_1 = a_2$, then f is one-one and if we get $a_1 \neq a_2$, then f is many-one.

e.g. Let $f: A \rightarrow B$ and $g: X \rightarrow Y$ be two functions represented by the following diagrams.





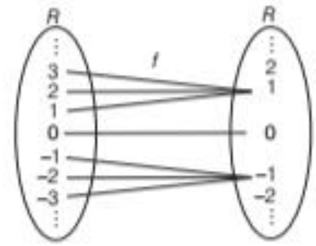
Here, f is an onto function and g is an into function because y_1 and y_4 are elements of Y, which do not have pre-image in X.

METHOD TO CHECK WHETHER THE FUNCTION IS ONTO OR INTO

Sometimes, a function is given to us and we have to check whether this function is onto or into. For this, we use the following steps

$f(0) = 0$, where $0 \in R$

and $f(-1) = f(-2) = f(-3) = -1$, where $-1, -2, -3 \in R$



We can observe that all positive real numbers have same image 1 and all negative real numbers have same image −1.

Hence, it is a many-one function.

Note In order to prove f is not one-one, it is sufficient to show that $f(1) = 1$ and $f(2) = 1$.

Onto (Surjective) and Into Function

A function $f: A \rightarrow B$ is said to be **onto** (or surjective), if every element of B is the image of some elements of A under f, i.e. for every $b \in B$, there exists an element a in A such that $f(a) = b$. In other words, $f: A \rightarrow B$ is onto if and only if

Range of $f = B$

i.e. Range = Codomain

A function $g: X \rightarrow Y$ is an **into function**, if there exists one element in Y, which have no pre-image in X. In other words, if a function is not onto, then it is an into function.

Then, $y = f(x) \Rightarrow y = 3x + 2 \Rightarrow x = \frac{y-2}{3}$

Now, when $y = 0$, then $x = \frac{0-2}{3} = \frac{-2}{3} \notin Z$

Thus, $y = 0 \in Z$ (codomain of f) does not has pre-image in Z(domain). Hence, f is not an onto function, i.e. it is an into function.

One-One and Onto Function (Bijective)

A function $f: A \rightarrow B$ is said to be **one-one and onto (or bijective)**, if f is both one-one and onto, i.e. all elements of A have distinct images in B and every element of B is the image of some elements of A.

e.g. Let $f: A \rightarrow B$ be a function represented by the following diagram.

I. Let $f: A \rightarrow B$ be the given function, then first consider y be an arbitrary element of B.

II. Put $y = f(x)$ and simplify it to obtain x in terms of y.

III. Now, if for any $y \in B$, the corresponding value of x does not belongs to A, then f is not onto, i.e. f is into but if for all $y \in B$, the corresponding value of x belongs to A, then f is onto.

Note Sometimes, we simply find the range of function. If range of function = Codomain of function, then the function is onto, otherwise it is an into function.

EXAMPLE |4| Check which of the following function is onto or into.

(i) $f: A \rightarrow B$, given by $f(x) = 3x$, where $A = \{0, 1, 2\}$ and $B = \{0, 3, 6\}$.

(ii) $f: Z \rightarrow Z$, given by $f(x) = 3x + 2$, where Z = set of integers.

Sol. (i) We have a function $f: A \rightarrow B$, given by $f(x) = 3x$, where $A = \{0, 1, 2\}$ and $B = \{0, 3, 6\}$.

Let $y \in B$ be any arbitrary element.

Then, $y = f(x) \Rightarrow y = 3x \Rightarrow x = \dfrac{y}{3}$

Now, at $y = 0$, $x = \dfrac{0}{3} = 0 \in A$

At $y = 3$, $x = \dfrac{3}{3} = 1 \in A$

At $y = 6$, $x = \dfrac{6}{3} = 2 \in A$

Thus, for each element y of B, there is a pre-image in A.

So, $f: A \rightarrow B$ is an onto function.

(ii) We have a function $f: Z \rightarrow Z$, given by $f(x) = 3x + 2$

Let $y \in Z$, (codomain of f) be any arbitrary element.

EXAMPLE |6| Show that the function $f: R \rightarrow R$ defined as $f(x) = x^2$ is neither one-one nor onto.

Sol. Given, a function $f: R \rightarrow R$ defined as $f(x) = x^2$

For one-one Here, at $x = 1$, $f(1) = 1$

and at $x = -1$, $f(-1) = (-1)^2 = 1$

Thus, $f(1) = f(-1) = 1$, but $1 \neq -1$

So, f is not one-one.

For onto Let $y \in R$ (codomain) be any arbitrary element.

Then, $y = f(x)$

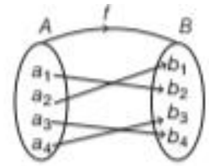
$\Rightarrow \quad y = x^2$

$\Rightarrow \quad x = \pm\sqrt{y}$

Now, for $y = -2 \in R$, $x = \pm\sqrt{-2} \notin R$

So, f is not onto.

Hence, given function is neither one-one nor onto.



Here, f is bijective function because each element have distinct image and every element of B have pre-image.

EXAMPLE |5| Let R be the set of all non-zero real numbers. Then, show that $f: R \rightarrow R$, given by $f(x) = \dfrac{1}{x}$ is one-one and onto.

Sol. Given, $f(x) = \dfrac{1}{x}$

For one-one Let $x_1, x_2 \in R$, such that $f(x_1) = f(x_2)$

$\Rightarrow \quad \dfrac{1}{x_1} = \dfrac{1}{x_2}$ $\quad\left[\text{put } x_1 \text{ and } x_2 \text{ in } f(x) = \dfrac{1}{x}\right]$

$\Rightarrow \quad x_1 = x_2$

So, f is one-one.

For onto Let $y \in R$ be any arbitrary element.

Then, $y = f(x)$

$\Rightarrow \quad y = \dfrac{1}{x}$

$\Rightarrow \quad x = \dfrac{1}{y}$ $\quad$ [expressing x in terms of y]

It is clear that for every $y \in R$ (codomain), $x \in R$ (domain)

Thus, for each $y \in R$ (codomain), there exist $x = \dfrac{1}{y} \in R$ (domain), such that $f(x) = f\left(\dfrac{1}{y}\right) = \dfrac{1}{\left(\dfrac{1}{y}\right)} = y$

[i.e. every element of codomain has pre-image in domain]

So, f is onto.

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 $f: X \rightarrow Y$ is onto, if and only if

(a) range of $f = Y$

(b) range of $f \neq Y$

(c) range of $f < Y$

(d) range of $f \geq Y$

2 The number of all one-one functions from set $A = \{1, 2, 3\}$ to itself is

(a) 2

(b) 6

(c) 3

(d) 1

EXAMPLE |7| Show that the function $f : N \to N$, given by $f(x) = 2x$ is one-one but not onto.

Sol. Given a function $f : N \to N$, defined by $f(x) = 2x$

For one-one Let $x_1, x_2 \in N$, such that $f(x_1) = f(x_2)$

$\Rightarrow \quad 2x_1 = 2x_2$

$\Rightarrow \quad x_1 = x_2$

So, f is one-one.

For onto Let $y \in N$ (codomain) be any arbitrary element.

Then, $\quad y = f(x)$

$\Rightarrow \quad y = 2x$

$\Rightarrow \quad x = \frac{y}{2}$

Now, for $y = 1$, $x = \frac{1}{2} \notin N$.

Thus, $y = 1 \in N$ (codomain) does not have a pre-image in domain (N). So, f is not onto.

EXAMPLE |8| Show that the function $f: N \to N$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for every $x > 2$, is onto but not one-one.

Sol. We have a function $f : N \to N$, defined as

$$f(1) = f(2) = 1 \text{ and } f(x) = x - 1, \text{ for every } x > 2$$

For one-one Since, $f(1) = f(2) = 1$, therefore 1 and 2 have same image, namely 1. So, f is not one-one.

For onto Note that $y = 1$ has two pre-images, namely 1 and 2. Now, let $y \in N$, $y \neq 1$ be any arbitrary element.

Then, $y = f(x) \Rightarrow y = x - 1$

$\Rightarrow \quad x = y + 1 > 2$ for every $y \in N$, $y \neq 1$.

Thus, for every $y \in N$, $y \neq 1$, there exists $x = y + 1$ such that

$$f(x) = f(y+1) = y + 1 - 1 = y$$

Hence, f is onto.

SHORT ANSWER Type I Questions

8 State whether the function $f : R \to R$, defined by $f(x) = 3 - 4x$ is onto or not.

9 Let $f : R \to R$ be defined by $f(x) = x^2 + 1$. Then, find pre-images of 17 and − 3.

10 Check the injectivity of the function $f : R \to R$ given by $f(x) = x^3$. **[NCERT]**

11 Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$. Then, what value should be assigned to α and β? **[NCERT Exemplar]**

12 Show that the function $f : R \to R$, given by $f(x) = \cos x, \forall x \in R$ is neither one-one nor onto.

SHORT ANSWER Type I Questions

13 Consider a function $f : [0, \pi/2] \to R$, given by $f(x) = \sin x$ and $g : [0, \pi/2] \to R$, given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one. **[NCERT]**

3 Let $A = \{1, 2, 3, \ldots, n\}$ and $B = \{a, b\}$. Then, the number of surjections from A into B is

(a) ${}^nP_2$

(b) $2^n - 2$

(c) $2^n - 1$

(d) None of these

4 If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is **[NCERT Exemplar]**

(a) 720

(b) 120

(c) 0

(d) None of these

5 The greatest integer function $f : R \to R$, given by $f(x) = [x]$ is

(a) one-one

(b) onto

(c) both one-one and onto

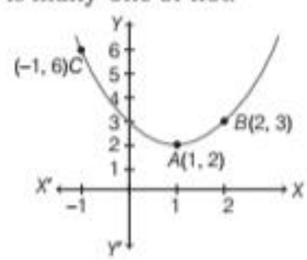
(d) neither one-one nor onto

VERY SHORT ANSWER Type Questions

6 If $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and $f = \{(1, 4), (2, 5), (3, 6)\}$ is a function from A to B. State whether f is one-one or not. **[All India 2011]**

7 Find whether the function $f : Z \to Z$, defined by $f(x) = x^2 + 5, \forall x \in Z$ is one-one or not.

20 With the help of following graph, find the equation of function. Also, check whether the function is many-one or not. **[NCERT]**



HINTS & SOLUTIONS

1. (a) A function $f : A \to B$ is said to be onto, if for every $b \in B$, there exists an element a in A such that $f(a) = b$

2. (b) If $n(A) = x$ and $n(B) = y$, then number of one-one functions from A to B is given by ${}^yP_x$, where $x \leq y$.

3. (b) Total number of functions $= (n(B))^{n(A)} = 2^n$. Clearly a function will not be onto if all elements of A map to either a or b.

14 Let $f : N \to N$ be defined by

$$f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Show that f is many-one and onto function. **[NCERT]**

15 Show that $f : N \to N$, given by $f(x) = \begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$ is bijective (both one-one and onto). **[All India 2012]**

16 Let A and B be sets. Show that $f : A \times B \to B \times A$, such that $f(a, b) = (b, a)$ is bijective function. **[NCERT]**

17 Show that the function $f : R \to R$, defined by $f(x) = \dfrac{x}{x^2+1}, \forall x \in R$ is neither one-one nor onto. **[Delhi 2020, NCERT Exemplar]**

LONG ANSWER Type Questions

18 Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}, x \in R$ is one-one and onto function. **[NCERT]**

19 Given a function defined by $f(x) = \sqrt{4 - x^2}$; $0 \le x \le 2, 0 \le f(x) \le 2$. Show that f is bijective function.

10. Let $x_1, x_2 \in R$, such that $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^3 = x_2^3 \Rightarrow x_1^3 - x_2^3 = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_2^2 + x_1x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)\left[x_1^2 + 2 \cdot x_1\left(\dfrac{x_2}{2}\right) + \left(\dfrac{x_2}{2}\right)^2 - \left(\dfrac{x_2}{2}\right)^2 + x_2^2\right] = 0$

$\Rightarrow \quad (x_1 - x_2)\left[\left(x_1 + \dfrac{x_2}{2}\right)^2 + \dfrac{3}{4}x_2^2\right] = 0$

$\Rightarrow \quad (x_1 - x_2)\left[\left(x_1 + \dfrac{x_2}{2}\right)^2 + \dfrac{3}{4}x_2^2\right] = 0$

$\Rightarrow \quad x_1 - x_2 = 0 \qquad \left[\because \left(x_1 + \dfrac{x_2}{2}\right)^2 + \dfrac{3}{4}x_2^2 = 0 \text{ only for } x_1 = x_2 = 0\right]$

$\Rightarrow \quad x_1 = x_2$

Hence, f is one-one, i.e. f is injective.

11. **Hint** According to the given information, we can have $1 = \alpha + \beta$ and $3 = 2\alpha + \beta$. [**Ans.** Yes; $\alpha = 2$ and $\beta = -1$]

12. We have a function $f : R \to R$, defined by

$f(x) = \cos x$

$\Rightarrow \quad f(0) = \cos 0 = 1$

$f(2\pi) = \cos 2\pi = 1$

So, f is not one-one.

Also, range $f = [-1, 1] \ne R$

Hence, f is not onto.

13. Let $x_1, x_2 \in \left[0, \dfrac{\pi}{2}\right]$, such that $x_1 \ne x_2$.

4. (c) One-one onto mapping is possible only if $n(A) = n(B)$

5. (d) Range (f) = Integers $\ne R$ and $[2 \cdot 3] = [2 \cdot 4] = 2$

$\Rightarrow f$ is not one-one.

6. Given, $f = \{(1, 4), (2, 5), (3, 6)\}$. Here, we see that each element of A have distinct image. So, it is one-one.

7. Let $x_1, x_2 \in Z$, such that $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^2 + 5 = x_2^2 + 5$

$\Rightarrow \quad x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

$\therefore f$ is not a one-one function, as if we take $x_1 = 1$ and $x_2 = -1$, then $f(1) = 6 = f(-1)$, but $1 \ne -1$.

8. Let $y \in R$ (codomain) be any real number and let $f(x) = y$.

$\therefore \quad y = 3 - 4x \Rightarrow 4x = 3 - y$

$\Rightarrow \quad x = \dfrac{3-y}{4} \in R$, for each y [write x in terms of y]

So, for any real number $y \in R$ (codomain), there exists $\dfrac{3-y}{4}$ in R (domain), such that

$$f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = 3 - 3 + y = y$$

Hence, f is onto.

9. To find the pre-images, consider $x^2 + 1 = 17$ and $x^2 + 1 = -3$

$\Rightarrow \quad x^2 = 16$ and $x^2 = -4$, which is not possible.

$\Rightarrow \quad x = \pm 4$

Hence, pre-images of 17 are ± 4 and pre-image(s) of -3 does not exist.

***Case* II** Let $m = f(n) = n/2$

$\Rightarrow n = 2m$, which is even for each $m \in N$.

Thus, for each $m \in N$ (codomain), there exists $n = 2m \in N$, such that $f(2m) = \dfrac{2m}{2} = m$.

Since, from both cases, we have pre-images. So, we can choose either of the case to get the pre-image.

Therefore, f is onto.

Hence, f is many-one and onto function.

choose either of the case to get the pre-image.

Therefore, f is onto.

Hence, f is many-one and onto function.

15. Solve as Question 14.

16. We have a mapping $f : A \times B \to B \times A$, defined as

$f(a, b) = (b, a)$

For one-one Let (a_1, b_1) and $(a_2, b_2) \in A \times B$, such that $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow \quad (b_1, a_1) = (b_2, a_2)$

$\Rightarrow \quad b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow \quad (a_1, b_1) = (a_2, b_2)$

So, f is one-one.

For onto Let $(b', a') \in B \times A$ be any arbitrary element, then there exists $(a', b') \in A \times B$, such that

$f(a', b') = (b', a')$

Since, (b', a') was arbitrary, so f is onto.

Hence, f is bijective function.

17. Given, $f : R \to R$, defined by $f(x) = \dfrac{x}{x^2+1}, \forall x \in R$

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

Then, $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$

[$\because$ for any two distinct values x in $\left[0, \frac{\pi}{2}\right]$, sine function cannot give same value. This is also true for cosine function]

Hence, f and g must be one-one. $[\because a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)]$

But $(f+g)(0) = \sin 0 + \cos 0 = 1$

and $(f+g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$

$\therefore f+g$ is not one-one.

14. Given, $f: N \to N$, defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Here, $f(1) = \frac{1+1}{2} = 1$ and $f(2) = \frac{2}{2} = 1$

So, f is not one-one, i.e. f is many-one.

Now, consider a natural number m in codomain N.

***Case* I** Let $m = f(n) = \frac{n+1}{2} \Rightarrow 2m = n+1$

$\Rightarrow n = 2m-1$, which is odd for each $m \in N$.

Thus, for each $m \in N$ (codomain), there exists

$n = 2m - 1 \in N$, such that $f(2m-1) = \frac{2m-1+1}{2} = m$

18. We have a function $f: R \to \{x \in R: -1 < x < 1\}$, defined as

$$f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x}, & \text{if } x \geq 0 \\ \frac{x}{1-x}, & \text{if } x < 0 \end{cases}$$

For one-one Let $x_1, x_2 \in R$. Then, the following cases arise

***Case* I** *When both are less than* 0

Let $x_1, x_2 \in R$ such that $x_1 < 0$, $x_2 < 0$ and $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 - x_1 x_2 = x_2 - x_1 x_2$

$\Rightarrow x_1 = x_2$

***Case* II** *When both are greater than or equal to* 0

Let $x_1, x_2 \in R$ such that $x_1 \geq 0$, $x_2 \geq 0$ and $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \Rightarrow x_1 + x_1 x_2 = x_2 + x_1 x_2$

$\Rightarrow x_1 = x_2$

***Case* III** *When one is non-negative and other is negative*

Let $x_1 \geq 0$ and $x_2 < 0$.

Now, if $f(x_1) = f(x_2)$, then $\frac{x_1}{1+x_1} = \frac{x_2}{1-x_2}$

$\Rightarrow x_1 - x_1 x_2 = x_2 + x_1 x_2$

$\Rightarrow x_1 - x_2 = 2x_1 x_2$, which is not possible as LHS > 0 and RHS ≤ 0. Thus, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

From cases I, II and III, we get f is one-one.

$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$

$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$

$\Rightarrow x_1 x_2^2 - x_2 x_1^2 + x_1 - x_2 = 0$

$\Rightarrow x_1 x_2 (x_2 - x_1) - 1(x_2 - x_1) = 0$

$\Rightarrow (x_2 - x_1)(x_1 x_2 - 1) = 0$

$\Rightarrow x_2 = x_1$ or $x_1 x_2 = 1$

$\Rightarrow x_1 = x_2$ or $x_1 = \frac{1}{x_2}$

$\therefore f$ is not one-one, as if we take

$x_1 = 3$ and $x_2 = \frac{1}{3}$, then

$f(3) = \frac{3}{10} = f\left(\frac{1}{3}\right)$ but $3 \neq \frac{1}{3}$

Now, let $k \in R$ be any arbitrary element

and let $f(x) = k$

$\Rightarrow \frac{x}{x^2+1} = k$ $\left[\because f(x) = \frac{x}{x^2+1}\right]$

$\Rightarrow kx^2 + k = x \Rightarrow kx^2 - x + k = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1-4k^2}}{2k} \notin R$, if $1 - 4k^2 < 0$

or $(1-2k)(1+2k) < 0$, i.e. $k > 1/2$ or $k < -1/2$

So, f is not onto.

Hence, f is neither one-one nor onto.

19. We have, $y = f(x) = \sqrt{4-x^2}$, $0 \leq x \leq 2$, $0 \leq y \leq 2$

For one-one Let x_1, x_2 be any two elements of the interval $0 \leq x \leq 2$, such that $f(x_1) = f(x_2)$

$\Rightarrow \sqrt{4-x_1^2} = \sqrt{4-x_2^2} \Rightarrow 4 - x_1^2 = 4 - x_2^2$

$\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2 \Rightarrow x_1 = x_2$

[$x_1 \neq -x_2$, since x_1 and x_2 are non-negative]

So, $f(x)$ is one-one.

For onto Let $k \in [0, 2]$ be any arbitrary element

and let $f(x) = k \Rightarrow \sqrt{4-x^2} = k$

On squaring both sides, we get $4 - x^2 = k^2$

$\Rightarrow x^2 = 4 - k^2 \Rightarrow x = \pm\sqrt{4-k^2} \Rightarrow x = \sqrt{4-k^2}$

[$\because x$ is non-negative, so we take positive]

Also, for $0 \leq k \leq 2$, we have

$0 \leq \sqrt{4-k^2} \leq 2 \Rightarrow 0 \leq x \leq 2$, which is true.

Thus, for each $k \in [0, 2]$, there exists $x = \sqrt{4-k^2} \in [0, 2]$ such that $f(x) = k$. So, $f(x)$ is onto.

Hence, f is bijective function.

20. It is clear from the graph that it is a parabolic curve. So, let the equation of the function is

$$y = ax^2 + bx + c \quad \text{...(i)}$$

Points on the curve are $A(1, 2)$, $B(2, 3)$ and $C(-1, 6)$.

On putting the points $A(1, 2)$, $B(2, 3)$ and $C(-1, 6)$ in the Eq. (i) one-by-one, we get

For onto Let $y \in (-1, 1)$ be any arbitrary element. Then, the following cases arise

***Case* I** *When* $y \geq 0$, *i.e.* $0 \leq y < 1$

Consider, $y = f(x) = \dfrac{x}{1+x}$ $\quad \left[\because f(x) = \dfrac{x}{1+x} \geq 0\right]$

$\Rightarrow \quad y + yx = x \Rightarrow y = x - yx$

$\Rightarrow \quad x = \dfrac{y}{1-y} \geq 0$, for $0 \leq y < 1$

Thus, for each $y \in [0, 1)$, there exists $x = \dfrac{y}{1-y} \in R$ such that $f(x) = y$.

***Case* II** *When* $y < 0$, *i.e.* $-1 < y < 0$

Consider, $y = f(x) = \dfrac{x}{1-x}$ $\quad \left[\because f(x) = \dfrac{x}{1-x} < 0\right]$

$\Rightarrow \quad y - yx = x \Rightarrow y = yx + x = x(y+1)$

$\Rightarrow \quad x = \dfrac{y}{y+1} < 0$, for $-1 < y < 0$

Thus, for each $y \in (-1, 0)$, there exist $x = \dfrac{y}{y+1} \in R$ such that $f(x) = y$.

From cases I and II, we get f is onto.

At $A(1, 2)$, $2 = a(1)^2 + b(1) + c \Rightarrow 2 = a + b + c$ ...(ii)

At $B(2, 3)$, $3 = a(2)^2 + 2b + c \Rightarrow 3 = 4a + 2b + c$ ...(iii)

and at $C(-1, 6)$, $6 = a(-1)^2 + b(-1) + c$

$\Rightarrow \quad 6 = a - b + c$ ...(iv)

On solving Eqs. (ii), (iii) and (iv), we get

$$a = 1, b = -2 \text{ and } c = 3$$

On putting the values of a, b and c in Eq. (i), we get

$$y = x^2 - 2x + 3$$

$\therefore$ Required equation of function is $f(x) = y = x^2 - 2x + 3$

Now, $\quad y = (x-1)^2 + 2$

$\Rightarrow \quad (x-1)^2 = y - 2$

$\Rightarrow \quad (x-1) = \pm\sqrt{y-2}$

$\Rightarrow \quad x = 1 \pm \sqrt{y-2}$

Let $y = 3$, then $x = 1 \pm \sqrt{3-2}$

$\Rightarrow \quad x = 1 \pm 1 \Rightarrow x = 2, 0$

Here, we see that for two different values of domain (i.e. $x = 0, 2$). We get same image (i.e. $y = 3$).

Hence, $f(x)$ is many-one function.

SUMMARY

- **Relation on Sets A and B** Let A and B be two non-empty sets, then a relation R from set A to set B is a subset of cartesian product $A \times B$, i.e. $R \subseteq A \times B$.
- **Relation on a Set** Let A be a non-empty set. Then, a relation from A to itself, i.e. a subset of $A \times A$ is called a relation on set A.
- **Domain, Range and Codomain of Relation** Let R be a relation from set A to set B, such that $R = \{(a, b) : a \in A \text{ and } b \in R\}$. The set of all first and second elements of the ordered pairs in R is called the domain and range respectively, i.e. Domain $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$. The set B is called the codomain of relation R.
- **Types of Relations**
  - (i) A relation R in a set A is called **reflexive**, if $(a, a) \in R$, for every $a \in A$, i.e. $aRa, \forall a \in A$.
  - (ii) A relation R in set A is called **symmetric**, if $(a, b) \in R \Rightarrow (b, a) \in R$ for every $a, b \in A$, i.e. $aRb \Rightarrow bRa, \forall a, b \in A$
  - (iii) A relation R in set A is called **transitive**, if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$, i.e. aRb and $bRc \Rightarrow aRc, \forall a, b, c \in A$.
- **Equivalence Relation** A relation R on a set A is said to be an equivalence relation on A iff it is reflexive, symmetric and transitive.
- **Equivalence Classes** Consider an arbitrary equivalence relation R in an arbitrary set X. R divides X into mutually disjoint subsets A_i called partitions or sub-divisions of X, satisfying
  - (i) for each i, all elements of A_i are related to each other, i.e. $A_i \cup A_j = X$.
  - (ii) no element of A_i is related to any element of A_j, $i \neq j$.
  - (iii) $\cup A_i = x$ and $A_i \cap A_j = \phi, i \neq j$.

  Then, subsets A_i are called equivalence classes.
- **Function** (Mapping) For any two non-empty sets A and B, a function f from A to B is a rule or mapping which associates each element of set A to a unique element in set B.
- **Domain, Codomain and Range of a Function** Let $f : A \rightarrow B$, then elements of set A are called domain of f and the elements of set B are called codomain of f, and the set of all f-images obtained in set B corresponding to each element belongs to A is called range of f.
- **Types of Functions**
  - (i) A function $f : A \rightarrow B$ is called a **one-one** or **injective function**, if distinct elements of A have distinct images in B.
  - (ii) A function $f : A \rightarrow B$ is called a **many-one function**, if there exist atleast two distinct elements in A, whose images are same in B.
  - (iii) A function $f : A \rightarrow B$ is said to be **onto** or **surjective function**, if every element of B is the image of some elements of A under f.
  - (iv) A function $f : A \rightarrow B$ is an **into function**, if there exists an element in B which have no pre-image in A.
  - (v) A function $f : A \rightarrow B$ is said to be **one-one** and **onto** or **bijective function**, if f is both one-one and onto.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 Let R be the relation in the set N given by $R = \{(a, b) : a = b - 2, b > 6\}$. Then, the correct option is **[NCERT]**

(a) $(2,4) \in R$ (b) $(3, 8) \in R$
(c) $(6, 8) \in R$ (d) $(8, 7) \in R$

2 Let R be a relation from R to R the set of real numbers defined by $R = \{(x, y) : x, y \in R$ and $x - y + \sqrt{3}$ is an irrational number$\}$. Then, R is

(a) reflexive
(b) transitive
(c) symmetric
(d) an equivalence relation

3 A relation R in set $A = \{1, 2, 3\}$ is defined as $R = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$. Which of the following ordered pair in R shall be removed to make it an equivalence relation in A? **[CBSE 2021 (Term I)]**

(a) (1, 1) (b) (1, 2)
(c) (2, 2) (d) (3, 3)

4 The maximum number of equivalence relations on the set $A = \{1, 2, 3\}$ are **[NCERT Exemplar]**

(a) 1 (b) 2 (c) 3 (d) 5

5 Let $f : R \to R$ be defined as $f(x) = x^4$. Then, the correct option is **[NCERT]**

(a) f is one-one onto
(b) f is many-one onto
(c) f is one-one but not onto
(d) f is neither one-one nor onto.

6 Set A has 3 elements and the set B has 4 elements. Then, the number of injective mappings that can be defined from A to B is **[NCERT Exemplar]**

(a) 144 (b) 12 (c) 24 (d) 64

VERY SHORT ANSWER Type Questions

7 If the relation R is defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b) : |a^2 - b^2| < 8\}$. Then, find the relation R. **[NCERT Exemplar]**

8 Let $A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. State whether f is one-one or not. **[All India 2011]**

SHORT ANSWER Type II Questions

9 If $A = \{1, 2, 3, 4\}$ define relations on A which have properties of being

(i) reflexive, transitive but not symmetric.
(ii) symmetric but neither reflexive nor transitive. **[NCERT Exemplar]**

10 Show that relation R in the set of real numbers, defined as $R = \{(a, b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive. **[NCERT]**

11 Let n be a fixed positive integer. Define a relation R in Z for all $a, b \in Z, aRb$, if and only if $a - b$ is divisible by n. Show that R is an equivalence relation. **[NCERT Exemplar]**

12 If $f : R \to R$ is the function, defined by $f(x) = 4x^3 + 7$, then show that f is a bijection. **[Delhi 2011C]**

LONG ANSWER Type Questions

13 Show that the relation R, defined in the set of A all triangles as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5; T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10, which triangle among T_1, T_2 and T_3 are related? **[NCERT]**

14 Show that the function $f : N \to N$, given by $f(n) = n - (-1)^n, \forall n \in N$ is a bijection.

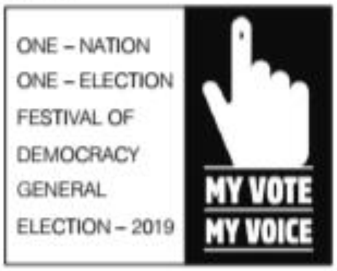
15 Let $A = [-1, 1]$, then discuss whether the following functions defined on A are one-one and onto or bijective.

(i) $f(x) = x/2$ (ii) $g(x) = |x|$
(iii) $h(x) = x\,|x|$ (iv) $k(x) = x^2$

[NCERT Exemplar]

CASE BASED Questions

16 A general election of Lok Sabha is a gigantic exercise. About 911 million people were eligible to vote and voter turnout was about 67%, the highest ever.



Let I be the set of all citizens of India who were eligible to exercise their voting right in general election held in 2019. A relation 'R' is defined on I as follows

$$R = \{(V_1, V_2) : V_1, V_2 \in I$$

and both use their voting right in general election-2019} **[CBSE Question Bank]**

Answer the following questions using the above information.

(i) Two neighbors X and $Y \in I$. X exercised his voting right while Y did not cast her vote in general election-2019. Which of the following is true?

(a) $(X, Y) \in R$ (b) $(Y, X) \in R$
(c) $(X, X) \notin R$ (d) $(X, Y) \notin R$

(ii) Mr. X and his wife W both exercised their voting right in general election-2019. Which of the following is true?

(a) Both (X, W) and $(W, X) \in R$
(b) $(X, W) \in R$ but $(W, X) \notin R$
(c) Both $(X, W) \in R$ and $(W, X) \notin R$
(d) $(W, X) \in R$ but $(X, W) \notin R$

(iii) Three friends F_1, F_2 and F_3 exercised their voting right in general election-2019, then which of the following is true?

(a) $(F_1, F_2) \in R$, $(F_2, F_3) \in R$ and $(F_1, F_3) \in R$
(b) $(F_1, F_2) \in R$, $(F_2, F_3) \in R$ and $(F_1, F_3) \notin R$
(c) $(F_1, F_2) \in R$, $(F_2, F_2) \in R$ but $(F_3, F_3) \notin R$
(d) $(F_1, F_2) \notin R$, $(F_2, F_3) \notin R$ and $(F_1, F_3) \notin R$

(iv) The above defined relation R is

(a) symmetric and transitive but not reflexive
(b) universal relation
(c) equivalence relation
(d) reflexive but not symmetric and transitive

(v) Mr. Shyam exercised his voting right in general election-2019, then Mr. Shyam is related to which of the following?

(a) All those eligible voters who cast their votes
(b) Family members of Mr.Shyam
(c) All citizens of India
(d) Eligible voters of India

17 An organization conducted bike race under 2 different categories- boys and girls. Totally there were 250 participants. Among all of them finally three from Category 1 and two from Category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project.

Let $B = \{b_1, b_2, b_3\}$ and $G = \{g_1, g_2\}$, where B represents the set of boys selected and G the set of girls who were selected for the final race.

Ravi decides to explore these sets for various types of relations and functions.

[Latest CBSE Sample Paper (Term I)]

Answer the following questions using the above information.

(i) Ravi wishes to form all the relations possible from B to G. How many such relations are possible?

(a) 2^6 (b) 2^5
(c) 0 (d) 2^3

(ii) Let $R : B \to B$ be defined by $R = \{(x, y) : x$ and y are students of same sex$\}$, then this relation R is

(a) equivalence

(b) reflexive only

(c) reflexive and symmetric but not transitive

(d) reflexive and transitive but not symmetric

(iii) Ravi wants to know among those relations, how many functions can be formed from B to G?

(a) 2^2 (b) 2^{12} (c) 3^2 (d) 2^3

(iv) Let $R : B \to G$ be defined by $R = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$, then R is

(a) injective

(b) surjective

(c) neither surjective nor injective

(d) surjective and injective

(v) Ravi wants to find the number of injective functions from B to G. How many numbers of injective functions are possible?

(a) 0 (b) 2! (c) 3! (d) 0!

ANSWERS

1. (c) 2. (a) 3. (b) 4. (d) 5. (d) 6. (c)

7. $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3) (4, 4), (5, 5)\}$

8. One-one 9. (i) $\{(1, 1), (1, 2), (2, 3), (2, 2), (1, 3), (3, 3), (4, 4)\}$ (ii) $\{(2, 2), (1, 2), (2, 1)\}$

13. T_1 and T_3 15. (i) f is one-one but not onto (ii) g is neither one-one nor onto (iii) h is bijective (iv) k is neither one-one nor onto

16. (i) → (d), (ii) → (a), (iii) → (a), (iv) → (c), (v) → (a) 17. (i) → (a), (ii) → (b), (iii) → (d), (iv) → (b), (v) → (a)