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This chapter is an introduction to calculus, a branch of Mathematics, which mainly deals with the study of change in the value of a function as the points in the domain change.

LIMITS AND DERIVATIVES

|TOPIC 1|

Fundamental of Limits

DEFINITION OF LIMIT

If $f(x)$ approaches to a real number l, when x approaches to a (through lesser or greater values to a) i.e. if $f(x) \to l$ when $x \to a$, then l is called limit of the function $f(x)$. In symbolic form, it can be written as $\lim_{x \to a} f(x) = l$.

Concept of Left Hand and Right Hand Limit

LEFT HAND LIMIT

A real number l_1 is the left hand limit of function $f(x)$ at $x = a$, if the values of $f(x)$ can be made as close as l_1 at points closed to a and on the left of a. Symbolically, it is written as

$$\text{LHL} = \lim_{x \to a^-} f(x) = l_1$$

In other words, we can say that $\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$, when we have the values of f near x to the left of a. This value is called the left hand limit of $f(x)$ at a.

RIGHT HAND LIMIT

A real number l_2 is the right hand limit of function $f(x)$ at $x = a$, if the values of $f(x)$ can be made as close as l_2 at points closed to a and on the right of a. Symbolically, it is written as $\text{RHL} = \lim_{x \to a^+} f(x) = l_2$

CHAPTER CHECKLIST

- Fundamental of Limits
- Limits of Rational Functions
- Limits of Trigonometric Functions
- Limits of Exponential Functions and Logarithmic Functions
- Derivative and First Principle of Derivative
- Algebra of Derivative of Functions
- Derivative of Trigonometric Functions

In other words, we can say that $\lim\limits_{x\to a^+} f(x)$ is the expected value of f at $x = a$, when we have the values of f near x to the right of a.

This value is called the right hand limit of $f(x)$ at a.

EXISTENCE OF LIMIT

If the right hand limit and left hand limit coincide (i.e. same), then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and denoted it by $\lim\limits_{x\to a} f(x)$.

Note

(i) (a) $x \to a - 0$ or $x \to a^-$ is read as x tends to a from left and it means that x is very close to a but it is always less than a.

(b) $x \to a + 0$ or $x \to a^+$ is read as x tends to a from right and it means that x is very close to a but it is always greater than a.

(c) $x \to a$ is read as x tends to a and it means that x is very close to a but it is never equal to a.

(ii) $\lim\limits_{x\to a^-}$ (Constant) = Constant; $\lim\limits_{x\to a^+}$ (Constant) = Constant and $\lim\limits_{x\to a}$ (Constant) = Constant. e.g. $\lim\limits_{x\to 0^+} 2 = 2$

METHOD TO SOLVE THE LEFT HAND AND RIGHT HAND LIMITS OF A FUNCTION

With the help of following steps, we can find the left hand and right hand limits of a function easily

***Step* I** For left hand limit, write the given function as $\lim\limits_{x\to a^-} f(x)$ and for right hand limit, write the given function as $\lim\limits_{x\to a^+} f(x)$.

***Step* II** For left hand limit, put $x = a - h$ and change the limit $x \to a^-$ by $h \to 0$. Then, limit obtained from step I is $\lim\limits_{h\to 0} f(a-h)$.

Similarly, for right hand limit, put $x = a + h$ and change the limit $x \to a^+$ by $h \to 0$. Then, limit obtained from step I is $\lim\limits_{h\to 0} f(a+h)$.

***Step* III** Now, simplify the result obtained in step II i.e. $\lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$.

EXAMPLE |1| Suppose the function is defined by

$$f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases}$$

(i) Find the left hand limit of $f(x)$ at $x = 3$.

(ii) Find the right hand limit of $f(x)$ at $x = 3$.

Sol. (i) Given, $f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases}$

$\therefore$ Left hand limit at $x = 3$ is

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x-3|}{x-3} \quad \text{...(i)}$$

On putting $x = 3 - h$ and changing the limit $x \to 3^-$ by $h \to 0$ in Eq. (i), we get

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{h\to 0} \frac{|-h|}{-h}$$

$$\Rightarrow \quad \lim_{x\to 3^-} f(x) = \lim_{h\to 0} \frac{h}{(-h)} \quad [\because |x| = x]$$

$$= -1$$

(ii) Right hand limit at $x = 3$ is

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{|x-3|}{x-3} \quad \text{...(ii)}$$

On putting $x = 3 + h$ and changing the limit $x \to 3^+$ by $h \to 0$ in Eq. (ii), we get

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{|x-3|}{x-3} = \lim_{h\to 0} \frac{|h|}{h}$$

$$\Rightarrow \quad \lim_{x\to 3^+} f(x) = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \quad [\because |x| = x]$$

Alternate Method

Given, $f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases}$

$$= \begin{cases} \dfrac{-(x-3)}{(x-3)}, & \text{if } x < 3 \\ \dfrac{x-3}{x-3}, & \text{if } x > 3 \\ 0, & \text{if } x = 3 \end{cases} = \begin{cases} -1, & \text{if } x < 3 \\ 1, & \text{if } x < 3 \\ 0, & \text{if } x < 3 \end{cases}$$

(i) LHL $= \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (-1)$ $\quad [\because f(x) = -1, \text{ for } x < 3]$

$= -1$

(ii) RHL $= \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (1)$ $\quad [\because f(x) = 1, \text{ for } x > 3]$

$= 1$

EXAMPLE |2| Find the left hand limit and right hand limit of the greatest integer function $f(x) = [x]$ = greatest integer less than or equal to x at $x = k$, where k is an integer. Also, show that $\lim\limits_{x\to k} f(x)$ does not exist.

Sol. We have, $f(x) = [x]$

LHL of f at $x = k$, $= \lim\limits_{x\to k^-} f(x)$

$= \lim\limits_{h\to 0} f(k-h)$ [putting $x = k - h$ and when $x \to k^-$, then $h \to 0$]

$= \lim\limits_{h\to 0} [k-h] = \lim\limits_{h\to 0} (k-1)$

$[\because k - 1 < k - h < k \Rightarrow (k-h) = k - 1]$

$= k - 1$

RHL of f at $x = k$, $= \lim_{x \to k^+} f(x)$

$= \lim_{h \to 0} f(k+h)$ [putting $x = k + h$ and when $x \to k^+$, then $h \to 0$]

$= \lim_{h \to 0} [k+h]$

$= \lim_{h \to 0} k$ $[\because k < k+h < k+1 \Rightarrow (k+h) = k]$

$= k$

Here, LHL $\neq$ RHL

$\therefore \lim_{x \to k} f(x)$ does not exist.

ALGEBRA OF LIMITS

Sometimes two or more functions involving algebraic operations, addition, subtraction, multiplication and division are given, then to find the limit of these functions involving algebraic operations,

we use the following theorem

Let f and g be two real functions with common domain D, such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists. Then,

(i) Limit of sum of two functions is sum of the limits of the functions. i.e.

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

(ii) Limit of difference of two functions is difference of the limits of the function. i.e.

$$\lim_{x \to a} (f-g)(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

(iii) Limit of product of a constant and one function is the product of that constant and limit of a function,

i.e.

$\lim_{x \to a} [c \cdot f(x)] = c \lim_{x \to a} f(x)$, where c is a constant.

(iv) Limit of product of two functions is product of the limits of the function, i.e.

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

(v) Limit of quotient of two functions is quotient of the limits of the functions, i.e.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)},$$

where $\lim_{x \to a} g(x) \neq 0$

EXAMPLE |3| Let us consider two functions $f(x) = x^2 + 4$ and $g(x) = x - 3$ such that $f(x)$ and $g(x)$ exist at $x = 5$. Find the limit of the following functions at $x = 5$.

(i) $f(x) + g(x)$ (ii) $f(x) - g(x)$

(iii) $f(x) \times g(x)$ (iv) $\frac{f(x)}{g(x)}$

Sol. Given functions are $f(x) = x^2 + 4$ and $g(x) = x - 3$

Clearly, $\lim_{x \to 5} f(x) = \lim_{x \to 5} x^2 + 4$

$= (5)^2 + 4$

$= 25 + 4 = 29$ ...(i)

and $\lim_{x \to 5} g(x) = \lim_{x \to 5} (x-3)$

$= (5-3) = 2$ ...(ii)

(i) $\lim_{x \to 5} [f(x) + g(x)] = \lim_{x \to 5} f(x) + \lim_{x \to 5} g(x)$

$= 29 + 2$ [from Eqs. (i) and (ii)]

$= 31$

(ii) $\lim_{x \to 5} [f(x) - g(x)] = \lim_{x \to 5} f(x) - \lim_{x \to 5} g(x)$

$= 29 - 2$

$= 27$ [from Eqs. (i) and (ii)]

(iii) $\lim_{x \to 5} [f(x) \times g(x)] = \lim_{x \to 5} f(x) \times \lim_{x \to 5} g(x)$

$= 29 \times 2$

$= 58$ [from Eqs. (i) and (ii)]

(iv) $\lim_{x \to 5} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 5} f(x)}{\lim_{x \to 5} g(x)}$ [from Eqs. (i) and (ii)]

$= \frac{29}{2} = 14.5$

LIMITS OF POLYNOMIAL FUNCTION

A function f is said to be a **polynomial function**, if $f(x)$ is zero function or if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, where a_i's are real numbers and $a_n \neq 0$.

METHOD TO FIND LIMIT OF A POLYNOMIAL

To find the limit of given polynomial, we use the algebra of limits and then put the limit and simplify. It can be understand in the following way

We know that, $\lim_{x \to a} x = a$ .Then

$$\lim_{x \to a} x^2 = \lim_{x \to a} (x \cdot x)$$

$$= \lim_{x \to a} x \cdot \lim_{x \to a} x$$

$$= a \cdot a = a^2$$

Similarly, $\lim_{x \to a} x^n = a^n$

Now, let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial function.

Then, limit of a polynomial function $f(x)$

$$= \lim_{x \to a} f(x) = \lim_{x \to a} [a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n]$$

$$= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + \ldots + \lim_{x \to a} a_n x^n$$

$$= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \ldots + a_n \lim_{x \to a} x^n$$

$$= a_0 + a_1 a + a_2 a^2 + \ldots + a_n a^n = f(a)$$

EXAMPLE |4| Evaluate the limits $\lim_{x \to 3} (4x^3 - 2x^2 - x + 1)$.

Sol. $\lim_{x \to 3} (4x^3 - 2x^2 - x + 1)$

$$= 4 \lim_{x \to 3} x^3 - 2 \lim_{x \to 3} x^2 - \lim_{x \to 3} x + \lim_{x \to 3} 1$$

$$= 4(3)^3 - 2(3)^2 - 3 + 1 = 108 - 18 - 2 = 88$$

EXAMPLE |5| Evaluate the left hand and right hand limits of the following function at $x = 2$.

$$f(x) = \begin{cases} 2x + 3, \text{ if } x \le 2 \\ x + 5, \text{ if } x > 2 \end{cases}$$

Does $\lim_{x \to 2} f(x)$ exist?

Sol. Given, $f(x) = \begin{cases} 2x + 3, \text{ if } x \le 2 \\ x + 5, \text{ if } x > 2 \end{cases}$

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x + 3$$

$[\because f(x) = 2x + 3, \text{ if } x \le 2]$

$$= \lim_{h \to 0} [2(2 - h) + 3] = 2(2 - 0) + 3$$

[putting $x = 2 - h$ and when $x \to 2^-$, then $h \to 0$]

$$= 4 + 3 = 7$$

and $\text{RHL} = \lim_{x \to 2^+} f(x)$

$$= \lim_{x \to 2^+} (x + 5) \quad [\because f(x) = x + 5, \text{ if } x > 2]$$

$$= \lim_{h \to 0} (2 + h + 5) = 2 + 0 + 5 = 7$$

[putting $x = 2 + h$ and $x \to 2^+$, then $h \to 0$]

$\because$ LHL of $f(at\ x = 2)$ = RHL of $f(at\ x = 2)$

$\therefore \lim_{x \to 2} f(x)$ exists and it is equal to 7.

EXAMPLE |6| Evaluate the left hand and right hand limits of the function defined by

$$f(x) = \begin{cases} 1 + x^2, \text{ if } 0 \le x \le 1 \\ 2 - x^2, \text{ if } x > 1 \end{cases} \text{ at } x = 1.$$

Also, show that $\lim_{x \to 1} f(x)$ does not exist.

Sol. We have, $f(x) = \begin{cases} 1 + x^2, \text{ if } 0 \le x \le 1 \\ 2 - x^2, \text{ if } x > 1 \end{cases}$

At $x = 1$, $\text{LHL} = \lim_{x \to 1^-} f(x)$

$$= \lim_{x \to 1^-} (1 + x^2) \quad [\because f(x) = 1 + x^2, \text{ if } 0 \le x \le 1]$$

$$= \lim_{h \to 0} [1 + (1 - h)^2] = 1 + (1 - 0)^2 = 2$$

[putting $x = 1 - h$ and when $x \to 1^-$, then $h \to 0$]

$\text{RHL} = \lim_{x \to 1^+} f(x)$

$$= \lim_{x \to 1^+} (2 - x^2) \quad [\because f(x) = 2 - x^2, \text{ if } x > 1]$$

$$= \lim_{h \to 0} [2 - (1 + h)^2]$$

[putting $x = 1 + h$ and when $x \to 1^+$, then $h \to 0$]

$$= 2 - 1 = 1$$

Since, LHL $\ne$ RHL

Therefore, $\lim_{x \to 1} f(x)$ does not exist.

EXAMPLE |7| For what integers m and n does $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist, if

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^2 + m, & x > 1 \end{cases}$$

[NCERT]

Sol. Given, $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^2 + m, & x > 1 \end{cases}$

Limit at $x = 0$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} mx^2 + n$$

$$= \lim_{h \to 0} m(0 - h)^2 + n$$

[putting $x = 0 - h$ and as $x \to 0^-$, then $h \to 0$]

$$= n$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} nx^2 + m$$

$$= \lim_{h \to 0} n(0 + h)^2 + m = m$$

[putting $x = 0 + h$ and as $x \to 0^+$, then $h \to 0$]

Now, for $\lim_{x \to 0} f(x)$ to be exists.

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \quad n = m$$

Hence, $\lim_{x \to 0} f(x)$ exist when $m = n$

Limit at $x = 1$

Here, $\text{LHL} = \lim_{x \to 1^-} f(x)$

$$= \lim_{x \to 1^-} (nx + m)$$

$= \lim_{h \to 0} [n(1-h)+m]$

[putting $x = 1-h$ and as $x \to 1^-$, then $h \to 0$]

$= n + m$

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (nx^2 + m)$

$= \lim_{h \to 0} n(1+h)^2 + m$

[putting $x = 1+h$ and as $x \to 1^+$, then $h \to 0$]

$= n + m$

Now, for $\lim_{x \to 1} f(x)$ to be exists,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

$\Rightarrow h + m = h + m$, which is true for all $n, m \in Z$.

Hence, $\lim_{x \to 1} f(x)$ exists for all $n, m \in Z$.

EXAMPLE |8| If $f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0, \\ |x|-1, & x > 0 \end{cases}$ for what value(s) of a does $\lim_{x \to a} f(x)$ exist? **[NCERT]**

Sol. We have, $f(x) = \begin{cases} -x+1, & x < 0 \\ 0, & x = 0 \\ x-1, & x > 0 \end{cases}$ $\left[\because |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}\right]$

***Case* I When $a = 0$**

In this case, LHL $\lim_{h \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} -(0-h)+1$

[$\because$ For $x < 0$, $f(x) = -x+1$]

$= \lim_{h \to 0} (h+1) = \lim_{h \to 0} h + \lim_{h \to 0} 1$

$= 0 + 1 = 1$

and RHL $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} f(0+h) - 1 = -1$

[$\because$ For $x > 0$, $f(x) = x - 1$]

$\because$ LHL $\neq$ RHL

$\because$ For $a = 0$, $\lim_{x \to a} f(x)$ does not exists.

***Case* II When $a < 0$**

In this case, $\lim_{x \to a} f(x) = \lim_{x \to a} (-x+1)$

[$\because$ For $x < 0$, $f(x) = -x+1$]

$= -a + 1$, which is a fixed real number.

$\therefore$ For $a < 0$, $\lim_{x \to a} f(x)$ exists.

***Case* III When $a > 0$**

In this case, $\lim_{x \to a} f(x) = \lim_{x \to a} (x-1)$

[$\because$ For $x > 0$, $f(x) = x - 1$]

$= a - 1$, which is a fixed real number.

$\therefore$ For $a > 0$, $\lim_{x \to a} f(x)$ exists.

Hence, from case I, II and III, we conclude that $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The limit of $f(x) = x^2$ when x tends to zero equals

(a) zero (b) one

(c) two (d) three

2 The right hand limit and left hand limit of a function $f(x)$ at a given point $x = a$ is the value of $f(x)$ which is dictated by the values of $f(x)$ when x tends to a from ...A... and ...B..., respectively. Here, A and B refer to

(a) left, right

(b) left, left

(c) right, left

(d) right, right

3 The process of finding the limit follows addition, subtraction, multiplication and division as long as the limits and functions under consideration are

(a) undefined

(b) well defined

(c) Both (a) and (b) are correct

(d) Neither (a) nor (b) is correct

4 Let f be a function such that $\lim_{x \to a} f(x)$ exist. Then for any real number λ, we have

(a) $\lim_{x \to a} [(\lambda \cdot f)(x)] = \lambda \cdot \lim_{x \to a} f(x)$

(b) $\lim_{x \to a} \left[\left(\frac{f}{\lambda}\right)(x)\right] = \frac{1}{\lambda} \cdot \lim_{x \to a} f(x)$

(c) Both (a) and (b) are correct

(d) Neither (a) nor (b) is correct

5 If f is an odd function, then $\lim_{x \to 0} f(x)$, when exists, is equal to

(a) $f(0)$

(b) 0

(c) any real value

(d) None of these

VERY SHORT ANSWER Type Questions

6 Find $\lim_{x \to 2^-} (x^2 - x + 1)$.

7 Evaluate LHL and RHL of the $\lim_{x \to 0} \frac{|x|}{x}$.

8 Evaluate $\lim_{x\to 1}[x-1]$, where $[\cdot]$ is greatest integer function.

9 Find the limits (Each part carries 1 mark)

(i) $\lim_{x\to 1}[x^2-x-1]$ (ii) $\lim_{x\to 3}[x^2(x+1)]$

(iii) $\lim_{x\to -1}(1+x+x^2+\ldots+x^9)$

SHORT ANSWER Type I Questions

10 If $f(x)=\begin{cases}x+2, x\le -1\\ cx^2, x>-1\end{cases}$, then find c when $\lim_{x\to -1} f(x)$ exists. **[NCERT Exemplar]**

11 Let $f(x)$ be a function defined by

$$f(x)=\begin{cases}6x-6, \text{if } x\le 3\\ 2x-k, \text{if } x>3\end{cases}$$

Find the value of k, if $\lim_{x\to 3} f(x)$ exists.

12 Show that $\lim_{x\to 4}\frac{|x-4|}{x-4}$ does not exist. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

13 Evaluate the left hand and right hand limits of the following functions at $x=1$.

$$f(x)=\begin{cases}5x-4, & \text{if } 0<x\le 1\\ 4x^2-3x, & \text{if } 1<x<2\end{cases}$$

Does $\lim_{x\to 1} f(x)$ exist?

14 Let $f(x)=\begin{cases}2x+3, \text{if } x\le 0\\ 3(x+1), \text{if } x>0\end{cases}$, then

(i) evaluate $\lim_{x\to 0} f(x)$. (ii) evaluate $\lim_{x\to 1} f(x)$.

15 Suppose $f(x)=\begin{cases}a+bx, x<1\\ 4, \quad x=1\\ b-ax, x>1\end{cases}$, and if $\lim_{x\to 1} f(x)=f(1)$,

then what are the possible values of a and b? **[NCERT]**

LONG ANSWER Type Question

16 Evaluate $\lim_{x\to 0} f(x)$, where $f(x)=\begin{cases}\frac{x}{|x|}, x\neq 0\\ 0, \quad x=0\end{cases}$. **[NCERT]**

17 If $f(x)=\begin{cases}\frac{x-|x|}{x}, \text{if } x\neq 0\\ 2 \quad, \text{if } x=0\end{cases}$, then show that $\lim_{x\to 0} f(x)$ does not exist.

18 Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where

$$f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$$

[NCERT]

HINTS & ANSWERS

1. (a) Given function is $f(x)=x^2$. Observe that when x takes values very close to 0, the value of $f(x)$ also approaches towards 0.
We say $\lim_{x\to 0} f(x)=0$

2. (c) There are essentially two ways x could approach to a number a either from left or from right. This naturally leads to two limits-the right hand limit and the left hand limit.

3. (b) The limiting process respects addition, subtraction multiplication and division as long as the limits and functions under consideration are well defined.

4. (a) For any real number λ, we have
$$\lim_{x\to a}[(\lambda\cdot f)(x)]=\lambda\cdot\lim_{x\to a} f(x)$$

5. (b) It is given that, $\lim_{x\to 0} f(x)$ exists.
$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$$
$$\Rightarrow \lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(0+h)$$
$$\Rightarrow -\lim_{h\to 0} f(h)=\lim_{h\to 0} f(h)$$
$[\because f(x) \text{ is odd} \Rightarrow f(-h)=-f(h)]$
$$\Rightarrow 2\lim_{h\to 0} f(h)=0 \Rightarrow \lim_{h\to 0} f(h)=0$$
$$\therefore \lim_{x\to 0} f(x)=0$$

6. $\lim_{x\to 2^-}(x^2-x+1)=\lim_{h\to 0}\{(2-h)^2-(2-h)+1\}$ **Ans.** 3

7. LHL $=\lim_{x\to 0^-}\frac{|x|}{x}=\frac{-x}{x}$ and RHL $=\lim_{x\to 0^+}\frac{|x|}{x}=\frac{x}{x}$
Ans. LHL $=-1$ and RHL $=1$

8. LHL $=\lim_{x\to 1^-}[x-1]=\lim_{h\to 0}[1-h-1]=\lim_{h\to 0}[-h]=-1$
and RHL $=\lim_{x\to 1^+}[x-1]=\lim_{h\to 0}[1+h-1]=\lim_{h\to 0}[h]=0$
Ans. Does not exist

9. (i) $\lim_{x\to 1}[x^2-x-1]=[1^2-1-1]=-1$
(ii) Solve as part (i). **Ans.** 36
(iii) Solve as part (ii). **Ans.** 0

10. LHL $=\lim_{x\to -1^-}(x+2)=\lim_{h\to 0}(-1-h+2)=1$
RHL $=\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+} cx^2=\lim_{h\to 0} c(-1+h)^2$
Ans. $c=1$

11. We have, $f(x)=\begin{cases}6x-6 \text{ if } x\le 3\\ 2x-k, \text{ if } x>3\end{cases}$

$\therefore$ LHL $=\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(6x-6)$

$=\lim_{h\to 0}[6(3-h)-6]=12$

and RHL $=\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(2x-k)$

$=\lim_{h\to 0}[2(3+h)-k]=6-k$ **Ans.** $k=-6$

12. Given, $\lim_{x\to 4}\frac{|x-4|}{x-4}$

LHL $=\lim_{x\to 4^-}\frac{-(x-4)}{x-4}=-1$ $[\because |x-4|=-(x-4), x<4]$

RHL $=\lim_{x\to 4^+}\frac{(x-4)}{x-4}=1$ $[\because |x-4|=(x-4), x>4]$

13. LHL $=\lim_{x\to 1^-}(5x-4)=\lim_{h\to 0}[5(1-h)-4]=1$

and RHL $=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4x^2-3x)$

$=\lim_{h\to 0}[4(1+h)^2-3(1+h)]=1$

Ans. $\lim_{x\to 1} f(x)$ exists and it is equal to 1.

14. (i) LHL $=\lim_{x\to 0^-}(2x+3)=\lim_{h\to 0}[2(0-h)+3]=3$

RHL $=\lim_{x\to 0^+}3(x+1)=\lim_{h\to 0}[3(0+h+1)]=3$ **Ans.** 3

(ii) $\lim_{x\to 1} f(x)=\lim_{x\to 1}3(x+1)$ **Ans.** 6

15. LHL $=\lim_{x\to 1^-}(a+bx)=\lim_{h\to 0}[a+b(1-h)]=a+b$

RHL $=\lim_{x\to 1^+}(b-ax)=\lim_{h\to 0}[b-a(1+h)]=b-a$

$\because$ LHL = RHL $=f(1)\Rightarrow a+b=b-a=4$

Ans. $a=0, b=4$

16. LHL $=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$

$=\lim_{h\to 0}\frac{(0-h)}{|0-h|}=\lim_{h\to 0}\frac{(0-h)}{-(0-h)}=-1$

RHL $=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)$

$=\lim_{h\to 0}\frac{0+h}{|0+h|}=\lim_{h\to 0}\frac{0+h}{(0+h)}=1$

Ans. At $x=0$, limit does not exist.

17. LHL $=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x-|x|}{x}=\lim_{h\to 0}\frac{(0-h)-|0-h|}{(0-h)}$

$=\lim_{h\to 0}\frac{-h-h}{-h}=2$

and RHL $=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x-|x|}{x}=\lim_{h\to 0}\frac{(0+h)-|0+h|}{0+h}=0$

Ans. $\lim_{x\to 0} f(x)$ does not exist.

18. At $x=0$, LHL $=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$

$=\lim_{h\to 0}2(0-h)+3=3$

RHL $=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}3(0+h+1)=3$

LHL $=\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}3(1-h+1)=6$

At $x=1$, RHL $=\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0}3(1+h+1)=6$

Ans. $\lim_{x\to 0} f(x)=3$ and $\lim_{x\to 1} f(x)=6$

|TOPIC 2|
Limits of Rational Functions

A function f is said to be a rational function, if $f(x)=\frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions such that $h(x)\neq 0$.

Then, $\lim_{x\to a} f(x)=\lim_{x\to a}\frac{g(x)}{h(x)}=\frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)}=\frac{g(a)}{h(a)}$

However, if $h(a)=0$, then there are two cases arise,
(i) $g(a)\neq 0$ (ii) $g(a)=0$.
In the first case, we say that the limit does not exist.

In the second case, we can find limit.

Limit of a rational function can be find with the help of following methods

Direct Substitution Method

In this method, we substitute the point, to which the variable tends to in the given limit. If it give us a real number, then the number so obtained is the limit of the function and if it does not give us a real number, then use other methods.

EXAMPLE |1| Find the limits of the following.

(i) $\lim_{x\to 2} \frac{x^2-4}{x+3}$ (ii) $\lim_{x\to -1} \frac{(x-1)^2+3x^2}{(x^4+1)^2}$

(iii) $\lim_{x\to 0} \frac{\sqrt{2+x}+\sqrt{2-x}}{2+x}$ **[NCERT]**

Sol. (i) $\lim_{x\to 2} \frac{x^2-4}{x+3} = \frac{4-4}{2+3} = \frac{0}{5} = 0$ [here, $h(a) \neq 0$]

(ii) $\lim_{x\to -1} \frac{(x-1)^2+3x^2}{(x^4+1)^2} = \frac{(-1-1)^2+3(-1)^2}{((-1)^4+1)^2} = \frac{(-2)^2+3(1)}{(1+1)^2}$

$= \frac{4+3}{2^2} = \frac{7}{4}$

(iii) $\lim_{x\to 0} \frac{\sqrt{2+x}+\sqrt{2-x}}{2+x} = \frac{\sqrt{2+0}+\sqrt{2-0}}{2+0}$

$= \frac{\sqrt{2}+\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

EXAMPLE |2| Evaluate $\lim_{x\to 1} \frac{x-4}{3-\sqrt{13-x}}$.

Sol. $\lim_{x\to 1} \frac{x-4}{3-\sqrt{13-x}} = \frac{-3}{3-\sqrt{12}}$

$= \frac{-3}{3-2\sqrt{3}} = \frac{-3(3+2\sqrt{3})}{(3-2\sqrt{3})(3+2\sqrt{3})} = \frac{-3(3+2\sqrt{3})}{9-12}$

$= \frac{-3(3+2\sqrt{3})}{-3} = 3+2\sqrt{3}$

Factorisation Method

Let $\lim_{x\to a} \frac{f(x)}{g(x)}$ reduces to the form $\frac{0}{0}$, when we substitute $x = a$. Then, we factorise $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

METHOD TO DETERMINE THE LIMIT BY USING FACTORISATION METHOD

***Step* I** Write the given limit as $\lim_{x\to a} \frac{f(x)}{g(x)}$.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then go to next step, otherwise use direct substitution method.

***Step* II** Factorise $f(x)$ and $g(x)$, such that $(x-a)$ is a common factor and write given limit as

$$\lim_{x\to a} \frac{(x-a)\, f_1(x)}{(x-a)\, g_1(x)}.$$

***Step* III** Cancel the common factor(s), then limit obtained in Step III becomes $\lim_{x\to a} \frac{f_1(x)}{g_1(x)}$.

***Step* IV** Use direct substitution method to obtain limit.

USEFUL FORMULAE FOR FACTORISATION

(i) $(a^2-b^2) = (a-b)(a+b)$

(ii) $a^3-b^3 = (a-b)(a^2+ab+b^2)$

(iii) $a^3+b^3 = (a+b)(a^2-ab+b^2)$

(iv) $a^4-b^4 = (a^2-b^2)(a^2+b^2)$

$= (a-b)(a+b)(a^2+b^2)$

(v) If $f(\alpha) = 0$, then $x-\alpha$ is a factor of $f(x)$.

EXAMPLE |3| Evaluate $L = \lim_{x\to 2} \frac{x^3-8}{(x-2)}$.

Sol. We have, $L = \lim_{x\to 2} \frac{x^3-8}{x-2}$.

Let $f(x) = x^3-8$ and $g(x) = x-2$

Here, $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^3-8 = 2^3-8 = 0$

and $\lim_{x\to 2} g(x) = \lim_{x\to 2} x-2 = 2-2 = 0$

Thus, we get $\frac{0}{0}$ form.

Now, factorise $f(x)$ and $g(x)$ such that $(x-2)$ is a common factor.

Here, $f(x) = x^3-8 = (x^3-2^3)$

$= (x-2)(x^2+4+2x)$ and $g(x) = x-2$

$\therefore\quad L = \lim_{x\to 2} \frac{(x-2)(x^2+4+2x)}{(x-2)}$

On cancelling the common factor $(x-2)$, we get

$L = \lim_{x\to 2} (x^2+4+2x)$

$= (2)^2+4+2(2) = 4+4+4 = 12$

Hence, $\lim_{x\to 2} \frac{x^3-8}{x-2} = 12$

Note We cancelled the term $(x-2)$ in the above calculation, because $x \neq 2$.

EXAMPLE |4| Evaluate $\lim_{x\to \frac{1}{2}} \frac{4x^2-1}{2x-1}$.

Sol. On putting $x = \frac{1}{2}$, we get the form $\frac{0}{0}$.

So, let us first factorise it.

Consider, $\lim_{x\to \frac{1}{2}} \frac{4x^2-1}{2x-1} = \lim_{x\to \frac{1}{2}} \frac{(2x+1)(2x-1)}{(2x-1)}$

[using factorisation method]

$= \lim_{x\to \frac{1}{2}} (2x+1)$

$= 2\left(\frac{1}{2}\right)+1 = 2$

EXAMPLE |5| Evaluate $\lim_{x\to 2}\left[\frac{x^2-4}{x^3-4x^2+4x}\right]$.

Sol. On putting $x=2$, we get the form $\frac{0}{0}$. So, let us first factorise it.

Consider, $\lim_{x\to 2}\frac{x^2-4}{x^3-4x^2+4x}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x(x-2)^2}$

$$=\lim_{x\to 2}\frac{(x+2)}{x(x-2)}=\frac{2+2}{2(2-2)}=\frac{4}{0}$$

which is not defined.

$\therefore\quad \lim_{x\to 2}\left[\frac{x^2-4}{x^3-4x^2+4x}\right]$ does not exist.

EXAMPLE |6| Evaluate $\lim_{x\to\sqrt{3}}\frac{x^2-3}{x^2+3\sqrt{3}\,x-12}$.

Sol. $\lim_{x\to\sqrt{3}}\frac{x^2-3}{x^2+3\sqrt{3}\,x-12}=\lim_{x\to\sqrt{3}}\frac{(x-\sqrt{3})(x+\sqrt{3})}{x^2+4\sqrt{3}x-\sqrt{3}x-12}$

$$=\lim_{x\to\sqrt{3}}\frac{(x-\sqrt{3})(x+\sqrt{3})}{(x+4\sqrt{3})(x-\sqrt{3})}$$

$$=\lim_{x\to\sqrt{3}}\frac{(x+\sqrt{3})}{(x+4\sqrt{3})}=\frac{\sqrt{3}+\sqrt{3}}{\sqrt{3}+4\sqrt{3}}=\frac{2\sqrt{3}}{5\sqrt{3}}=\frac{2}{5}$$

EXAMPLE |7| Evaluate $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$.

When $x=1$, the expression $\frac{2}{1-x^2}-\frac{1}{1-x}$ becomes the form $\infty-\infty$. So, we need to simplify it to express it in the form $\frac{0}{0}$.

Sol. We have,

$$\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)=\lim_{x\to 1}\left(\frac{2}{1-x^2}-\frac{1}{1-x}\right)$$

[$\infty-\infty$ form]

$$=\lim_{x\to 1}\frac{2-(1+x)}{1-x^2}$$

$\left[\frac{0}{0}\text{ form}\right]$

$$=\lim_{x\to 1}\frac{1-x}{1-x^2}=\lim_{x\to 1}\frac{1}{1+x}=\frac{1}{2}.$$

EXAMPLE |8| Find the limit

$$\lim_{x\to 1}\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right].$$

[NCERT]

When $x=1$, the expression $\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}$ becomes the form $\infty-\infty$. So, we need to simplify it to express it in the form $\frac{0}{0}$.

Sol. Here, $\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x^2-3x+2)}\right]$$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x-1)(x-2)}\right]$$

$$=\frac{(x-2)^2-1}{x(x-1)(x-2)}$$

$$=\left[\frac{x^2-4x+4-1}{x(x-1)(x-2)}\right]$$

$$=\left[\frac{x^2-4x+3}{x(x-1)(x-2)}\right]=\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$\therefore\quad \lim_{x\to 1}\left[\frac{x^2-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$

$$=\lim_{x\to 1}\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{x-3}{x(x-2)}$$

$$=\frac{1-3}{1(1-2)}=\frac{-2}{-1}=2$$

EXAMPLE |9| Prove that $\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{3-x}-3^{x/2}}=-\frac{4}{3}$.

Sol. We have, $\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{3-x}-3^{x/2}}=\lim_{x\to 2}\frac{(3^x)^2-12(3^x)+27}{3^3-(3^{x/2})^3}$

$$=\lim_{x\to 2}\frac{(3^x-3)(3^x-9)}{(3-3^{x/2})(9+3\times 3^{x/2}+3^x)}$$

$[\because a^3-b^3=(a-b)(a^2+ab+b^2)]$

$$=-\lim_{x\to 2}\frac{(3^x-3)(3^{x/2}-3)(3^{x/2}+3)}{(3^{x/2}-3)(3^x+3\times 3^{x/2}+9)}$$

$$=-\lim_{x\to 2}\frac{(3^x-3)(3^{x/2}+3)}{(3^x+3\times 3^{x/2}+9)}$$

$$=-\frac{(9-3)(3+3)}{(9+9+9)}=-\frac{6\times 6}{27}=-\frac{4}{3}$$

Hence proved.

Rationalisation Method

If we get $\frac{0}{0}$ form and numerator or denominator or both have radical sign, then we rationalise the numerator or denominator or both by multiplying their conjugate to remove $\frac{0}{0}$ form and then find limit by direct substitution method.

EXAMPLE |10| Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$.

[NCERT Exemplar]

Sol. When $x = 0$, then the expression $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ becomes of the form $\frac{0}{0}$. So,we will rationalising the numerator by multiplying and dividing its conjugate i.e. $\sqrt{2+x}+\sqrt{2}$.

$$\therefore \lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})}$$

[multiplying numerator and denominator by $\sqrt{2+x}+\sqrt{2}$]

$$= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})} = \lim_{x \to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

[using direct substitution method]

EXAMPLE |11| Evaluate $\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$.

[NCERT Exemplar]

Sol. We have, $\lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

$$= \lim_{x \to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}$$

[multiplying numerator and denominator by $\sqrt{a+2x}+\sqrt{3x}$]

$$= \lim_{x \to a} \frac{a+2x-3x}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{a+2x}+\sqrt{3x})}$$

$[\because (A-B)(A+B) = A^2 - B^2]$

$$= \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})(\sqrt{3x+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})}$$

[multiplying numerator and denominator by $\sqrt{3a+x}+2\sqrt{x}$]

$$= \lim_{x \to a} \frac{(a-x)[\sqrt{3a+x}+2\sqrt{x}]}{(\sqrt{a+2x}+\sqrt{3x})(3a+x-4x)}$$

$$= \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})(3a-3x)}$$

$$= \frac{\sqrt{3a+a}+2\sqrt{a}}{3(\sqrt{a+2a}+\sqrt{3a})}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3}\sqrt{a}}$$

$$= \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$$

EXAMPLE |12| Evaluate $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$.

Sol. When $x = 1$, the expression $\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ becomes of the form $\frac{0}{0}$. So, rationalising $(\sqrt{x}-1)$ in the numerator. by multiplying and dividing its conjugate i.e. $\sqrt{x}+1$.

We have, $\lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

$$= \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)(2x^2+x-3)}$$

$$= \lim_{x \to 1} \frac{(2x-3)(x-1)}{(\sqrt{x}+1)(2x^2+x-3)} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x \to 1} \frac{(2x-3)(x-1)}{(\sqrt{x}+1)(2x+3)(x-1)}$$

$$= \lim_{x \to 1} \frac{2x-3}{(\sqrt{x}+1)(2x+3)} = -\frac{1}{10}$$

By using Some Standard Limits

If the given limit is of the form $\lim_{x \to a} \frac{x^n - a^n}{x - a}$, then we can find the limit directly by using the following theorem

Theorem *Let n be any positive integer. Then,*

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$$

Proof We know that, $x^n - a^n$

$$= (x-a)(x^{n-1} + ax^{n-2} + \ldots + a^{n-2}x + a^{n-1})$$

On dividing both sides by $(x - a)$, we get

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + \ldots + a^{n-2}x + a^{n-1}$$

Thus, $\lim_{x \to a} \frac{x^n - a^n}{x - a}$

$$= \lim_{x \to a}(x^{n-1} + ax^{n-2} + \ldots + a^{n-2}x + a^{n-1})$$

$$= a^{n-1} + a(a^{n-2}) + \ldots + a^{n-2}a + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} \quad [n \text{ terms}]$$

$$= na^{n-1}$$

Note

The above theorem i.e. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ is also true, if n is any rational number and a is positive.

EXAMPLE |13| Evaluate $\lim_{x\to 2} \dfrac{x^{10}-1024}{x-2}$.

Sol. When $x = 2$, the expression $\dfrac{x^{10}-1024}{x-2}$ becomes of the form $\dfrac{0}{0}$.

Now, $\lim_{x\to 2} \dfrac{x^{10}-1024}{x-2} = \lim_{x\to 2} \dfrac{x^{10}-2^{10}}{x-2} = 10(2^{10-1}) = 5120$

EXAMPLE |14| Evaluate $\lim_{x\to 2} \dfrac{x-2}{\sqrt[3]{x}-\sqrt[3]{2}}$.

Sol. $\lim_{x\to 2} \dfrac{x-2}{x^{1/3}-2^{1/3}} = \dfrac{1}{\lim_{x\to 2} \dfrac{x^{1/3}-2^{1/3}}{x-2}}$

$$= \frac{1}{\frac{1}{3}(2^{1/3-1})} = \frac{1}{\frac{1}{3}\times(2^{-2/3})} = 3(2^{2/3})$$

EXAMPLE |15| Find all the possible values of a, if

$$\lim_{x\to a} \frac{x^9-a^9}{x-a} = \lim_{x\to 5} (4+x).$$

Sol. We have, $\lim_{x\to a} \dfrac{x^9-a^9}{x-a} = \lim_{x\to 5} (4+x)$ ...(i)

Now, $\lim_{x\to a} \dfrac{x^9-a^9}{x-a} = 9(a)^{9-1} = 9a^8$

$\left[\because \lim_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}\right]$

and $\lim_{x\to 5} (4+x) = 4+5 = 9$

From Eq. (i), $9a^8 = 9 \Rightarrow a^8 = 1 \Rightarrow a = \pm 1$

EXAMPLE |16| Evaluate $\lim_{x\to 2} \dfrac{x^5-32}{x^3-8}$.

First, write the given function in the form of $\dfrac{x^n-a^n}{x-a}$ and apply the theorem $\lim_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}$ to get the required value and then putting $x = 2$, the expression $\dfrac{x^5-32}{x^3-8}$ becomes of the form 0/0.

Sol. $\lim_{x\to 2} \dfrac{x^5-32}{x^3-8} = \lim_{x\to 2} \dfrac{x^5-2^5}{x^3-2^3}$ $[\because 2^5 = 32 \text{ and } 2^3 = 8]$

$$= \lim_{x\to 2} \frac{\frac{x^5-2^5}{x-2}}{\frac{x^3-2^3}{x-2}}$$ [dividing numerator and denominator by $(x-2)$]

$$= \lim_{x\to 2} \frac{x^5-2^5}{x-2} \div \lim_{x\to 2} \frac{x^3-2^3}{x-2}$$

$\left[\because \lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)\right]$

$$= 5\times 2^{5-1} \div 3\times 2^{3-1}$$ $\left[\because \lim_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}\right]$

$$= 5\times 2^4 \div 3\times 2^2 = \frac{5\times 2^4}{3\times 2^2} = \frac{5}{3}\times 2^2 = \frac{5}{3}\times 4 = \frac{20}{3}$$

EXAMPLE |17| If $\lim_{x\to 1} \dfrac{x^4-1}{x-1} = \lim_{x\to k} \dfrac{x^3-k^3}{x^2-k^2}$, then find the value of k. **[NCERT Exemplar]**

Sol. Given, $\lim_{x\to 1} \dfrac{x^4-1}{x-1} = \lim_{x\to k} \dfrac{x^3-k^3}{x^2-k^2}$

$$\Rightarrow 4(1)^{4-1} = \lim_{x\to k} \frac{\frac{x^3-k^3}{x-k}}{\frac{x^2-k^2}{x-k}}$$ $\left[\because \lim_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}\right]$

$$\Rightarrow 4 = \frac{\lim_{x\to k} \frac{x^3-k^3}{x-k}}{\lim_{x\to k} \frac{x^2-k^2}{x-k}}$$ $\left[\because \lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\right]$

$$\Rightarrow 4 = \frac{3k^2}{2k} \Rightarrow 4 = \frac{3}{2}k, \therefore k = \frac{4\times 2}{3} = \frac{8}{3}$$

EXAMPLE |18| Evaluate $\lim_{x\to 1} \dfrac{x^7-2x^5+1}{x^3-3x^2+2}$. **[NCERT Exemplar]**

Sol. $\lim_{x\to 1} \dfrac{x^7-2x^5+1}{x^3-3x^2+2}$ $\left[\dfrac{0}{0} \text{ from}\right]$

$$= \lim_{x\to 1} \frac{x^7-x^5-x^5+1}{x^3-x^2-2x^2+2} = \lim_{x\to 1} \frac{x^5(x^2-1)-1(x^5-1)}{x^2(x-1)-2(x^2-1)}$$

[dividing numerator and denominator by $(x-1)$]

$$= \lim_{x\to 1} \frac{\frac{x^5(x^2-1)}{(x-1)} - \frac{1(x^5-1)}{(x-1)}}{\frac{x^2(x-1)}{(x-1)} - \frac{2(x^2-1)}{(x-1)}}$$

$$= \frac{\lim_{x\to 1} x^5(x+1) - \lim_{x\to 1} 2\left(\frac{x^5-1}{x-1}\right)}{\lim_{x\to 1} x^2 - \lim_{x\to 1}{}^2 (x+1)}$$

$$= \frac{1\times 2 - 5\times(1)^4}{1-2\times 2} = \frac{2-5}{1-4} = \frac{-3}{-3} = 1$$

EXAMPLE |19| Find the limit $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$. **[NCERT]**

To evaluate such limit, we replace the function under radical sign by y and change the limit. So firstly, put $x+1 = y$ and change the limit, then apply $\lim_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}$.

Sol. Put $y = 1 + x$, then $y \to 1$ as $x \to 0$.

$$\therefore \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{y \to 1} \frac{\sqrt{y}-1}{y-1} \quad [\because x+1=y \Rightarrow x = y-1]$$

$$= \lim_{y \to 1} \frac{y^{\frac{1}{2}} - 1^{\frac{1}{2}}}{y-1} = \frac{1}{2}(1)^{\frac{1}{2}-1} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$= \frac{1}{2}$$

EXAMPLE |20| Evaluate $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$.

Sol. Let $x + 2 = y$, then $y \to a + 2$ as $x \to a$

$$\therefore \lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x-a}$$

$$= \lim_{y \to (a+2)} \frac{y^{3/2} - (a+2)^{3/2}}{y-(a+2)} \quad [\because x+2 = y \Rightarrow x = y-2]$$

$$= \frac{3}{2}(a+2)^{\frac{3}{2}-1} = \frac{3}{2}(a+2)^{1/2} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

EXAMPLE |21| Evaluate $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$.

Sol. Put $1 + x = y$, then $y \to 1$ as $x \to 0$

$$\therefore \lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1} = \lim_{y \to 1} \frac{y^6 - 1}{y^2 - 1} = \lim_{y \to 1} \left(\frac{\frac{y^6-1}{y-1}}{\frac{y^2-1}{y-1}} \right)$$

[dividing numerator and denominator by $y - 1$]

$$= \lim_{y \to 1} \frac{y^6 - 1}{y-1} \div \lim_{y \to 1} \frac{y^2-1}{y-1}$$

$$= \frac{6(1)^{6-1}}{2(1)^{2-1}} = \frac{6}{2} = 3 \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

EXAMPLE |22| Evaluate $\lim_{x \to 2} \frac{(2x+4)^{1/3} - 2}{x-2}$.

Sol. Put $2x + 4 = y$, then $y \to 8$ as $x \to 2$

$$\therefore \lim_{x \to 2} \frac{(2x+4)^{1/3} - 2}{x-2} = \lim_{y \to 8} \frac{y^{1/3} - 2}{\frac{y-4}{2} - 2} \quad \left[\begin{array}{l}\because 2x+4 = y \\ \Rightarrow x = \frac{y-4}{2}\end{array}\right]$$

$$= 2 \lim_{y \to 8} \frac{y^{1/3} - (8)^{1/3}}{y-4-4} \quad [\because 2 = (8)^{1/3}]$$

$$= 2 \lim_{y \to 8} \frac{y^{1/3} - 8^{1/3}}{y-8}$$

$$= 2 \cdot \frac{1}{3}(8)^{\frac{1}{3}-1} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$= 2 \cdot \frac{1}{3}(2^3)^{-\frac{2}{3}} = \frac{2}{3} \cdot (2)^{-2} = \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$$

EXAMPLE |23| Evaluate

$$\lim_{x \to 1} \frac{x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x - n}{x-1}.$$

Sol. $$\lim_{x \to 1} \frac{x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x - n}{x-1}$$

$$= \lim_{x \to 1} \frac{(x^n - 1) + (x^{n-1} - 1) + \ldots + (x^2 - 1) + (x-1)}{(x-1)}$$

$[\because n = 1+1+1+\ldots \text{to } n \text{ terms}]$

$$= \lim_{x \to 1} \frac{x^n - 1^n}{x-1} + \lim_{x \to 1} \frac{x^{n-1} - 1^{n-1}}{x-1} + \ldots + \lim_{x \to 1} \frac{x^2 - 1^2}{x-1} + \lim_{x \to 1} \frac{x-1}{x-1}$$

$$= n(1)^{n-1} + (n-1)(1)^{n-2} + \ldots + 2(1)^{2-1} + 1$$

$$\left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$= n + (n-1) + \ldots + 2 + 1$$

$$= \frac{n(n+1)}{2} \quad \left[\because \Sigma n = \frac{n(n+1)}{2}\right]$$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 Which of the following is/are true?

I. $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$ (where, $a + b + c \neq 0$) is 1.

II. $\lim_{x \to -3} \frac{\frac{1}{x} + \frac{1}{3}}{x+3}$ is $\frac{1}{9}$.

(a) Both I and II are true

(b) Only I is true

(c) Only II is true

(d) Both I and II are false

2 $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$ is equal to **[NCERT Exemplar]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\sqrt{2}$

3 $\lim_{x \to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2 + x - 3}$ is equal to

(a) $\frac{1}{10}$ (b) $\frac{-1}{10}$

(c) 1 (d) None of these

4 $\lim_{x\to 2} \dfrac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ is equal to

(a) 2 (b) 4

(c) 8 (d) 16

5 If $\lim_{x\to(-a)} \dfrac{x^7+a^7}{x+a}=7$, then the value of a is

(a) 1 (b) –1

(c) ±1 (d) None of these

VERY SHORT ANSWER Type Questions

6 Find the following limits. (Each part carries 1 mark)

(i) $\lim_{x\to 0} \dfrac{ax+b}{cx+1}$ (ii) $\lim_{x\to 2} \dfrac{x^2-4}{x+3}$

(iii) $\lim_{x\to 3} \dfrac{\sqrt{2x+3}}{x+3}$ (iv) $\lim_{x\to 2} \dfrac{x^2-4}{(x-2)}$

7 Evaluate the following limits. (Each part carries 1 mark)

(i) $\lim_{x\to -1} \dfrac{x^{10}+x^5+1}{x-1}$ (ii) $\lim_{x\to 1} \dfrac{ax^2+bx+c}{cx^2+bx+a}$, $a+b+c\neq 0$

8 Evaluate the following limits. (Each part carries 1 mark)

(i) $\lim_{x\to b} \dfrac{x^{5/2}-b^{5/2}}{x-b}$ (ii) $\lim_{x\to 3} \dfrac{x^4-81}{x-3}$ (iii) $\lim_{x\to b} \dfrac{\sqrt{x}-\sqrt{b}}{x-b}$

9 Find n, $\lim_{x\to 2} \dfrac{x^n-2^n}{x-2}=80, n\in N$.

10 Evaluate $\lim_{x\to -2} \dfrac{\frac{1}{x}+\frac{1}{2}}{x+2}$.

SHORT ANSWER Type I Questions

Evaluate the following limits

11 $\lim_{x\to 2} \dfrac{x^2-3x+2}{x^2+x-6}$. **12** $\lim_{x\to 5} \dfrac{x^2-9x+20}{x^2-6x+5}$.

13 $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$. **14** $\lim_{x\to\sqrt{2}} \dfrac{x^4-4}{x^2+3\sqrt{2}x-8}$.

15 $\lim_{x\to 3} \dfrac{x^4-81}{2x^2-5x-3}$. **16** $\lim_{x\to 2} \dfrac{3x^2-x-10}{x^2-4}$.

17 $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$. **18** $\lim_{x\to 0} \dfrac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}}$.

19 $\lim_{x\to -1} \dfrac{x^3+1}{x^5+1}$. **20** $\lim_{x\to 1} \dfrac{x^{15}-1}{x^{10}-1}$

21 $\lim_{x\to 0} \dfrac{(1+x)^n-1}{x}$. **22** $\lim_{x\to 1} \dfrac{x^m-1}{x^n-1}$.

23 $\lim_{x\to 0} \dfrac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$. **24** $\lim_{x\to b} \dfrac{(x+3)^{5/2}-(b+3)^{5/2}}{x-b}$.

25 $\lim_{x\to 1} \dfrac{(\sqrt{3+x}-\sqrt{5-x})}{x^2-1}$. **26** $\lim_{x\to 2}\left(\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right)$.

27 $\lim_{x\to 0} \dfrac{(\sqrt{1+x+x^2}-\sqrt{x+1})}{2x^2}$.

SHORT ANSWER Type II Questions

28 $\lim_{x\to 2} \dfrac{x^3-7x^2+14x-8}{x^2+2x-8}$. **29** $\lim_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$.

30 $\lim_{x\to 0}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\right)$. **31** $\lim_{x\to 1} \dfrac{x^4-\sqrt{x}}{\sqrt{x}-1}$.

32 $\lim_{x\to 1}\left(\dfrac{x^4-3x^2+2}{x^3-5x^2+3x+1}\right)$. **33** $\lim_{x\to 1/2}\left(\dfrac{8x-3}{2x-1}-\dfrac{4x^2+1}{4x^2-1}\right)$.

34 $\lim_{x\to\sqrt{2}} \dfrac{x^9-3x^8+x^6-9x^4-4x^2-16x+84}{x^5-3x^4-4x+12}$

35 If $\lim_{x\to b} \dfrac{x^3-b^3}{x-b}=\lim_{x\to 1} \dfrac{x^4-1}{x-1}$, find all possible values of b.

HINTS & ANSWERS

1. (b) I. Given, $\lim_{x\to 1} \dfrac{ax^2+bx+c}{cx^2+bx+a}$

$$= \frac{a\times(1)^2+b\times 1+c}{c\times(1)^2+b\times 1+a}=1$$

II. $\lim_{x\to -3} \dfrac{\frac{1}{x}+\frac{1}{3}}{x+3}$

$$= \lim_{x\to -3} \frac{(3+x)}{3x(x+3)}$$

$$= \lim_{x\to -3} \frac{1}{3x}=-\frac{1}{9}$$

2. (c) Put $y=2+x$ so that when $x\to 0$, $y\to 2$.

$$\Rightarrow \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$$

$$= \lim_{y\to 2} \frac{y^{\frac{1}{2}}-2^{\frac{1}{2}}}{y-2}=\frac{1}{2\sqrt{2}}$$

3. (b) Given, $\lim_{x\to 1}\frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}=\lim_{x\to 1}\frac{(\sqrt{x}-1)(2x-3)}{(2x+3)(x-1)}$

$$=\frac{-1}{10}$$

4. (c) $\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ $\left[\text{form } \frac{0}{0}\right]$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{(\sqrt{3x-2}-\sqrt{x+2})(\sqrt{3x-2}+\sqrt{x+2})}\quad\left[\text{form } \frac{0}{0}\right]$$

$$=\lim_{x\to 2}\frac{(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2}=8$$

5. (c) $\lim_{x\to(-a)}\frac{x^7-(-a)^7}{x-(-a)}=7$

$\Rightarrow\quad 7(-a)^{7-1}=7\ \Rightarrow a=\pm 1$

6. (i) b (ii) 0 (iii) $\frac{1}{2}$ (iv) 4

7. (i) $-\frac{1}{2}$ (ii) 1

8. (i) $\frac{5}{2}b^{3/2}$

(ii) Given, limit $=\lim_{x\to 3}\frac{x^4-3^4}{x-3}$ **Ans.** 108

(iii) $\frac{1}{2\sqrt{b}}$

9. $n2^{n-1}=5\times 2^{5-1}$ **Ans.** 5

10. $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{(2+x)}{2x(x+2)}$ **Ans.** $-\frac{1}{4}$

11. $\lim_{x\to 2}\frac{x^2-3x+2}{x^2+x-6}=\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+3)}$ **Ans.** $\frac{1}{5}$

12. Given limit $=\lim_{x\to 5}\frac{(x-4)(x-5)}{(x-1)(x-5)}$ **Ans.** $\frac{1}{4}$

13. $\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)}$ **Ans.** 3

14. $\lim_{x\to\sqrt{2}}\frac{x^4-4}{x^2+3\sqrt{2}x-8}=\lim_{x\to\sqrt{2}}\frac{(x+\sqrt{2})(x-\sqrt{2})(x^2+2)}{(x-\sqrt{2})(x+4\sqrt{2})}$

Ans. $\frac{8}{5}$

15. $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$

Ans. $\frac{108}{7}$

16. $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$ **Ans.** $\frac{11}{4}$

17. Put $x+h=y$, then $y\to x$ as $h\to 0$.

Now, $\lim_{h\to 0}\frac{(x+h)^{1/2}-(x)^{1/2}}{h}$

$$=\lim_{y\to x}\frac{y^{1/2}-(x)^{1/2}}{y-x}=\frac{1}{2}x^{\frac{1}{2}-1}$$

Ans. $\frac{1}{2\sqrt{x}}$

18. Given limit $=\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a}}{x\sqrt{a^2+ax}}\times\frac{\sqrt{a+x}+\sqrt{a}}{\sqrt{a+x}+\sqrt{a}}$

$$=\lim_{x\to 0}\frac{x}{x(\sqrt{a^2+ax})(\sqrt{a+x}+\sqrt{a})}$$

Ans. $\frac{1}{2a\sqrt{a}}$

19. $\lim_{x\to -1}\frac{x^3+1}{x^5+1}=\lim_{x\to -1}\frac{\frac{x^3+1}{(x+1)}}{\frac{x^5+1}{x+1}}=\lim_{x\to -1}\frac{x^3+1}{x+1}\div\lim_{x\to -1}\frac{x^5+1}{x+1}$

$$=\lim_{x\to -1}\frac{x^3-(-1)^3}{x-(-1)}\div\lim_{x\to -1}\frac{x^5-(-1)^5}{x-(-1)}$$ **Ans.** $\frac{3}{5}$

20. $\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1}=\lim_{x\to 1}\frac{\frac{x^{15}-1}{x-1}}{\frac{x^{10}-1}{x-1}}=\lim_{x\to 1}\frac{x^{15}-1}{x-1}\div\lim_{x\to 1}\frac{x^{10}-1}{x-1}$

$$=\lim_{x\to 1}\frac{x^{15}-1^{15}}{x-1}\div\lim_{x\to 1}\frac{x^{10}-1^{10}}{x-1}$$ **Ans.** $\frac{3}{2}$

21. Put $1+x=y$, then $y\to 1$ as $x\to 0$.

Now, $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{y\to 1}\frac{y^n-1}{y-1}=\lim_{y\to 1}\frac{y^n-1^n}{y-1}$

Ans. n

22. Given limit $=\lim_{x\to 1}\frac{\frac{x^m-1}{x-1}}{\frac{x^n-1}{x-1}}=\frac{\lim_{x\to 1}\frac{x^m-1^m}{x-1}}{\lim_{x\to 1}\frac{x^n-1^n}{x-1}}$ **Ans.** $\frac{m}{n}$

23. Given limit $=\lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}\cdot\frac{\sqrt{1+x^3}+\sqrt{1-x^3}}{\sqrt{1+x^3}+\sqrt{1-x^3}}$

$$=\lim_{x\to 0}\frac{(1+x^3)-(1-x^3)}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$$

$$=\lim_{x\to 0}\frac{2x^3}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$$ **Ans.** 0

24. Given, limit $=\lim_{y\to(b+3)}\frac{y^{5/2}-(b+3)^{5/2}}{y-(b+3)}$,

where $y=x+3$

Ans. $\frac{5}{2}(b+3)^{3/2}$

25. Given limit $= \lim_{x\to1} \frac{(\sqrt{3+x}-\sqrt{5-x})}{(x-1)(x+1)} \times \frac{(\sqrt{3+x}+\sqrt{5-x})}{(\sqrt{3+x}+\sqrt{5-x})}$

$$= \lim_{x\to1} \frac{2x-2}{(x-1)(x+1)(\sqrt{3+x}+\sqrt{5-x})}$$

Ans. $\frac{1}{4}$

26. Given limit

$$\lim_{x\to2}\left(\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right)\times\frac{(\sqrt{x+2}+\sqrt{3x-2})}{(\sqrt{x+2}+\sqrt{3x-2})}$$

$$= \lim_{x\to2} \frac{(x-2)(x+2)(\sqrt{x+2}+\sqrt{3x-2})}{-2x+4}$$

Ans. -8

27. Multiplying numerator and denominator by $\sqrt{1+x+x^2}+\sqrt{x+1}$ **Ans.** $\frac{1}{4}$

28. Given limit $= \lim_{x\to2} \frac{(x-1)(x-2)(x-4)}{(x-2)(x+4)}$

Ans. $-\frac{1}{3}$

29. Given limit $= \lim_{x\to2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x(x-1)(x-2)}\right]$

$$= \lim_{x\to2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right]$$ **Ans.** $\frac{-1}{2}$

30. Given limit $= \lim_{x\to0}\left\{\frac{\sqrt{1+x^2}-\sqrt{1+x}}{(\sqrt{1+x^3}-\sqrt{1+x})}\times\right.$

$$\frac{(\sqrt{1+x^3}+\sqrt{1+x})}{(\sqrt{1+x^3}+\sqrt{1+x})}\times\frac{(\sqrt{1+x^2}+\sqrt{1+x})}{(\sqrt{1+x^2}+\sqrt{1+x})}$$

$$= \lim_{x\to0}\frac{\{(1+x^2)-(1+x)\}\times(\sqrt{1+x^3}+\sqrt{1+x})}{\{(1+x^3)-(1+x)\}\times(\sqrt{1+x^2}+\sqrt{1+x})}$$

$$= \lim_{x\to0}\frac{x(x-1)(\sqrt{1+x^3}+\sqrt{1+x})}{x(x-1)(x+1)(\sqrt{1+x^2}+\sqrt{1+x})}$$

Ans. 1

31. $\lim_{x\to1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to1}\frac{\sqrt{x}[(x)^{7/2}-1]}{\sqrt{x}-1}$

$$= \lim_{x\to1}\frac{(x)^{7/2}-1}{\sqrt{x}-1}\cdot\lim_{x\to1}\sqrt{x} = \lim_{x\to1}\frac{\frac{x^{7/2}-1}{x-1}}{\frac{(x)^{1/2}-1}{x-1}}\cdot1$$

$$= \frac{\lim_{x\to1}\frac{x^{7/2}-1}{x-1}}{\lim_{x\to1}\frac{(x)^{1/2}-1}{x-1}}$$

Ans. 7

32. Given limit $= \lim_{x\to1}\frac{(x-1)(x+1)(x^2-2)}{(x-1)(x^2-4x-1)}$ **Ans.** $\frac{1}{2}$

33. $\lim_{x\to1/2}\left(\frac{8x-3}{2x-1}-\frac{4x^2+1}{4x^2-1}\right)$

$$= \lim_{x\to1/2}\left[\frac{(8x-3)(2x+1)-(4x^2+1)}{(4x^2-1)}\right]$$

$$= \lim_{x\to1/2}\left[\frac{12x^2+2x-4}{4x^2-1}\right]$$

$$= \lim_{x\to1/2}\frac{2(6x^2+x-2)}{4x^2-1}$$

$$= \lim_{x\to1/2}\frac{2[(3x+2)(2x-1)]}{(2x)^2-(1)^2}$$

$$= \lim_{x\to1/2}\frac{2(3x+2)(2x-1)}{(2x-1)(2x+1)}$$ **Ans.** $\frac{7}{2}$

34. When $x=\sqrt{2}$, the given limit assume the form $\frac{0}{0}$.

Therefore, $(x-\sqrt{2})$ is a factor of both numerator and denominator. But, irrational roots occur in pairs. So, $(x+\sqrt{2})$ will also be a factor of both numerator and denominator, consequently, (x^2-2) will be a common factor.

$$\therefore \lim_{x\to\sqrt{2}}\frac{x^9-3x^8+x^6-9x^4-4x^2-16x+84}{x^5-3x^4-4x+12}$$

$$= \lim_{x\to\sqrt{2}}\frac{x^7-3x^6+2x^5-5x^4+4x^3-19x^2+8x+42}{x^3-3x^2+2x-6}$$

Ans. $\frac{8\sqrt{2}-31}{\sqrt{2}-3}$

35. $3b^{3-1}=4(1)^{4-1}$ **Ans.** $b=\pm\frac{2}{\sqrt{3}}$

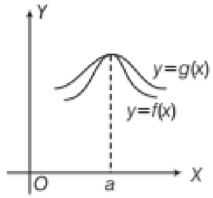
|TOPIC 3|
Limits of Trigonometric Functions

To find the **limits of trigonometric functions**, *we use the following theorems*

Theorem 1 Let f and g be two real valued functions with the same domain such that $f(x) \le g(x)$ for all x in the domain of definition. For some real number a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists, then

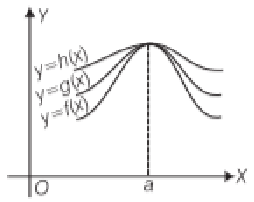
$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x).$$

This is shown in the adjoining figure



Theorem 2 (**Sandwich theorem**) Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if **$\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$.**

This is shown in the adjoining figure



Theorem 3 Three important limits are

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (ii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

A GENERAL RULE

Let given limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and we want to evaluate this. First, we check the values of $f(a)$ and $g(a)$. If both are 0, then we try to vanish those terms which make it zero. For this, we write $f(x) = f_1(x) f_2(x)$, such that $f_1(a) = 0$ and $f_2(a) \ne 0$ and $g(x) = g_1(x) g_2(x)$, such that $g_1(a) = 0$ and $g_2(a) \ne 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and get

$$\frac{f(x)}{g(x)} = \frac{p(x)}{q(x)}$$

Then, $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{p(a)}{q(a)}, \text{ where } q(a) \ne 0.$$

METHOD TO DETERMINE THE LIMIT OF TRIGONOMETRIC FUNCTIONS

Step I First, check that given variable tends to zero or not. If yes, then go to Step II, otherwise put $x = a + h$ in the given function such that as $x \to a$, then $h \to 0$.

Step II Put the limit in given function, if $\frac{0}{0}$ form is obtained, then we go to next step. Otherwise, we get the required answer.

Step III Simplify the numerator and denominator to eliminate those factors which becomes 0 (zero) on putting the limit.

Step IV Now, convert the result obtained in step III, in the form of $\frac{\sin \theta}{\theta}$ or $\frac{\tan \theta}{\theta}$.

Step V Substitute the value of standard limit of trigonometric function as obtained in step IV and simplify it.

EXAMPLE |1| Evaluate $\lim_{\theta \to 0} \theta \operatorname{cosec} \theta$.

Sol. $\lim_{\theta \to 0} \theta \operatorname{cosec} \theta = \lim_{\theta \to 0} \frac{\theta}{\sin \theta}$

$$= \lim_{\theta \to 0} \frac{1}{\frac{\sin \theta}{\theta}} = \frac{1}{1} = 1 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin x}{x} = 1\right]$$

EXAMPLE |2| Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$.

Sol. $\lim_{x \to 0} \frac{\sin 3x}{5x} = \lim_{x \to 0} \frac{\sin 3x}{5x \times \frac{3}{3}}$

$$= \lim_{x \to 0} \frac{3}{5} \cdot \frac{\sin 3x}{3x}$$

$$= \frac{3}{5} \cdot \lim_{3x \to 0} \frac{\sin 3x}{3x} \quad [\text{as } x \to 0, \text{ therefore } 3x \to 0]$$

$$= \frac{3}{5} \times 1 = \frac{3}{5} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

EXAMPLE |3| Evaluate $\lim_{\theta\to 0}\frac{\sin a\theta}{\sin b\theta}$.

Sol. We have, $\lim_{\theta\to 0}\frac{\sin a\theta}{\sin b\theta} = \lim_{\theta\to 0}\frac{\sin a\theta}{\theta} \div \lim_{\theta\to 0}\frac{\sin b\theta}{\theta}$

$$= a\lim_{a\theta\to 0}\frac{\sin a\theta}{a\theta} \div b\lim_{b\theta\to 0}\frac{\sin b\theta}{b\theta}$$

[as $\theta\to 0$, then $a\theta\to 0$ and as $\theta\to 0$, then $b\theta\to 0$]

$$= a(1)\div b(1) = \frac{a}{b}$$

EXAMPLE |4| Evaluate $\lim_{\theta\to b}\frac{\tan(\theta-b)}{\theta-b}$.

Sol. We have $\lim_{\theta\to b}\frac{\tan(\theta-b)}{\theta-b}$

Put $\theta-\beta = h \Rightarrow \theta = h+b$

Also, when $\theta\to b$, then $h\to b$

$\therefore \lim_{h\to b}\frac{\tan(\theta-b)}{\theta-b} = \lim_{h\to 0}\frac{\tan h}{h} = 1$

EXAMPLE |5| Evaluate $\lim_{x\to 0}\frac{\tan x^\circ}{x^\circ}$.

First, change the angle in radian and then use $\lim_{x\to 0}\frac{\tan x}{x} = 1$.

Sol. $\lim_{x\to 0}\frac{\tan x^\circ}{x^\circ} = \lim_{x\to 0}\frac{\tan\frac{\pi x}{180}}{\frac{\pi x}{180}} = 1$ $\left[\because 1^\circ = \frac{\pi}{180}\text{ rad}\right]$

EXAMPLE |6| Evaluate $\lim_{x\to 0}\frac{\sin 7x}{\tan 5x}$.

Sol. $\lim_{x\to 0}\frac{\sin 7x}{\tan 5x} = \lim_{x\to 0}\frac{7x\left(\frac{\sin 7x}{7x}\right)}{5x\left(\frac{\tan 5x}{5x}\right)} = \frac{7}{5}\frac{\lim_{x\to 0}\frac{\sin 7x}{7x}}{\lim_{x\to 0}\frac{\tan 5x}{5x}}$

$$= \frac{7}{5}\times\frac{1}{1} = \frac{7}{5} \quad \left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1\right]$$

EXAMPLE |7| Evaluate $\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$.

Sol. $\lim_{x\to 0}\left(\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$

$$= \lim_{x\to 0}\left(\frac{2\sin 4x\cos 2x}{2\cos 4x\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin 4x\cos 2x}{\cos 4x\sin x}\right)$$

$$= \lim_{x\to 0}\left\{\frac{\sin 4x}{4x}\times\frac{x}{\sin x}\times\cos 2x\times\frac{1}{\cos 4x}\times 4\right\}$$

$$= 4\times\lim_{4x\to 0}\frac{\sin 4x}{4x}\times\lim_{x\to 0}\left(\frac{x}{\sin x}\right)$$

$$\times\lim_{2x\to 0}\cos 2x\times\frac{1}{\lim_{4x\to 0}\cos 4x} = \left(4\times 1\times 1\times 1\times\frac{1}{1}\right) = 4$$

EXAMPLE |8| Evaluate $\lim_{x\to 0}\frac{1-\cos 4x}{x}$.

Sol. $\lim_{x\to 0}\frac{1-\cos 4x}{x} = \lim_{x\to 0}\frac{2\sin^2 2x}{x}\times\frac{x}{x}$ $[\because 1-\cos 2\theta = 2\sin^2\theta]$

$$= \lim_{x\to 0} 2\left(\frac{\sin 2x}{2x}\right)^2\times 4x$$

$$= 2\times 1\times 0 = 0 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

EXAMPLE |9| Evaluate

$$\lim_{x\to 0}\frac{\sin ax + bx}{ax+\sin bx};\ a, b, a+b\neq 0.$$

[NCERT]

Sol. $\lim_{x\to 0}\frac{\sin ax + bx}{ax+\sin bx} = \lim_{x\to 0}\frac{\frac{\sin ax}{x}+\frac{bx}{x}}{\frac{ax}{x}+\frac{\sin bx}{x}}$

[dividing both numerator and denominator by x]

$$= \frac{\lim_{x\to 0}\frac{\sin(ax)}{ax}\times a + \lim_{x\to 0} b}{\lim_{x\to 0} a + \lim_{x\to 0}\frac{\sin bx}{bx}\times b}$$

$$\left[\because \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ and } \lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\right]$$

$$= \frac{a\lim_{x\to 0}\frac{\sin ax}{ax} + \lim_{x\to 0} b}{\lim_{x\to 0} a + b\lim_{x\to 0}\frac{\sin bx}{bx}}$$

$$= \frac{a(1)+b}{a+b(1)} \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= \frac{a+b}{a+b} = 1$$

EXAMPLE |10| Evaluate $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$.

Sol. $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3} = \lim_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x}-\sin 2x}{x^3}$

$$= \lim_{x\to 0}\frac{\sin 2x-\sin 2x\cdot\cos 2x}{x^3\cdot\cos 2x} = \lim_{x\to 0}\frac{\sin 2x(1-\cos 2x)}{x^3\cdot\cos 2x}$$

$$= \lim_{x\to 0}\frac{\tan 2x}{x}\cdot\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

[by using product of limits and $\cos 2\theta = 1-2\sin^2\theta$]

$$= 2\cdot\lim_{x\to 0}\frac{\tan 2x}{2x}\times 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$= 2(1)\times 2(1)^2 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = 1\right]$$

$$= 4$$

EXAMPLE |11| Evaluate $\lim_{x\to 0} \dfrac{\cos ax - \cos bx}{\cos cx - 1}$. **[NCERT Exemplar]**

Sol. $\lim_{x\to 0} \dfrac{\cos ax - \cos bx}{\cos cx - 1}$

$$= \lim_{x\to 0} \frac{-2\sin\left[\left(\frac{a+b}{2}\right)x\right]\sin\left(\frac{a-b}{2}\right)x}{-2\sin^2\frac{cx}{2}}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \text{ and } \cos 2\theta = 1 - 2\sin^2\theta\right]$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{a+b}{2}\right)x \cdot \sin\left(\frac{a-b}{2}\right)x}{x^2} \cdot \frac{x^2}{\sin^2\frac{cx}{2}}$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x \cdot \left(\frac{2}{a+b}\right)} \cdot \frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x \cdot \left(\frac{2}{a-b}\right)} \cdot \frac{\left(\frac{cx}{2}\right)^2 \times \frac{4}{c^2}}{\sin^2\frac{cx}{2}}$$

$$= \left(\frac{a+b}{2} \times \frac{a-b}{2} \times \frac{4}{c^2}\right) = \frac{a^2 - b^2}{c^2} \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

EXAMPLE |12| Evaluate $\lim_{x\to \pi/4} \left(\dfrac{\sin x - \cos x}{x - \pi/4}\right)$.

Sol. Let $L = \lim_{x\to \pi} \left(\dfrac{\sin x - \cos x}{x - \pi/4}\right)$

Here, given variable is not tending to zero i.e. limit is not of the form $x \to 0$.

So, put $x - \dfrac{\pi}{4} = h$

$\Rightarrow \quad x = \dfrac{\pi}{4} + h$ Now, as $x \to \dfrac{\pi}{4}$, then $h \to 0$.

$$\therefore \quad L = \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{4} + h\right) - \cos\left(\frac{\pi}{4} + h\right)}{h} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h\to 0} \frac{\left[\left(\sin\frac{\pi}{4}\cos h + \cos\frac{\pi}{4}\sin h\right) - \left(\cos\frac{\pi}{4}\cos h - \sin h \sin\frac{\pi}{4}\right)\right]}{h}$$

$$\lim_{h\to 0} \frac{\left[\left(\frac{1}{\sqrt{2}}\cos h + \frac{1}{\sqrt{2}}\sin h\right) - \left(\frac{1}{\sqrt{2}}\cos h - \frac{1}{\sqrt{2}}\sin h\right)\right]}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2}{\sqrt{2}}\sin h}{h} = \sqrt{2}(1) = \sqrt{2} \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

EXAMPLE |13| Evaluate $\lim_{x\to \pi/2} \dfrac{1 + \cos 2x}{(\pi - 2x)^2}$.

Sol. $$\lim_{x\to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h\to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h\right)}{\left[\pi - 2\left(\frac{\pi}{2} + h\right)\right]^2}$$

$$\left[\text{putting } x = \frac{\pi}{2} + h, \text{as } x \to \frac{\pi}{2}, \text{ then } h \to 0\right]$$

$$= \lim_{h\to 0} \frac{1 + \cos(\pi + 2h)}{(\pi - \pi - 2h)^2}$$

$$= \lim_{h\to 0} \frac{1 - \cos 2h}{4h^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h\to 0} \frac{1 - (1 - 2\sin^2 h)}{4h^2}$$

$$= \lim_{h\to 0} \frac{2\sin^2 h}{4h^2} \qquad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$= \frac{2}{4} \lim_{h\to 0} \left(\frac{\sin h}{h}\right)^2 = \frac{2}{4} \times 1 = \frac{1}{2} \qquad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

EXAMPLE |14| Evaluate

$\lim_{y\to 0} \dfrac{(x + y)\sec(x + y) - x\sec x}{y}$. **[NCERT Exemplar]**

Sol. $\lim_{y\to 0} \dfrac{(x + y)\sec(x + y) - x\sec x}{y}$

$$\lim_{y\to 0} x \frac{x(\sec(x + y) - \sec x) + y\sec(x + y)}{y}$$

$$\Rightarrow \lim_{y\to 0} x\left\{\frac{\sec(x + y) - \sec x}{y}\right\} + \lim_{y\to 0} \frac{y\sec(x + y)}{y}$$

$$= \lim_{y\to 0} \frac{\frac{x}{\cos(x + y)} - \frac{x}{\cos x}}{y} + \lim_{y\to 0} \sec(x + y)$$

$$= \lim_{y\to 0} x\left\{\frac{\cos x - \cos(x + y)}{y\cos x\cos(x + y)}\right\} + \lim_{y\to 0} \sec(x + y)$$

$$= \lim_{y\to 0} \left\{\frac{\cos x - \cos(x + y)}{y} \times \frac{x}{\cos x\cos(x + y)}\right\} + \lim_{y\to 0} \sec(x + y)$$

$$= \lim_{y\to 0}\left\{\frac{2\sin\left(x+\frac{y}{2}\right)\sin\left(\frac{y}{2}\right)}{2\left(\frac{y}{2}\right)}\times\frac{x}{\cos x\cos(x+y)}\right\} + \lim_{y\to 0}\sec(x+y)$$

$$= \lim_{y\to 0}\sin\left(x+\frac{y}{2}\right)\times\lim_{y\to 0}\frac{\sin\left(\frac{y}{2}\right)}{\frac{y}{2}}\times\lim_{y\to 0}\frac{x}{\cos x\cos(x+y)} + \lim_{y\to 0}\sec(x+y)$$

$$= \sin x\times 1\times\frac{x}{\cos^2 x}+\sec x = x\tan x\sec x+\sec x$$

EXAMPLE |15| Evaluate $\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$.

[NCERT Exemplar]

First, use the formula, $\sin C-\sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$ and then apply the limit to get the required value.

Sol. $\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$

$$= \lim_{x\to 0}\frac{2\cos\left(\frac{2+x+2-x}{2}\right)\sin\left(\frac{2+x-2+x}{2}\right)}{x}$$

$$\left[\because \sin C-\sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{x\to 0}\frac{2\cos 2\,\sin x}{x}$$

$$= 2\cos 2\lim_{x\to 0}\frac{\sin x}{x} = 2\cos 2 \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

EXAMPLE |16| Evaluate $\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$.

Sol. $\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

$$= \lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\times\frac{1+\cos x\sqrt{\cos 2x}}{1+\cos x\sqrt{\cos 2x}}$$

[rationalising the numerator]

$$= \lim_{x\to 0}\frac{1-\cos^2 x\cos 2x}{x^2(1+\cos x\sqrt{\cos 2x})} = \lim_{x\to 0}\frac{1-\cos^2 x(2\cos^2 x-1)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{1-2\cos^4 x+\cos^2 x}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{(1-\cos^2 x)(1+2\cos^2 x)}{x^2(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\times\lim_{x\to 0}\frac{1+2\cos^2 x}{\{1+\cos x\sqrt{\cos 2x}\}} = 1\times\left(\frac{1+2}{1+1}\right)=\frac{3}{2}$$

EXAMPLE |17| Evaluate $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 5\theta}$.

Sol. $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 5\theta} = \lim_{\theta\to 0}\frac{2\sin^2(2\theta)}{2\sin^2\left(\frac{5\theta}{2}\right)}$ $[\because 1-\cos 2\theta = 2\sin^2\theta]$

$$= \lim_{\theta\to 0}\frac{\frac{\sin^2(2\theta)}{4\theta^2}\times 4\theta^2}{\frac{\sin^2\frac{5\theta}{2}}{\frac{25\theta^2}{4}}\times\frac{25\theta^2}{4}} = \frac{16}{25}\times\frac{\lim_{2\theta\to 0}\left(\frac{\sin 2\theta}{2\theta}\right)^2}{\lim_{\frac{5\theta}{2}\to 0}\left(\frac{\sin 5\theta/2}{5\theta/2}\right)^2}$$

[as $\theta\to 0$, then $2\theta\to 0$ and $5\theta/2\to 0$]

$$= \frac{16}{25}\times\frac{(1)^2}{(1)^2}=\frac{16}{25} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

EXAMPLE |18| Evaluate $\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x}$.

Sol. On putting $x=\pi+h$, we get

$$\lim_{x\to\pi}\frac{1+\cos x}{\tan^2 x} = \lim_{h\to 0}\frac{1+\cos(\pi+h)}{\tan^2(\pi+h)}$$

[$\because$ when $x\to\pi$, then $h\to 0$]

$$= \lim_{h\to 0}\frac{2\cos^2\left(\frac{\pi+h}{2}\right)}{\tan^2 h} \qquad \left[\begin{array}{l}\because \tan(\pi+h)=\tan h\\ \text{and } 1+\cos 2\theta = 2\cos^2\theta\end{array}\right]$$

$$= \lim_{h\to 0}\frac{2\cos^2\left(\frac{\pi}{2}+\frac{h}{2}\right)}{\tan^2 h} = \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\tan^2 h}$$

$$\left[\because \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta \;\therefore \cos^2\left(\frac{\pi}{2}+\theta\right)=\sin^2\theta\right]$$

$$= \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\sin^2 h}\times\cos^2 h$$

$$= \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\left(2\sin\frac{h}{2}\cdot\cos\frac{h}{2}\right)^2}\cos^2 h \qquad \left[\because \sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{2\cos^2\frac{h}{2}}\times\cos^2 h = \frac{1}{2\times 1}\times(1)^2=\frac{1}{2} \qquad [\because \cos 0 = 1]$$

EXAMPLE |19| Evaluate $\lim_{x\to\pi/2}(\sec x-\tan x)$.

[NCERT Exemplar]

Sol. Put $y=\frac{\pi}{2}-x$, as $x\to\frac{\pi}{2}$, then $y\to 0$

$$\therefore \lim_{x\to\pi/2}(\sec x-\tan x)$$

$$= \lim_{y\to 0}\left[\sec\left(\frac{\pi}{2}-y\right)-\tan\left(\frac{\pi}{2}-y\right)\right]$$

$$= \lim_{y\to 0}(\operatorname{cosec} y - \cot y) \quad \left[\because \left(\sec\frac{\pi}{2}-\theta\right) = \operatorname{cosec}\theta \text{ and } \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta\right]$$

$$= \lim_{y\to 0}\left(\frac{1}{\sin y} - \frac{\cos y}{\sin y}\right) = \lim_{y\to 0}\left(\frac{1-\cos y}{\sin y}\right)$$

$$= \lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{2\sin\frac{y}{2}\cos\frac{y}{2}} \quad \left[\because \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]$$

$$= \lim_{\frac{y}{2}\to 0}\tan\frac{y}{2} = 0 \quad \left[\text{as } y\to 0, \text{ then } \frac{y}{2}\to 0\right]$$

EXAMPLE |20| Evaluate

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h) - a^2\sin a}{h}.$$

[NCERT Exemplar]

Sol. $$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h) - a^2\sin a}{h}$$

$$= \lim_{h\to 0}\frac{(a^2+h^2+2ah)[\sin a\cos h + \cos a\sin h] - a^2\sin a}{h}$$

$$[\because \sin(C+D) = \sin C\cos D + \cos C\sin D]$$

$$= \lim_{h\to 0}\left[\frac{a^2\sin a(\cos h - 1)}{h} + \frac{a^2\cos a\sin h}{h} + (h+2a)(\sin a\cos h + \cos a\sin h)\right]$$

$$= \lim_{h\to 0}\left[\frac{a^2\sin a\left(-2\sin^2\frac{h}{2}\right)}{\frac{h^2}{2}}\cdot\frac{h}{2}\right] + \lim_{h\to 0}\frac{a^2\cos a\sin h}{h} + \lim_{h\to 0}(h+2a)\sin(a+h)$$

$$\left[\because \cos h - 1 = -2\sin^2 h/2 \text{ and } \sin a\cos h + \cos a\sin h = \sin(a+h)\right]$$

$$= a^2\sin a\times 0 + a^2\cos a(1) + 2a\sin a \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= a^2\cos a + 2a\sin a$$

EXAMPLE |21| Evaluate $\lim_{x\to\pi/6}\dfrac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}}$.

[NCERT Exemplar]

Sol. $$\lim_{x\to\pi/6}\frac{\sqrt{3}\sin x - \cos x}{x-\frac{\pi}{6}}$$

$$= \lim_{x\to\pi/6}\frac{2\left(\sin x\cos\frac{\pi}{6} - \cos x\sin\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)}$$

$$= 2\lim_{x\to\pi/6}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)}$$

$$[\because \sin A\cos B - \cos A\sin B = \sin(A-B)]$$

$$= 2 \quad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \text{ and } x\to\frac{\pi}{6} \Rightarrow \left(x-\frac{\pi}{6}\right)\to 0\right]$$

EXAMPLE |22| Evaluate $\lim_{x\to 0}\dfrac{x\tan 4x}{1-\cos 4x}$.

Sol. $$\lim_{x\to 0}\frac{x\tan 4x}{1-\cos 4x} = \lim_{x\to 0}\frac{x\cdot\frac{\sin 4x}{\cos 4x}}{1-\cos 2\cdot(2x)}$$

$$= \lim_{x\to 0}\frac{x\cdot\frac{\sin 4x}{\cos 4x}}{1-(1-2\sin^2 2x)} \quad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$= \lim_{x\to 0}\frac{x\cdot\sin 4x}{\cos 4x\cdot 2\sin^2 2x}$$

$$= \lim_{x\to 0}\frac{x\cdot\sin 2(2x)}{\cos 4x\,(2\cdot\sin 2x\cdot\sin 2x)}$$

$$= \lim_{x\to 0}\frac{x\,(2\sin 2x\cdot\cos 2x)}{(2\sin 2x\cdot\sin 2x)\cos 4x} \quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \lim_{x\to 0}\frac{x\cdot\cos 2x}{(\sin 2x)\cdot\cos 4x} = \lim_{x\to 0}\frac{\cos 2x}{\cos 4x}\cdot\left(\frac{x}{\sin 2x}\right)$$

$$= \lim_{x\to 0}\frac{1}{\left(\frac{\sin 2x}{x}\times\frac{2}{2}\right)}\times\lim_{x\to 0}\frac{\cos 2x}{\cos 4x}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{1}{\left(\frac{\sin 2x}{2x}\right)}\times\lim_{x\to 0}\frac{\cos 2x}{\cos 4x}$$

$$= \frac{1}{2}\times\frac{1}{1}\times\frac{\cos 0}{\cos 0} = \frac{1}{2}\times\frac{1}{1}\times\frac{1}{1} = \frac{1}{2} \quad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

EXAMPLE |23| Evaluate $\lim_{x\to 0}\dfrac{\sin x - 2\sin 3x + \sin 5x}{x}$.

[NCERT Exemplar]

Sol. $$\lim_{x\to 0}\frac{\sin x - 2\sin 3x + \sin 5x}{x}$$

$$= \lim_{x\to 0}\frac{(\sin 5x + \sin x) - 2\sin 3x}{x}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right) - 2\sin 3x}{x}$$

$$\left[\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{x\to 0} \frac{2\sin 3x \cos 2x - 2\sin 3x}{x}$$

$$= \lim_{x\to 0} \frac{2\sin 3x(\cos 2x - 1)}{x}$$

$$= \lim_{x\to 0} 2\left(\frac{\sin 3x}{3x}\right) \times 3\left[\frac{-(1-\cos 2x)}{1}\right]$$

$$= -2 \times 1 \times 3(1-\cos 0) \qquad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= -6(1-1) = -6 \times 0 = 0$$

EXAMPLE |24| Evaluate $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

Sol. $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0}\left(\frac{\sin x - \sin x \cos x}{x^3 \cos x}\right)$

$$= \lim_{x\to 0}\left\{\frac{\sin x(1-\cos x)}{x^3 \cos x}\right\} = \lim_{x\to 0}\left\{\frac{\sin x}{x} \times \frac{1-\cos x}{x^2} \times \frac{1}{\cos x}\right\}$$

$$= \left\{\lim_{x\to 0} \frac{\sin x}{x}\right\} \times \left\{\lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \times 4}\right\} \times \lim_{x\to 0} \frac{1}{\cos x}$$

$$= \left\{\lim_{x\to 0} \frac{\sin x}{x}\right\} \times \frac{1}{2} \times \left\{\lim_{x\to 0}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2\right\} \times \lim_{x\to 0} \frac{1}{\cos x}$$

$$= 1 \times \frac{1}{2}(1)^2 \times \frac{1}{1} = \frac{1}{2}$$

Evaluation of Trigonometric Limits by Factorisation

Sometimes, trigonometric limits can be evaluated by factorisation method.

EXAMPLE |25| Evaluate $\lim_{x\to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$. **[NCERT Exemplar]**

Sol. $\lim_{x\to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$

$$= \lim_{x\to \pi/6} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2} \qquad [\because \operatorname{cosec}^2 x - \cot^2 x = 1]$$

$$= \lim_{x\to \pi/6} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2}$$

$$= \lim_{x\to \pi/6} \frac{(\operatorname{cosec} x - 2)(\operatorname{cosec} x + 2)}{(\operatorname{cosec} x - 2)} \qquad \text{[by factorisation]}$$

$$= \lim_{x\to \pi/6} (\operatorname{cosec} x + 2)$$

$$= \operatorname{cosec}\frac{\pi}{6} + 2 = 2 + 2 = 4$$

EXAMPLE |26| Evaluate $\lim_{x\to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$ by using factorisation method. **[NCERT Exemplar]**

Sol. $\lim_{x\to \pi/6} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$

$$= \lim_{x\to \pi/6} \frac{(2\sin x - 1)(\sin x + 1)}{(2\sin x - 1)(\sin x - 1)}$$

$$= \lim_{x\to \pi/6} \frac{\sin x + 1}{\sin x - 1}$$

$$= \frac{1 + \sin\frac{\pi}{6}}{\sin\frac{\pi}{6} - 1} = \frac{1 + \frac{1}{2}}{\frac{1}{2} - 1} = -3$$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 $\lim_{x\to 0} x \sin\frac{1}{x}$ is equal to **[NCERT Exemplar]**

(a) 0 (b) 1
(c) $\frac{1}{2}$ (d) Does not exist

2 $\lim_{\theta\to 0} \frac{1-\cos 4\theta}{1-\cos 6\theta}$ is equal to

(a) $\frac{4}{9}$ (b) $\frac{1}{2}$
(c) $\frac{-1}{2}$ (d) -1

3 If α and β are the roots of $ax^2 + bx + c$, then the value of $\lim_{x\to \beta} \frac{1-\cos(ax^2+bx+c)}{(x-\beta)^2}$ is

(a) $\frac{b^2(\beta-\alpha)^2}{2}$ (b) $\frac{a^2(\beta-\alpha)^2}{2}$
(c) $\frac{b^2(\beta-\alpha)^2}{4}$ (d) $\frac{a^2(\beta-\alpha)^2}{4}$

4 $\lim_{x\to 0} \frac{\tan x + 4\tan 2x - 3\tan 3x}{x}$ is equal to

(a) 1 (b) 8
(c) 9 (d) 0

5 $\lim_{x\to 0} \frac{8}{x^8}\left(1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$
(c) $\frac{1}{16}$ (d) $\frac{1}{32}$

Evaluate the following limits.

VERY SHORT ANSWER Type Questions

6 $\lim_{x\to 0}\frac{\sin 5x}{x}$

7 $\lim_{x\to 0}\frac{\tan 3x}{x}$

8 $\lim_{x\to 0} x\,\text{cosec}\,3x$

9 $\lim_{x\to 0} x\cot 4x$

10 $\lim_{x\to 0} x\sin\frac{1}{x}$

11 $\lim_{x\to 0} x\cos\frac{1}{x}$

12 $\lim_{x\to \pi}\frac{\sin x}{x-\pi}$

13 $\lim_{x\to 0}\frac{\cos x}{\pi - x}$

14 $\lim_{x\to \pi}\frac{\sin x^\circ}{x}$

15 $\lim_{x\to 0}\frac{\tan 2x^\circ}{x^\circ}$

SHORT ANSWER Type Questions

16 $\lim_{x\to \pi}\frac{\sin(\pi - x)}{\pi(\pi - x)}$.

17 $\lim_{x\to 0}\frac{\sin 3x}{\sin 7x}$

18 $\lim_{x\to 0}\frac{\sin^2 5x}{x^2}$

19 $\lim_{x\to 0}\frac{ax + x\cos x}{b\sin x}$

20 $\lim_{x\to 0}\frac{\sin 4x}{\tan 4x}$

21 $\lim_{x\to \pi/2}\frac{\tan 2x}{x-\frac{\pi}{2}}$

22 $\lim_{x\to \pi/2}\left(\frac{\pi}{2}-x\right)\tan x$

23 $\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x}$

24 $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

25 $\lim_{x\to 0}\frac{1-\cos 5x}{x}$

26 $\lim_{x\to \pi/2}\frac{\cos^2 x}{1-\sin x}$

27 $\lim_{x\to 0}(\text{cosec}\,x - \cot x)$

28 $\lim_{x\to 0}\frac{\sin x}{x(1+\cos x)}$

29 $\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$

30 $\lim_{x\to 0}\frac{\tan x - \sin x}{x}$

31 $\lim_{x\to a}\frac{\cos x - \cos a}{x - a}$

32 $\lim_{x\to 0}\frac{\tan 2x - x}{3x - \sin x}$

33 $\lim_{x\to 0}\frac{x^2\cos x}{1-\cos x}$

34 $\lim_{x\to 0}\frac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$

35 $\lim_{x\to \pi/4}\frac{\sec^2 x - 2}{\tan x - 1}$

36 $\lim_{x\to \infty}\frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$

LONG ANSWER Type I Questions

37 $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$

38 $\lim_{x\to a}\frac{\sin x - \sin a}{\sqrt{x}-\sqrt{a}}$

39 $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

40 $\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1}$

41 $\lim_{x\to 0}\frac{2\sin x - \sin 2x}{x^3}$

42 $\lim_{x\to \pi/2}\frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$

43 $\lim_{x\to 0}\frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

44 $\lim_{x\to 0}\frac{\text{cosec}\,x - \cot x}{x}$

45 $\lim_{x\to 0}\frac{\sin 2x + 3x}{2x + \tan 3x}$

46 $\lim_{x\to 0}\frac{(\sin 3x + \sin 5x)}{(\sin 6x - \sin 4x)}$

47 $\lim_{x\to \pi/8}\frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

48 $\lim_{x\to a}\frac{\sin\sqrt{x} - \sin\sqrt{a}}{x-a}$

49 $\lim_{x\to a}\frac{\cos x - \cos a}{\sqrt{x}-\sqrt{a}}$

LONG ANSWER Type II Questions

50 $\lim_{x\to \pi/6}\frac{2-\sqrt{3}\cos x - \sin x}{(6x-\pi)^2}$

51 $\lim_{x\to 0}\frac{\tan x + 4\tan 2x - 3\tan 3x}{x^2\tan x}$

52 $\lim_{x\to \pi/4}\frac{\tan^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

53 $\lim_{x\to \pi}\frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$

54 $\lim_{x\to \pi}\frac{\sqrt{2+\cos x}-1}{(\pi - x)^2}$

55 $\lim_{x\to \pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin x - 2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}$

56 If $f(x) = \begin{cases}\frac{\sin[x]}{[x]}, & [x]\neq 0\\ 0, & [x]=0\end{cases}$, where [] denotes the greatest integer function, then find $\lim_{x\to 0} f(x)$.

57 If $f(x) = \begin{cases}\frac{k\cos x}{\pi - 2x}, & \text{when } x\neq\frac{\pi}{2}\\ 3, & \text{when } x=\frac{\pi}{2}\end{cases}$ and $\lim_{x\to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, then find the value of k.

HINTS & ANSWERS

1. (a) Since, $\lim_{x\to 0} x = 0$ and $-1 \le \sin\frac{1}{x} \le 1$, therefore

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

2. (a) Given, $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta} = \lim_{\theta\to 0}\frac{2\sin^2 2\theta}{2\sin^2 3\theta}$

$$= \frac{4}{9}\cdot\frac{\lim_{\theta\to 0}\left(\frac{\sin 2\theta}{2\theta}\right)^2}{\lim_{\theta\to 0}\left(\frac{\sin 3\theta}{3\theta}\right)^2}$$

$$= \frac{4}{9}$$

3. (b) It is given that α and β are the roots of the given equation $ax^2 + bx + c = 0$.

$\therefore \quad ax^2 + bx + c = a(x-\alpha)(x-\beta)$

Now, given limit $= \lim_{x\to\beta}\frac{1-\cos\{a(x-\alpha)(x-\beta)\}}{(x-\beta)^2}$

$$= 2\lim_{x\to\beta}\left[\frac{\sin\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{\frac{a(x-\alpha)(x-\beta)}{2}}\right]^2 \times \frac{a^2(x-\alpha)^2}{4}$$

$$= 2\times\frac{a^2(\beta-\alpha)^2}{4} = \frac{a^2(\beta-\alpha)^2}{2}$$

4. (d) We have,

$$\lim_{x\to 0}\frac{\tan x + 4\tan 2x - 3\tan 3x}{x}$$

$$= \lim_{x\to 0}\left[\frac{\tan x}{x} + \frac{4\tan 2x}{x} - \frac{3\tan 3x}{x}\right]$$

$$= \lim_{x\to 0}\frac{\tan x}{x} + 8\lim_{x\to 0}\frac{\tan 2x}{2x} - 9\lim_{x\to 0}\frac{\tan 3x}{3x}$$

$$= 1 + 8 - 9 = 0$$

5. (d) We have,

$$\lim_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$$

$$= \lim_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)$$

$$= \lim_{x\to 0} 32\left(\frac{\sin\frac{x^2}{4}}{x^2}\right)^2\left(\frac{\sin\frac{x^2}{8}}{x^2}\right)^2$$

$$= 32\times\left(\frac{1}{4}\right)^2\times\left(\frac{1}{8}\right)^2 = \frac{1}{32}$$

6. 5 **7.** 3

8. Given limit $= \lim_{x\to 0}\frac{x}{\sin 3x}$ **Ans.** $\frac{1}{3}$

9. Given limit $= \lim_{x\to 0}\frac{x}{\tan 4x}$ **Ans.** $\frac{1}{4}$

10. $\lim_{x\to 0} x = 0$ and $-1\le\sin\frac{1}{x}\le 1$ and then use Sandwich theorem. **Ans.** 0

11. 0

12. $\lim_{x\to\pi}\frac{\sin x}{x-\pi} = \lim_{x\to\pi}\frac{\sin(\pi-x)}{(x-\pi)} = -\lim_{x\to\pi}\frac{\sin(\pi-x)}{(\pi-x)}$ **Ans.** -1

13. Substitute $x = 0$ **Ans.** $\frac{1}{\pi}$

14. Convert $x°$ in radian measure, i.e. write $x° = \frac{\pi x}{180}$ radian **Ans.** $\pi/180$

15. 2

16. Put $\pi - x = h$ **Ans.** $\frac{1}{\pi}$

17. Given limit $= \frac{3}{7}\cdot\frac{\lim_{x\to 0}\frac{\sin 3x}{3x}}{\lim_{x\to 0}\frac{\sin 7x}{7x}}$ **Ans.** $\frac{3}{7}$

18. Given limit $= \lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)^2\times 25$

$$= 25\lim_{5x\to 0}\frac{\sin 5x}{5x}\lim_{5x\to 0}\frac{\sin 5x}{5x}$$ **Ans.** 25

19. Divide numerator and denominator by x.

Ans. $\frac{a+1}{b}$

20. Given limit $= \lim_{x\to 0}\frac{\sin 4x/4x}{\tan 4x/4x}$ **Ans.** 1

21. Let $x - \frac{\pi}{2} = h$. Then,

Given limit $= \lim_{h\to 0}\frac{\tan 2\left(\frac{\pi}{2}+h\right)}{h} = \lim_{h\to 0}\frac{\tan(\pi+2h)}{h}$

$$= \lim_{h\to 0}\frac{\tan 2h}{h} \qquad [\because \tan(\pi+\theta) = \tan\theta]$$

Ans. 2

22. 1

23. $\lim_{x\to 0}\frac{\sin^2 2x}{[\sin 2(2x)]^2} = \lim_{x\to 0}\frac{\sin^2 2x}{(2\sin 2x\cos 2x)^2}$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{4\sin^2 2x\cos^2 2x} = \lim_{x\to 0}\frac{1}{4\cos^2 2x}$$ **Ans.** $\frac{1}{4}$

24. $\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{1-1+2\sin^2 x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0}\frac{\sin^2 x}{x^2}\ 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$ **Ans.** 2

25. Given limit $=\lim_{x\to0}\dfrac{2\sin^2\frac{5x}{2}}{\left(\frac{5x}{2}\right)^2}\times\left(\frac{5}{2}\right)^2\times x$ **Ans.** 0

26. Given limit $=\lim_{x\to\pi/2}\dfrac{1-\sin^2x}{1-\sin x}=\lim_{x\to\pi/2}1+\sin x$ **Ans.** 2

27. Given limit $=\lim_{x\to0}\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}\right)$

$$=\lim_{x\to0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to0}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

Ans. 0

28. $\lim_{x\to0}\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x\left(2\cos^2\frac{x}{2}\right)}=\dfrac{1}{2}\lim_{x\to0}\dfrac{\tan\frac{x}{2}}{\frac{x}{2}}$ **Ans.** $\frac{1}{2}$

29. $\lim_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{\cos x}x=\lim_{h\to0}\left[\dfrac{1-\sin\left(\frac{\pi}{2}-h\right)}{\cos\left(\frac{\pi}{2}-h\right)}\right]$

$$=\lim_{h\to0}\frac{1-\cos h}{\sin h}=\lim_{h\to0}\frac{2\sin^2\frac{h}{2}}{2\sin\frac{h}{2}\cos\frac{h}{2}}$$ **Ans.** 0

30. Given limit $=\lim_{x\to0}\left[\dfrac{\tan x}{x}-\dfrac{\sin x}{x}\right]$ **Ans.** 0

31. Given limit $=\lim_{x\to a}\dfrac{2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{a-x}{2}\right)}{(x-a)}$

$$=2\lim_{x\to a}\frac{\sin\left(\frac{a-x}{2}\right)}{-\left(\frac{a-x}{2}\right)}\times\frac{1}{2}\sin\left(\frac{x+a}{2}\right)$$

Ans. $-\sin a$

32. $\lim_{x\to0}\dfrac{\tan2x-x}{3x-\sin x}=\lim_{x\to0}\dfrac{x\left[\frac{\tan2x}{x}-1\right]}{x\left[3-\frac{\sin x}{x}\right]}=\dfrac{\lim_{x\to0}2\times\frac{\tan2x}{2x}-1}{3-\lim_{x\to0}\frac{\sin x}{x}}$

Ans. 1/2

33. Given limit $=\lim_{x\to0}\dfrac{x^2\cos x}{2\sin^2\frac{x}{2}}=2\lim_{x\to0}\dfrac{\left(\frac{x}{2}\right)^2}{\sin^2\frac{x}{2}}\cdot\lim_{x\to0}\cos x$

Ans. 2

34. $\lim_{x\to0}\dfrac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}\cdot\dfrac{\sqrt{x+1}+\sqrt{1-x}}{\sqrt{x+1}+\sqrt{1-x}}$

$$=\lim_{x\to0}\frac{\sin x(\sqrt{x+1}+\sqrt{1-x})}{(x+1)-(1-x)}$$

$$=\lim_{x\to0}\frac{\sin x(\sqrt{x+1}+\sqrt{1-x})}{x+1-1+x}$$

$$=\frac{1}{2}\lim_{x\to0}\frac{\sin x}{x}\lim_{x\to0}(\sqrt{x+1}+\sqrt{1-x})$$ **Ans.** 1

35. $\lim_{x\to\pi/4}\dfrac{\sec^2x-2}{\tan x-1}=\lim_{x\to\pi/4}\dfrac{1+\tan^2x-2}{\tan x-1}$

$$=\lim_{x\to\pi/4}\frac{\tan^2x-1}{\tan x-1}$$ **Ans.**. 2

36. Given limit $=\lim_{x\to0}\dfrac{\left[\sin\left(\frac{a}{2^n}\right)\Big/\frac{a}{2^n}\right]\times\frac{a}{2^n}}{\left[\sin\left(\frac{b}{2^n}\right)\Big/\frac{b}{2^n}\right]\times\frac{b}{2^n}}$ **Ans.** $\frac{a}{b}$

37. Given limit $=\lim_{x\to0}\dfrac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{x}}=\left(\dfrac{m}{n}\right)^2\left[\dfrac{\lim_{\frac{mx}{2}\to0}\dfrac{\sin\left(\frac{mx}{2}\right)}{\frac{mx}{2}}}{\lim_{\frac{mx}{2}\to0}\dfrac{\sin\left(\frac{nx}{2}\right)}{\frac{nx}{2}}}\right]^2$

Ans. $\frac{m^2}{n^2}$

38. Given limit $=\lim_{x\to a}\dfrac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}\times\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$

$$=\lim_{x\to a}\frac{(\sin x-\sin a)(\sqrt{x}+\sqrt{a})}{x-a}$$

On putting $x-a=h$, we get

Given limit $=\lim_{h\to0}\dfrac{[\sin(a+h)-\sin a][\sqrt{a+h}+\sqrt{a}]}{h}$

$[\because x\to a\Rightarrow h\to0]$

$$=\lim_{h\to0}\frac{2\cos\frac{a+h+a}{2}\sin\left(\frac{a+h-a}{2}\right)(\sqrt{a+h}+\sqrt{a})}{h}$$

$$=\lim_{h\to0}\frac{2\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)(\sqrt{a+h}+\sqrt{a})}{2\left(\frac{h}{2}\right)}$$

Ans. $2\sqrt{a}\cos a$

39. $\lim_{x\to0}\dfrac{\tan x-\sin x}{\sin^3x}=\lim_{x\to0}\dfrac{1-\cos x}{\cos x\sin^2x}$

$$=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{\cos x\left(4\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}\right)}$$ **Ans.** $\frac{1}{2}$

40. Given limit $= \lim_{x\to 0} \frac{1-\cos 2x}{1-\cos x}$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{2\sin^2 \frac{x}{2}} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \times \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 \times 4 \quad \textbf{Ans. } 4$$

41. Given limit $= \lim_{x\to 0} \frac{2\sin x - 2\sin x\cos x}{x^3}$

$$= 2\lim_{x\to 0}\frac{\sin x}{x}\cdot \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$$

$$= 2\cdot 1 \lim_{x\to 0}\frac{1-\cos x}{x^2} = 2\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{4\times\frac{x^2}{4}} \quad \textbf{Ans. } 1$$

42. Put $x = \frac{\pi}{2}+h$. Then, given limit $= \lim_{h\to 0}\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left\{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)\right\}^2}$

$$= \lim_{h\to 0}\frac{1-\cos h}{h^2} = \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}\times\frac{1}{4} \quad \textbf{Ans. } \frac{1}{2}$$

43. Given limit $= \lim_{x\to 0}\left(\frac{\frac{\cos 2x-\cos 4x}{\cos 2x\cos 4x}}{\frac{\cos x-\cos 3x}{\cos x\cos 3x}}\right)$

$$= \lim_{x\to 0}\left(\frac{\cos 2x-\cos 4x}{\cos x-\cos 3x}\times\frac{\cos x\cos 3x}{\cos 2x\cos 4x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin 3x\sin x}{2\sin 2x\sin x}\times\frac{\cos x\cos 3x}{\cos 2x\cos 4x}\right)$$

$$= \frac{3}{2}\,\frac{\lim_{x\to 0}\frac{\sin 3x}{3x}}{\lim_{x\to 0}\frac{\sin 2x}{2x}}\times\lim_{x\to 0}\frac{\cos x\cos 3x}{\cos 2x\cos 4x} \quad \textbf{Ans. } \frac{3}{2}$$

44. **Hint** Given limit $= \lim_{x\to 0}\frac{\frac{1}{\sin x}-\frac{\cos x}{\sin x}}{x} = \lim_{x\to 0}\frac{1-\cos x}{x\cdot\sin x}$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{x\to 0}\frac{\tan\frac{x}{2}}{\frac{x}{2}}\cdot\frac{1}{2} \quad \textbf{Ans. } \frac{1}{2}$$

45. $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\tan 3x} = \lim_{x\to 0}\frac{\left(\frac{\sin 2x}{2x}+\frac{3x}{2x}\right)2x}{\left(\frac{2x}{3x}+\frac{\tan 3x}{3x}\right)3x}$

$$= \left(\lim_{x\to 0}\frac{\frac{\sin 2x}{2x}+\frac{3}{2}}{\frac{2}{3}+\frac{\tan 3x}{3x}}\right)\cdot\frac{2}{3} \quad \textbf{Ans. } 1$$

46. Given limit $= \lim_{x\to 0}\frac{2\sin 4x\cos x}{2\cos 5x\sin x}$

$$= \lim_{x\to 0}\frac{\sin 4x}{\sin x}\cdot\lim_{x\to 0}\frac{\cos x}{\cos 5x} \quad \textbf{Ans. } 4$$

47. Given limit $= \lim_{x\to\pi/8}\frac{\frac{\cos 4x}{\sin 4x}-\cos 4x}{(\pi-8x)^3\sin 4x}$

$$= \lim_{x\to\pi/8}\frac{\cos 4x(1-\sin 4x)}{(\pi-8x)^3}\cdot\lim_{x\to\pi/8}\frac{1}{\sin 4x}$$

$$= \lim_{x\to\pi/8}\frac{\cos 4x(1-\sin 4x)}{(\pi-8x)^3}$$

On putting $x = \frac{\pi}{8}+h$, we get

Given limit $= \lim_{h\to 0}\frac{\cos 4\left(\frac{\pi}{8}+h\right)\left[1-\sin 4\left(\frac{\pi}{8}+h\right)\right]}{\left[\pi-8\left(\frac{\pi}{8}+h\right)\right]^3}$

$$= \lim_{h\to 0}\frac{-\sin 4h(1-\cos 4h)}{(-8h)^3} \quad \textbf{Ans. } \frac{1}{16}$$

48. Given limit $= \lim_{x\to a}\frac{2\cos\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}{(\sqrt{x})^2-(\sqrt{a})^2}$

$$= \lim_{x\to a}\frac{2\cos\left(\frac{\sqrt{x}+\sqrt{a}}{2}\right)}{(\sqrt{x}+\sqrt{a})}\cdot\lim_{x\to a}\frac{\sin\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{a}}{2}\right)}\times\frac{1}{2}$$

Ans. $\frac{1}{2\sqrt{a}}\cos\sqrt{a}$

49. Given limit $= \lim_{x\to a}\frac{2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{a-x}{2}\right)}{(\sqrt{x}-\sqrt{a})\times(\sqrt{x}+\sqrt{a})}\times(\sqrt{x}+\sqrt{a})$

$$= \lim_{x\to a}\frac{2\sin\left(\frac{x+a}{2}\right)}{1}\times\lim_{x\to a}\frac{\sin\left(\frac{a-x}{2}\right)}{\frac{x-a}{2}}\times\lim_{x\to a}\frac{1}{2}(\sqrt{x}+\sqrt{a})$$

Ans. $-2\sqrt{a}\sin a$

50. Given limit $= \lim_{h\to 0}\frac{2-\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)-\sin\left(\frac{\pi}{6}+h\right)}{\left[6\left(\frac{\pi}{6}+h\right)-\pi\right]^2}$

$$= \lim_{h\to 0}\frac{\left[2-\sqrt{3}\left(\cos\frac{\pi}{6}\cdot\cos h-\sin\frac{\pi}{6}\cdot\sin h\right)-\left(\sin\frac{\pi}{6}\cdot\cos h+\cos\frac{\pi}{6}\sin h\right)\right]}{(\pi+6h-\pi)^2}$$

$$= \lim_{h\to 0}\frac{2-\sqrt{3}\cdot\left(\frac{\sqrt{3}}{2}\cos h-\frac{1}{2}\sin h\right)-\left(\frac{1}{2}\cos h+\frac{\sqrt{3}}{2}\sin h\right)}{36h^2}$$

$$= \lim_{h\to 0}\frac{1-\cos h}{18h^2} = \lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{18h^2} \quad \textbf{Ans. } \frac{1}{36}$$

51. Given limit $= \lim_{x\to 0}\frac{\tan x+4\tan 2x-3\tan 3x}{x^2\tan x}$

$$= \lim_{x\to 0}\frac{\tan x+4\cdot\frac{2\tan x}{1-\tan^2 x}-3\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}}{x^2\tan x}$$

$$= \lim_{x\to 0}\frac{\left(1+\frac{8}{1-\tan^2 x}-\frac{9-3\tan^2 x}{1-3\tan^2 x}\right)}{x^2}$$

$$= \lim_{x\to 0}\left[\frac{(1-\tan^2 x)(1-3\tan^2 x)+8(1-3\tan^2 x)-(9-3\tan^2 x)(1-\tan^2 x)}{x^2(1-\tan^2 x)(1-3\tan^2 x)}\right]$$

$$= \lim_{x\to 0}\frac{-16\tan^2 x}{x^2(1-\tan^2 x)(1-3\tan^2 x)}$$

$$= -16\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2\frac{1}{(1-\tan^2 x)(1-3\tan^2 x)}$$

Ans. -16

52. Given limit $= \lim_{x\to \pi/4}\frac{\tan x(\tan^2 x-1)}{\cos\left(x+\frac{\pi}{4}\right)}$

$$= \lim_{x\to \pi/4}\tan x\cdot\lim_{x\to\pi/4}\left[\frac{-(1-\tan^2 x)}{\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$$= 1\times\left[-\lim_{x\to \pi/4}\frac{(1+\tan x)(1-\tan x)}{\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$$= -\lim_{x\to \pi/4}(1+\tan x)\lim_{x\to \pi/4}\left[\frac{\cos x-\sin x}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$$= -(1+1)\times\lim_{x\to \pi/4}\frac{\sqrt{2}\left[\frac{1}{\sqrt{2}}\cdot\cos x-\frac{1}{\sqrt{2}}\cdot\sin x\right]}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}$$

$$= -2\sqrt{2}\lim_{x\to \pi/4}\frac{\cos\left(x+\frac{\pi}{4}\right)}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)} \quad \textbf{Ans. } -4$$

53. Given limit $= \lim_{x\to \pi}\frac{\cos^2\frac{x}{4}+\sin^2\frac{x}{4}-2\sin\frac{x}{4}\cos\frac{x}{4}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$

$$= \lim_{x\to \pi}\frac{\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)^2}{\left(\cos^2\frac{x}{4}-\sin^2\frac{x}{4}\right)\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)} \quad \textbf{Ans. } \frac{1}{\sqrt{2}}$$

54. Given limit $= \lim_{x\to \pi}\frac{(\sqrt{2+\cos x}-1)}{(\pi-x)^2}\times\frac{(\sqrt{2+\cos}+1)}{(\sqrt{2+\cos x}+1)}$

$$= \lim_{x\to \pi}\frac{2+\cos x-1}{(\pi-x)^2(\sqrt{2+\cos x}+1)}$$

$$= \lim_{x\to \pi}\frac{2\cos^2\frac{x}{2}}{(\pi-x)^2(\sqrt{2+\cos x}+1)}$$

$$= \lim_{h\to 0}\frac{2\cos^2\left(\frac{\pi+h}{2}\right)}{(-h)^2(\sqrt{2+\cos(\pi+h)}+1)}$$

$$= \lim_{h\to 0}\frac{\sin^2\frac{h}{2}}{2\left(\frac{h}{2}\right)^2(\sqrt{2+\cos(\pi+h)}+1)} \quad \textbf{Ans. } \frac{1}{4}$$

55. Given limit $= \lim_{h\to 0}\frac{(-h)\sin\left(\frac{\pi}{2}+h\right)-2\cos\left(\frac{\pi}{2}+h\right)}{(-h)+\cot\left(\frac{\pi}{2}+h\right)}$

$$= \lim_{h\to 0}\frac{-h\cos h+2\sin h}{-h-\tan h} \quad \textbf{Ans. } -\frac{1}{2}$$

56. RHL $= \lim_{x\to 0}f(x) = \lim_{h\to 0}f(0+h) = \lim_{h\to 0}\frac{\sin[0+h]}{[0+h]} = \frac{\sin 0}{0}$

Ans. Does not exist

57. $\lim_{x\to\pi/2}f(x) = \lim_{x\to\pi/2}\frac{k\cos x}{\pi-2x} = \lim_{x\to\pi/2}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$

$$= \lim_{h\to 0}\frac{k\sin h}{2h} = \frac{k}{2}$$

$\because$ It is given that, $\lim_{x\to\pi/2}f(x) = f\left(\frac{\pi}{2}\right)$

$\therefore$ $\frac{k}{2} = f\left(\frac{\pi}{2}\right) = 3$ **Ans.** $k=6$

|TOPIC 4|

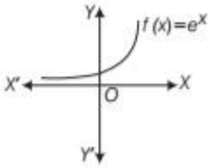
Limits of Exponential Functions and Logarithmic Functions

LIMITS OF EXPONENTIAL FUNCTIONS

The great Swiss Mathematician **Leonhard Euler** (1707-1783) introduced the number e, whose value lies between 2 and 3. This number is useful in defining exponential function.

A function of the form of $f(x) = e^x$ is called **exponential** function.

The graph of the function is given below



(i) Domain of $f(x) = (-\infty, \infty)$

(ii) Range of $f(x) = (0, \infty)$

To find the limit of a function involving exponential function, we use the following theorem

Theorem $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

Proof We have an inequality

$$\frac{1}{1+|x|} \le \frac{e^x - 1}{x} \le 1 + (e-2)|x|,$$

$$x \in [-1, 1] - \{0\} \qquad \ldots(\text{i})$$

Here, $\dfrac{e^x - 1}{x}$ is sandwiched between the functions $\dfrac{1}{1+|x|}$ and $1 + (e-2)|x|$.

Now, $\lim\limits_{x \to 0} \dfrac{1}{1+|x|} = \dfrac{\lim\limits_{x \to 0} 1}{\lim\limits_{x \to 0} (1+|x|)}$ [by quotient of limits]

$$= \frac{1}{1+|0|} = \frac{1}{1} = 1$$

and $\lim\limits_{x \to 0} [1 + (e-2)|x|] = 1 + (e-2)|0|$

$$= 1 + (e-2)(0) = 1$$

Thus, $\lim\limits_{x \to 0} \dfrac{1}{1+|x|} = 1 = \lim\limits_{x \to 0} [1 + (e-2x)|x|]$

So, by applying sandwich theorem in Eq. (i), we get

$\therefore \quad \lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ **Hence proved.**

METHOD TO FIND THE LIMIT OF EXPONENTIAL FUNCTIONS

If given function has exponential term, then we convert the given theorem in the form of $\dfrac{e^x - 1}{x}$ and then use the theorem $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

EXAMPLE |1| Find the value of $\lim\limits_{x \to 0} \dfrac{e^{3x} - 1}{x}$.

Sol. $\lim\limits_{x \to 0} \dfrac{e^{3x} - 1}{x} = \lim\limits_{x \to 0} \dfrac{e^{3x} - 1}{x} \times \dfrac{3}{3}$

[multiplying numerator and denominator by 3]

$$= 3 \lim_{x \to 0} \frac{e^{3x} - 1}{3x} \qquad \ldots(\text{i})$$

Let $h = 3x$. Then, $x \to 0 \Rightarrow h \to 0$

Now, from Eq. (i), we get

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = 3 \lim_{h \to 0} \frac{e^h - 1}{h} = 3(1) \quad \left[\because \lim_{\theta \to 0} \frac{e^\theta - 1}{\theta} = 1\right]$$

$$= 3$$

EXAMPLE |2| Evaluate $\lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$.

Sol. $\lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$

$$= \lim_{x \to 0} \frac{e^{2x} + 1 - 2e^x}{x^2 e^x}$$

$$= \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)^2 \times e^{-x}$$

$$= \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)^2 \times \lim_{x \to 0} e^{-x} = (1)^2 \times e^0 = 1$$

EXAMPLE |3| Evaluate $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$.

Sol. $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \to 0} \frac{e^{\sin x} - 1}{x} \times \frac{\sin x}{\sin x}$

[multiplying numerator and denominator by $\sin x$]

$= \lim_{x \to 0} \left[\frac{e^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x}\right] = \lim_{x \to 0} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \to 0} \frac{\sin x}{x}$

$= 1 \times 1 = 1$ $\left[\because \lim_{\theta \to 0} \frac{e^{\theta} - 1}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$

EXAMPLE |4| Evaluate $\lim_{x \to 3} \frac{e^x - e^3}{x - 3}$.

Sol. We have, $\lim_{x \to 3} \frac{e^x - e^3}{x - 3}$

On putting $h = x - 3$ we get

$\lim_{x \to 3} \frac{e^x - e^3}{x - 3} = \lim_{h \to 0} \frac{e^{h+3} - e^3}{h}$ $[\because x \to 3 \Rightarrow h \to 0]$

$= \lim_{h \to 0} \frac{e^h e^3 - e^3}{h}$

$= e^3 \lim_{h \to 0} \frac{e^h - 1}{h} = e^3 \times 1 = e^3$ $\left[\because \lim_{\theta \to 0} \frac{e^{\theta} - 1}{\theta} = 1\right]$

EXAMPLE |5| Evaluate $\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x}$.

Sol. $\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x} = \lim_{x \to 0} \frac{x(e^x - 1)}{2\sin^2 \frac{x}{2}}$ $\left[\because 1 - \cos x = 2\sin^2 \frac{x}{2}\right]$

$= \frac{1}{2} \lim_{x \to 0} \frac{x(e^x - 1)}{\sin^2 \frac{x}{2}} \times \frac{x}{x}$

[multiplying numerator and denominator by x]

$= \frac{1}{2} \lim_{x \to 0} \frac{(e^x - 1)}{x} \times \frac{1}{\frac{\sin^2 x/2}{x^2}}$

$= \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} \times \frac{1}{\frac{\sin^2 x/2}{\left(\frac{x}{2}\right)^2} \times 4}$

$= \frac{1}{2} \lim_{x \to 0} \frac{e^x - 1}{x} \times 4 \lim_{x \to 0} \frac{1}{\left(\frac{\sin x/2}{x/2}\right)^2}$

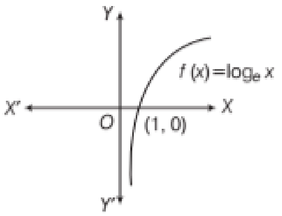
$= \frac{1}{2} \times 1 \times 4 \times \frac{1}{(1)^2}$ $\left[\because \lim_{\theta \to 0} \frac{e^{\theta} - 1}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$

$= \frac{4}{2} = 2$

LIMITS OF LOGARITHMIC FUNCTIONS

The logarithmic function expressed as $\log_e R^+ \to R$ and given by $\log_e x = y$ iff $e^y = x$.

The graph of the function is given below



(i) **Domain of $f(x) = (0, \infty)$ or R^+**

(ii) **Range of $f(x) = (-\infty, \infty)$ or R**

To find the limit of functions involving logarithmic function, we the following theorem

Theorem $\lim_{x \to 0} \frac{\log_e (1 + x)}{x} = 1$

Proof Let $\frac{\log_e (1 + x)}{x} = y$

Then, $\log_e (1 + x) = xy$

$\Rightarrow$ $1 + x = e^{xy}$ $[\because \log x = y \Rightarrow e^y = x]$

$\Rightarrow$ $\frac{e^{xy} - 1}{x} = 1 \Rightarrow \frac{e^{xy} - 1}{xy} \cdot y = 1$

On taking limit $xy \to 0$ both sides, we get

$\lim_{xy \to 0} \frac{e^{xy} - 1}{xy} \lim_{x \to 0} y = 1 \Rightarrow \lim_{x \to 0} y = 1$

$\left[\because \lim_{xy \to 0} \frac{e^{xy} - 1}{xy} = 1 \text{ and as } x \to 0, \text{ then } xy \to 0\right]$

$\Rightarrow$ $\lim_{x \to 0} \frac{\log_e (1 + x)}{x} = 1$

Note

$\lim_{x \to 0} \frac{\log_e (1 - x)}{-x} = 1$

COROLLARY

I. $\lim_{x \to 0} \frac{\log_e (1 - x)}{-x} = 1$

II. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$

METHOD TO FIND THE LIMIT OF LOGARITHMIC FUNCTION

If given function involves logarithmic function, then we convert the given function in the form of $\frac{\log_e(1+x)}{x}$ and then use the theorem $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1.$

EXAMPLE |6| Evaluate $\lim_{x\to 0}\frac{\log_e(1+2x)}{x}$. **[NCERT]**

Sol. We have, $\lim_{x\to 0}\frac{\log_e(1+2x)}{x}\times\frac{2}{2}$

[multiplying numerator and denominator by 2]

$$=2\lim_{x\to 0}\frac{\log_e(1+2x)}{2x}$$

On putting $h=2x$, we get

$$\lim_{x\to 0}\frac{\log_e(1+2x)}{x}=2\lim_{h\to 0}\frac{\log_e(1+h)}{h}\quad[\because x\to 0\Rightarrow h\to 0]$$

$$=2\,(1)\qquad\left[\because \lim_{x\to 0}\frac{\log_e(1+x)}{x}=1\right]$$

$$=2$$

EXAMPLE |7| Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\log(1+x)}$.

Sol. $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\log(1+x)}$

On multiplying numerator and denominator by $\sqrt{1+x}+1$, we get

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\log(1+x)}\times\frac{\sqrt{1+x}+1}{(\sqrt{1+x}+1)}$$

$$=\lim_{x\to 0}\frac{1+x-1}{(\sqrt{1+x}+1)\log(1+x)}$$

$$=\log_{x\to 0}\frac{x}{(\sqrt{1+x}+1)\log(1+x)}$$

$$=\frac{1}{(\sqrt{1+0}+1)}\lim_{x\to 0}\frac{1}{\frac{\log(1+x)}{x}}=\frac{1}{1+1}\times 1=\frac{1}{2}$$

EXAMPLE |8| Evaluate

$$\lim_{x\to 2}\frac{x^2-x\log x+2\log x-4}{x-2}.$$

Sol. $\lim_{x\to 2}\frac{x^2-x\log x+2\log x-4}{x-2}$

$$=\lim_{x\to 2}\frac{(x^2-4)-\log x(x-2)}{x-2}$$

$$=\lim_{x\to 2}\frac{(x+2)(x-2)-\log x\,(x-2)}{(x-2)}$$

$$=\lim_{x\to 2}\frac{(x-2)[x+2-\log x]}{(x-2)}$$

$$=\lim_{x\to 2}[x+2-\log x]=2+2-\log 2$$

$$=4-\log 2$$

EXAMPLE |9| Evaluate $\lim_{x\to e}\frac{\log x-1}{x-e}$.

Sol. We have, $\lim_{x\to e}\frac{\log x-1}{x-e}=\lim_{x\to e}\frac{\log x-\log e}{e\left(\frac{x}{e}-1\right)}\quad[\because \log e=1]$

$$=\lim_{x\to e}\frac{\log\frac{x}{e}}{e\left(\frac{x}{e}-1\right)}\qquad\left[\because \log m-\log n=\log\frac{m}{n}\right]$$

On putting $x=h+e$, we get

$$\lim_{x\to e}\frac{\log x-1}{x-e}=\frac{1}{e}\lim_{h\to 0}\frac{\log\left(\frac{h+e}{e}\right)}{e\left(\frac{h+e}{e}-1\right)}[\because x\to e\Rightarrow h\to 0]$$

$$=\frac{1}{e}\lim_{h\to 0}\frac{\log\left(1+\frac{h}{e}\right)}{\left(\frac{h}{e}\right)}\qquad\ldots(\text{i})$$

Again, putting $y=\frac{h}{e}$ in Eq. (i), we get

$$\lim_{x\to e}\frac{\log x-1}{x-e}=\frac{1}{e}\lim_{h\to 0}\frac{\log\left(1+\frac{h}{e}\right)}{e\left(\frac{h}{e}\right)}$$

$$=\frac{1}{e}\lim_{y\to 0}\frac{\log(1+y)}{y}\quad[\because h\to 0\Rightarrow y\to 0]$$

$$=\frac{1}{e}\times 1=\frac{1}{e}$$

EXAMPLE |10| Evaluate $\lim_{x\to 0}\frac{\log(6+x)-\log(6-x)}{x}$.

Sol. $\lim_{x\to 0}\frac{\log(6+x)-\log(6-x)}{x}$

$$=\lim_{x\to 0}\frac{\log 6\left(1+\frac{x}{6}\right)-\log 6\left(1-\frac{x}{6}\right)}{x}$$

$$=\lim_{x\to 0}\frac{\left[\log 6+\log\left(1+\frac{x}{6}\right)\right]-\left[\log 6+\log\left(1-\frac{x}{6}\right)\right]}{x}$$

$$= \lim_{x\to 0}\left[\frac{\log\left(1+\frac{x}{6}\right)}{x} - \frac{\log\left(1-\frac{x}{6}\right)}{x}\right]$$

$$= \lim_{x\to 0}\frac{1}{6}\frac{\log\left(1+\frac{x}{6}\right)}{\frac{x}{6}} + \lim_{x\to 0}\frac{1}{6}\frac{\log\left(1-\frac{x}{6}\right)}{-\frac{x}{6}}$$

$$\left[\because \lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{\log(1-x)}{-x} = 1\right]$$

$$= \frac{1}{6}\times 1 + \frac{1}{6}\times 1$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

EXAMPLE |11| Evaluate $\lim_{x\to 0}\left(\frac{3^{2x}-1}{2^{3x}-1}\right)$.

Sol. $$\lim_{x\to 0}\left(\frac{3^{2x}-1}{2^{3x}-1}\right) = \lim_{x\to 0}\frac{\left\{\left(\frac{3^{2x}-1}{2x}\right)\cdot 2x\right\}}{\left\{\left(\frac{2^{2x}-1}{3x}\right)\cdot 3x\right\}}$$

$$= \frac{2}{3}\cdot\frac{\lim_{2x\to 0}\left(\frac{3^{2x}-1}{2x}\right)}{\lim_{3x\to 0}\left(\frac{2^{3x}-1}{3x}\right)}$$

$$= \frac{2}{3}\cdot\frac{\log 3}{\log 2} \qquad \left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log a\right]$$

$$= \frac{2\log 3}{3\log 2} = \frac{\log(3^2)}{\log(2^2)} = \frac{\log 9}{\log 8}$$

EXAMPLE |12| Evaluate $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$.

Sol. $$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$$

$$= \lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0}\frac{2^x-1}{x}\times\{\sqrt{1+x}+1\}$$

$$= \lim_{x\to 0}\frac{2^x-1}{x}\times\lim_{x\to 0}(\sqrt{1+x}+1)$$

$$= (\log 2)2 = 2\log 2$$

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 $\lim_{x\to 0} e^x$ is equal to

(a) −1 (b) 0
(c) 1 (d) does not exist

2 $\lim_{x\to 0}\frac{3^x-2^x}{x}$ is equal to

(a) $\log\frac{3}{2}$ (b) $\log\frac{2}{3}$
(c) $\log\frac{1}{2}$ (d) $\log\frac{1}{3}$

3 $\lim_{x\to 0} e^{1/x}$ is equal to

(a) 0 (b) −1
(c) 1 (d) does not exist

4 $\lim_{x\to 0}\frac{5^x-1}{\sqrt{1+x}-1}$ is equal to

(a) log 5 (b) 2 log 5
(c) − log 5 (d) −2 log 5

5 The value of $\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$ is

(a) −1 (b) 1
(c) 0 (d) None of these

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6- 15) *Evaluate the following limits.*

6 $\lim_{x\to 0}\left(\frac{e^{4x}-1}{x}\right)$

7 $\lim_{x\to 0}\frac{e^{bx}-1}{x}$

8 $\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)$

9 $\lim_{x\to 0}\frac{\log(1+5x)}{x}$

10 $\lim_{x\to 0}\frac{\log(1+cx)}{x}$

11 $\lim_{x\to 0}\frac{a^x-b^x}{x}$

12 $\lim_{x\to 0}\left(\frac{e^{2+x}-e^2}{x}\right)$

13 $\lim_{x\to 0}\left(\frac{e^x-e^4}{x-4}\right)$

14 $\lim_{x\to 5}\frac{e^x-e^5}{x-5}$

15 $\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$

Directions (Q. Nos. 16-23) *Evaluate the following limits.*

16 $\lim_{x\to 0}\frac{a^x - a^{-x}}{x}$

17 $\lim_{x\to 0}\left(\frac{e^x - x - 1}{x}\right)$

18 $\lim_{x\to 0}\frac{e^{3x} - e^{2x}}{x}$

19 $\lim_{x\to 0}\frac{3^{2+x} - 9}{x}$

20 $\lim_{x\to 0}\frac{a^{\sin x} - 1}{\sin x}$

21 $\lim_{x\to 1}\frac{x - 1}{\log_e x}$

22 $\lim_{x\to 5}\frac{\log x - \log 5}{x - 5}$

23 $\lim_{x\to 0}\frac{\log(\sin x + 1)}{x}$

SHORT ANSWER Type Questions

Directions (Q.Nos. 24-32) *Evaluate the following limits*

24 $\lim_{x\to 0}\frac{\log(5+x) - \log(5-x)}{x}$

25 $\lim_{x\to 0}\frac{a^x - b^x}{\sin x}$

26 $\lim_{x\to 0}\frac{3^x - 2^x}{\tan x}$

27 $\lim_{x\to 0}\frac{e^{3+x} - \sin x - e^3}{x}$

28 $\lim_{x\to 0}\frac{\log(1 + x^3)}{\sin^3 x}$

29 $\lim_{x\to 0}\frac{e^x - 1}{\sqrt{1 - \cos 2x}}$

30 $\lim_{x\to 2}\frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$

31 $\lim_{x\to 0}\frac{e^x - e^{\sin x}}{x - \sin x}$

32 $\lim_{x\to 0}\left(\frac{3^{2x} - 2^{3x}}{x}\right)$

HINTS & ANSWERS

1. (c) $\lim_{x\to 0} e^x = e^0 = 1$

2. (a) $\lim_{x\to 0}\left(\frac{3^x - 2^x}{x}\right)$

$= \lim_{x\to 0}\left(\frac{3^x - 1}{x}\right) - \lim_{x\to 0}\left(\frac{2^x - 1}{x}\right)$

$= \log 3 - \log 2 = \log(3/2)$

3. (d) Let $f(x) = e^{1/x}$.

Then,

LHL at $(x = 0) = \lim_{x\to 0^-} f(x)$

$= \lim_{h\to 0} f(0 - h)$

$= \lim_{h\to 0} e^{-1/h} = e^{-\infty} = 0$

and RHL at $(x = 0) = \lim_{x\to 0^+} f(x)$

$= \lim_{h\to 0} f(0 + h)$

$= \lim_{h\to 0} e^{1/h}$

$= e^{\infty} = \infty$, which is not defined

So, $\lim_{x\to 0} f(x)$ does not exit.

4. (b) We have,

$\lim_{x\to 0}\frac{5^x - 1}{\sqrt{1+x} - 1} = \lim_{x\to 0}\frac{5^x - 1}{\sqrt{1+x} - 1}\times\frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$

$= \lim_{x\to 0}\frac{5^x - 1}{x}\times\lim_{x\to 0}(\sqrt{1+x} + 1)$

$= 2\log 5$

5. (b) Given limit $= \lim_{x\to 0}\frac{e^{\tan x} - 1}{\tan x}\cdot\frac{\tan x}{x}$

$= 1\cdot 1 = 1$

6. Given limit $= \lim_{x\to 0} 4\left(\frac{e^{4x} - 1}{4x}\right)$ **Ans.** 4

7. b

8. log3

9. Given limit $= \lim_{x\to 0}\frac{5\log(1 + 5x)}{5x}$ **Ans.** 5

10. c

11. Given limit $= \lim_{x\to 0}\frac{b^x\left[\left(\frac{a}{b}\right)^x - 1\right]}{x}$ **Ans.** $\log\left(\frac{a}{b}\right)$

12. Given limit $= \lim_{x\to 0}\frac{e^2(e^x - 1)}{x}$ **Ans.** e^2

13. Given limit $= \lim_{x\to 4} e^4\left(\frac{e^{x-4} - 1}{x - 4}\right)$ **Ans.** e^4

14. Put $x - 5 = h$ and as $x \to 5$, then $h \to 0$.

$\therefore$ We have

$\lim_{h\to 0}\frac{e^{h+5} - e^5}{h} = \lim_{h\to 0}\frac{e^h e^5 - e^5}{h}$

$= e^5\lim_{h\to 0}\frac{e^h - 1}{h}$ **Ans.** e^5

15. $L = \lim_{x\to 0}\frac{e^{2x} - 1}{xe^x} = \lim_{x\to 0}\frac{e^{2x} - 1}{2x}\times\lim_{x\to 0}\frac{2}{e^x}$ **Ans.** 2

16. $\lim_{x\to 0}\frac{a^{-x}(a^{2x} - 1)}{2x}\times 2 = a^0\log a^2$

Ans. $2\log a$

17. $\lim_{x\to 0}\left(\frac{e^x - x - 1}{x}\right) = \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) - \lim_{x\to 0}\frac{x}{x}$

Ans. 0

18. $\lim_{x\to 0}\frac{e^{3x}-1}{x}-\left(\lim_{x\to 0}\frac{e^{2x}-1}{x}\right)$

$$=\lim_{x\to 0}\frac{e^{3x}-1}{3x}\times 3-\left(\lim_{x\to 0}\frac{e^{2x}-1}{2x}\right)2$$

Ans. 1

19. $\lim_{x\to 0}\frac{3^{2+x}-9}{x}=\lim_{x\to 0}\frac{3^2(3^x-1)}{x}$

Ans. $9\log 3$

20. Let $y=\sin x$.

Then, $y\to 0$ as $x\to 0$.

$\therefore\ \lim_{x\to 0}\frac{a^{\sin x}-1}{\sin x}=\lim_{y\to 0}\frac{a^y-1}{y}$ **Ans.** $\log a$

21. $\lim_{h\to 0}\frac{1+h-1}{\log_e(1+h)}=\lim_{h\to 0}\frac{h}{\log_e(1+h)}$

$$=\frac{1}{\lim_{h\to 0}\frac{\log(1+h)}{h}}$$ **Ans.** 1

22. Put $x-5=h$ and as $x\to 5$, then $h\to 0$

$\therefore\ \lim_{h\to 0}\frac{\log(h+5)-\log 5}{h}=\lim_{h\to 0}\frac{\log\left(1+\frac{h}{5}\right)}{\frac{h}{5}\times 5}$ **Ans.** $\frac{1}{5}$

23. $\lim_{x\to 0}\frac{\log(\sin x+1)}{\sin x}\times\frac{\sin x}{x}$ **Ans.** 1

24. $\lim_{x\to 0}\frac{\log\left\{5\left(1+\frac{x}{5}\right)\right\}-\log\left\{5\left(1-\frac{x}{5}\right)\right\}}{x}$

$$=\lim_{x\to 0}\frac{\left\{\log 5+\log\left(1+\frac{x}{5}\right)\right\}-\left\{\log 5+\log\left(1-\frac{x}{5}\right)\right\}}{x}$$

$$=\lim_{x\to 0}\frac{1}{5}\frac{\log\left(1+\frac{x}{5}\right)}{\frac{x}{5}}-\lim_{x\to 0}\frac{\log\left(1-\frac{x}{5}\right)}{-\frac{x}{5}}\cdot\frac{1}{(-5)}$$

Ans. $\frac{2}{5}$

25. $\lim_{x\to 0}\left\{\left(\frac{a^x-1}{\sin x}\right)-\left(\frac{b^x-1}{\sin x}\right)\right\}$

$$=\lim_{x\to 0}\left(\frac{a^x-1}{x}\times\frac{x}{\sin x}\right)-\lim_{x\to 0}\left(\frac{b^x-1}{x}\times\frac{x}{\sin x}\right)$$

$$=\lim_{x\to 0}\frac{a^x-1}{x}\times\lim_{x\to 0}\frac{x}{\sin x}-\lim_{x\to 0}\frac{b^x-1}{x}\times\lim_{x\to 0}\frac{x}{\sin x}$$

Ans. $\log\left(\frac{a}{b}\right)$

26. $\lim_{x\to 0}\frac{(3^x-1)-(2^x-1)}{x}\times\frac{x}{\tan x}$

$$=\left(\lim_{x\to 0}\frac{3^x-1}{x}-\lim_{x\to 0}\frac{2^x-1}{x}\right)\times\lim_{x\to 0}\frac{x}{\tan x}$$

Ans. $\log\left(\frac{3}{2}\right)$

27. $\lim_{x\to 0}\left[\frac{e^3(e^x-1)}{x}-\frac{\sin x}{x}\right]$ **Ans.** (e^3-1)

28. $\lim_{x\to 0}\frac{\log(x^3+1)}{x^3}\times\frac{x^3}{\sin^3 x}$ **Ans.** 1

29. $\lim_{x\to 0}\frac{e^x-1}{\sqrt{2\sin^2 x}}$

$$=\frac{1}{\sqrt{2}}\lim_{x\to 0}\frac{e^x-1}{x}\times\frac{x}{\sin x}$$ **Ans.** $\frac{1}{\sqrt{2}}$

30. We have,

$$\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{3-x}-3^{x/2}}=\lim_{x\to 2}\frac{(3^x)^2-12(3^x)+27}{3^3-(3^{x/2})^3}$$

$$=\lim_{x\to 2}\frac{(3^x-3)(3^x-9)}{(3-3^{x/2})(9+3\times 3^{x/2}+3^x)}$$

$$=-\lim_{x\to 2}\frac{(3^x-3)(3^{x/2}+3)}{(3^x+3\times 3^{x/2}+9)}$$

$$=-\frac{4}{3}$$

31. $\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}=\lim_{x\to 0}e^{\sin x}\left(\frac{e^{x-\sin x}-1}{x-\sin x}\right)=e^{\sin 0}\times 1$

Ans. 1

32. $\lim_{x\to 0}\left\{\frac{(3^{2x}-1)-(2^{3x}-1)}{x}\right\}$

$$=2\cdot\lim_{2x\to 0}\left(\frac{3^{2x}-1}{2x}\right)-3\cdot\lim_{3x\to 0}\left(\frac{2^{3x}-1}{3x}\right)$$

Ans. $\log\left(\frac{9}{8}\right)$

|TOPIC 5|

Derivative and First Principle of Derivative

DERIVATIVE AT A POINT

Suppose f is a real valued function and a is a point in its domain. Then, derivative of f at a is defined by

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

provided this limit exists.

The derivative of $f(x)$ at a is denoted by $f'(a)$.

EXAMPLE |1| Find the derivative of $f(x) = 4x + 5$ at $x = 3$.

Sol. Given, $f(x) = 4x + 5$

We know that, at $x = a$, $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

$\therefore \; f'(3) = \lim_{h \to 0} \frac{f(3+h)-f(3)}{h}$

$= \lim_{h \to 0} \frac{4(3+h)+5-(4 \times 3+5)}{h}$

$= \lim_{h \to 0} \frac{12+4h+5-17}{h} = \lim_{h \to 0} \frac{4h}{h} = 4$

EXAMPLE |2| Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also, prove that $f'(0) + 3f'(-1) = 0$. **[NCERT]**

Use $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, to find $f'(0)$ and $f'(-1)$. Then, show that $f'(0) + 3f'(-1) = 0$.

Sol. Clearly, $f'(-1) = \lim_{h \to 0} \frac{f(-1+h)-f(-1)}{h}$

$\left[\because f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}\right]$

$= \lim_{h \to 0} \frac{[2(-1+h)^2 + 3(-1+h) - 5] - [2(-1)^2 + 3(-1) - 5]}{h}$

$= \lim_{h \to 0} \frac{[2(1+h^2-2h) - 3 + 3h - 5] - [2 - 3 - 5]}{h}$

$= \lim_{h \to 0} \frac{2h^2 - h}{h}$

$= \lim_{h \to 0} (2h - 1)$

$= 2(0) - 1 = -1$

and $f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$

$\left[\because f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}\right]$

$= \lim_{h \to 0} \frac{[2(0+h)^2 + 3(0+h) - 5] - [2(0)^2 + 3(0) - 5]}{h}$

$= \lim_{h \to 0} \frac{2h^2 + 3h}{h} = \lim_{h \to 0} (2h + 3) = 2(0) + 3 = 3$

Now, $f'(0) + 3f'(-1) = 3 - 3 = 0$ **Hence proved.**

FIRST PRINCIPLE OF DERIVATIVE

Suppose f is a real valued function, the function defined by $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is called the first principle of derivative.

Thus, $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Sometimes $f'(x)$ is denoted by $\frac{d}{dx}[f(x)]$ or if $y = f(x)$, then it is denoted by $\frac{dy}{dx}$ and referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by $D[f(x)]$.

Note

Derivative of f at $x = a$ is also given by substituting $x = a$ in $f'(x)$ and it is denoted by $\frac{d}{dx} f(x)\Big|_a$ or $\frac{df}{dx}\Big|_a$ or $\left(\frac{df}{dx}\right)_{x=a}$.

EXAMPLE |3| Find the derivative of $f(x) = \frac{1}{x}$ from first principle.

Sol. We have, $f(x) = \frac{1}{x}$

By using first principle, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$\therefore \; f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$ $\left[\because \; f(x) = \frac{1}{x}, \; \therefore f(x+h) = \frac{1}{x+h}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] = \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right]$

$= \lim_{h \to 0} \left[\frac{-1}{x(x+h)}\right] = \frac{-1}{x^2}$

EXAMPLE |4| Find the derivative of $f(x) = ax + b$, where a and b are non-zero constants, by first principle.

Sol. We have, $f(x) = ax + b$

By definition of first principle, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a(x+h) + b - (ax+b)}{h} = \lim_{h\to 0} \frac{ah}{h} = a$$

EXAMPLE |5| Find the derivative of $f(x) = x^n$, where n is positive integer, by first principle. **[NCERT]**

Sol. By definition of first principle, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \quad \text{...(i)}$$

By binomial theorem, we have

$$(x+h)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \dots + {}^nC_n h^n$$

$$\Rightarrow (x+h)^n = x^n + nhx^{n-1} + \frac{n(n-1)}{2} h^2 x^{n-2} + \dots + h^n$$

$$\Rightarrow (x+h)^n - x^n = nhx^{n-1} + \frac{n(n-1)}{2} h^2 x^{n-2} + \dots + h^n$$

On putting this value in Eq. (i), we get

$$f'(x) = \lim_{h\to 0} \frac{nhx^{n-1} + \frac{n(n-1)}{2} h^2 x^{n-2} + \dots + h^n}{h}$$

$$= \lim_{h\to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2} hx^{n-2} + \dots + h^{n-1}\right)}{h}$$

$$= \lim_{n\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} hx^{n-2} + \dots + h^{n-1}\right] = nx^{n-1}$$

Hence, $f'(x)$ or $\frac{d}{dx} f(x) = nx^{n-1}$

EXAMPLE |6| Find the derivative of $(x-1)(x-2)$ from first principle. **[NCERT]**

Sol. Let $f(x) = (x-1)(x-2) = x^2 - 3x + 2$

By first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2 - 3(x+h) + 2] - [x^2 - 3x + 2]}{h}$$

$$= \lim_{h\to 0} \frac{[x^2 + h^2 + 2xh - 3x - 3h + 2] - [x^2 - 3x + 2]}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 - 3h}{h} = \lim_{h\to 0} \frac{h(2x + h - 3)}{h} = 2x - 3$$

EXAMPLE |7| Find the derivative of $\left(\frac{x+1}{x-1}\right)$ from the first principle. **[NCERT]**

Sol. Let $f(x) = \frac{x+1}{x-1}$

By first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[\frac{(x+h)+1}{(x+h)-1} - \frac{x+1}{x-1}\right]}{h}$$

$$= \lim_{h\to 0} \frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{(x^2 - x + xh - h + x - 1) - (x^2 + xh - x + x + h - 1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-2h}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} = \frac{-2}{(x-1)^2}$$

EXAMPLE |8| Find the derivative of e^x, using first principle.

Sol. Let $f(x) = e^x$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{(e^h - 1)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\left(1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \infty\right) - 1\right]}{h}$$

$$= \lim_{h\to 0} \frac{h\left[1 + \frac{h}{2!} + \frac{h^2}{3!} + \dots \infty\right]}{h} = e^x \times 1 = e^x$$

EXAMPLE |9| Find the derivative of a^x from first principle.

Sol. Let $f(x) = a^x$.

By using first principle of derivative, we have

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h}$$

$$\Rightarrow \quad f'(x) = a^x \lim_{h\to 0} \left(\frac{a^h - 1}{h}\right) = a^x \log_e a$$

$$\left[\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a\right]$$

EXAMPLE |10| Find the derivative of e^{x^2} from first principle.

Sol Let $f(x) = e^{x^2}$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{(x+h)^2} - e^{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{e^{x^2+h^2+2hx} - e^{x^2}}{h}$$

$$= \lim_{h\to 0} e^{x^2}\left[\frac{e^{h(h+2x)}-1}{h}\right] \times \frac{(h+2x)}{(h+2x)}$$

$$= e^{x^2} \lim_{h\to 0}\left[\frac{e^{h(h+2x)}-1}{h(h+2x)}\right] \times \lim_{h\to 0}(h+2x)$$

$$= e^{x^2} \times 1 \times (0+2x) = 2x\, e^{x^2} \left[\because \lim_{y\to 0} \frac{e^y-1}{y} = 1\right]$$

EXAMPLE |11| Find the derivative of $e^{\sqrt{x}}$ from first principle.

Sol Let $f(x) = e^{\sqrt{x}}$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to 0} \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{e^{\sqrt{x}}(e^{\sqrt{x+h}-\sqrt{x}}-1)}{(x+h)-x}$$

$$= \lim_{h\to 0} \frac{e^{\sqrt{x}}(e^{\sqrt{x+h}-\sqrt{x}}-1)}{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}$$

$$= e^{\sqrt{x}} \lim_{h\to 0} \frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{(\sqrt{x+h}-\sqrt{x})} \times \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$= e^{\sqrt{x}} \times 1 \times \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \left[\because \lim_{y\to 0} \frac{e^y-1}{y} = 1\right]$$

EXAMPLE |12| Find the derivative of the function $\log x$, by using first principle.

Sol. Let $f(x) = \log x$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\log(x+h)-\log x}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \times \frac{1}{x}$$

$$= 1 \times \frac{1}{x} = \frac{1}{x} \quad \left[\because \lim_{y\to 0} \log\left(\frac{1+y}{y}\right) = 1\right]$$

EXAMPLE |13| Find the derivative of $\log_a x$ from first principle.

Sol. Let $f(x) = \log_a x$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\log_a\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\log_e\left(1+\frac{h}{x}\right)}{(\log_e a)\cdot h} \quad \left[\because \log_a \lambda = \frac{\log_e \lambda}{\log_e a}\right]$$

$$\Rightarrow f'(x) = \frac{1}{\log_e a} \lim_{h\to 0} \frac{\log_e\left(1+\frac{h}{x}\right)}{x\left(\frac{h}{x}\right)}$$

$$= \frac{1}{x \log_e a} \quad \left[\because \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = 1\right]$$

EXAMPLE |14| Find the derivative of the following function by using first principle. **[NCERT Exemplar]**

(i) $\sin x$ (ii) $\sec x$

(iii) $\tan x$

Sol. (i) Let $f(x) = \sin x$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right) \times \sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\cdot\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \cdot \lim_{\frac{h}{2}\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \quad \left[\therefore h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \times 1 \quad \left[\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \cos(x+0) = \cos x \quad [\text{putting } h = 0]$$

$$\therefore \frac{d}{dx}(\sin x) = \cos x$$

(ii) Let $f(x) = \sec x$

By using first principle of derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{\sec(x+h)-\sec x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \times \cos x \cdot \cos(x+h)}$$

$$= \lim_{h \to 0} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \cdot \sin \frac{(x - x - h)}{2}}{h \cdot \cos x \cdot \cos(x+h)} \right]$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h \to 0} \left[\frac{-2 \sin\left(x + \frac{h}{2}\right) \cdot \left(-\sin \frac{h}{2}\right)}{h \cdot \cos x \cos(x+h)} \right]$$

$$= \lim_{h \to 0} \frac{\sin\left(x + \frac{h}{2}\right)}{\cos(x+h) \cdot \cos x} \cdot \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}$$

$$= \frac{\sin x}{\cos^2 x} \times 1 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

(iii) Let $f(x) = \tan x$

Then, by first principle of derivative, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}$$

$$= 1 \cdot \frac{1}{\cos(x+0)\cos x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

Hence, $f'(x)$ or $\frac{d}{dx}(\tan x) = \sec^2 x$

EXAMPLE |15| Find the derivative of $f(x) = \tan(ax + b)$, by first principle. **[NCERT Exemplar]**

Sol. We have, $f(x) = \tan(ax + b)$

By first principle of derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\tan[a(x+h)+b] - \tan(ax+b)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\sin(ax+ah+b)}{\cos(ax+ah+b)} - \frac{\sin(ax+b)}{\cos(ax+b)}}{h}$$

$$= \lim_{h \to 0} \frac{\left[\frac{\sin(ax+ah+b)\cos(ax+b) - \sin(ax+b)}{\cos(ax+ah+b)}\right]}{h\cos(ax+b)\cos(ax+ah+b)}$$

$$= \lim_{h \to 0} \frac{a\sin(ah)}{a \cdot h\cos(ax+b)\cos(ax+ah+b)}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$= \lim_{h \to 0} \frac{a}{\cos(ax+b)\cos(ax+ah+b)} \lim_{ah \to 0} \frac{\sin ah}{ah}$$

$$[\because h \to 0, \Rightarrow ah \to 0]$$

$$= \frac{a}{\cos^2(ax+b)} \times 1 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= a \sec^2(ax+b)$$

EXAMPLE |16| Differentiate the function $\cos(x^2 + 1)$ by the first principle. **[NCERT Exemplar]**

Sol. Let $f(x) = \cos(x^2 + 1)$

We know by first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\cos[(x+h)^2 + 1] - \cos(x^2+1)}{h}$$

$$[\because f(x) = \cos(x^2+1)]$$

$$= \lim_{h \to 0} \frac{-2\sin\frac{(x+h)^2 + 1 + x^2 + 1}{2}\sin\frac{(x+h)^2 + 1 - (x^2+1)}{2}}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{-2\sin\frac{(x+h)^2 + x^2 + 2}{2}\sin\frac{(x+h)^2 - x^2}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\left\{-2\sin\frac{(x+h)^2 + x^2 + 2}{2}\left[\frac{\sin\frac{(x+h)^2 - x^2}{2}}{\left[\frac{(x+h)^2 - x^2}{2}\right]}\right]\right\}}{h \times \frac{(x+h)^2 - x^2}{2}}$$

$$= \lim_{h \to 0} \frac{-2\sin\frac{(x+h)^2 + x^2 + 2}{2} \times \frac{x^2 + h^2 + 2xh - x^2}{2}}{h}$$

$$\left[\because \lim_{h \to 0} \frac{\sin\left(\frac{(x+h)^2 - x^2}{2}\right)}{\frac{(x+h)^2 - x^2}{2}} = 1\right]$$

$$= \lim_{h \to 0} \frac{-2\sin\left[\frac{(x+h)^2 + x^2 + 2}{2}\right](2x+h)\frac{h}{2}}{h}$$

$$= -2\sin\left(\frac{2x^2+2}{2}\right) \times 2x \times \frac{1}{2} = -2x \sin(x^2+1)$$

EXAMPLE |17| By using first principle, find the derivative of function $f(x) = \sin^2 x$.

Sol. We have, $f(x) = \sin^2 x$

By using first principle of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin^2(x+h) - \sin^2 x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin(x+h) - \sin x)(\sin(x+h) + \sin x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]\left[2\sin\left(\frac{2x+h}{2}\right)\cos\left(\frac{h}{2}\right)\right]}{h}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2} \text{ and } \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}} \times \lim_{h \to 0} 2\sin\left(\frac{2x+h}{2}\right)\cos\frac{h}{2}$$

$$= \cos\left(\frac{2x+0}{2}\right) \times 1 \times 2\sin\left(\frac{2x+0}{2}\right)\cos 0$$

$$= \cos x \times 2\sin x \times 1 = \sin 2x$$

EXAMPLE |18| Find the derivative of $\sin\sqrt{x}$ functions by using first principle.

Sol. Let $f(x) = \sin\sqrt{x}$

By using first principle of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin\sqrt{x+h} - \sin\sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{(x+h)-x}$$

$$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{(\sqrt{x+h})^2 - (\sqrt{x})^2}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+h}-\sqrt{x}}{2}} \times \lim_{h \to 0} \cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right) \times \lim_{h \to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= 1 \times \cos(\sqrt{x}) \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}\cos\sqrt{x}$$

$$\left[\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1\right]$$

EXAMPLE |19| Find the derivative of $\sqrt{\cos x}$

Sol. (i) Let $f(x) = \sqrt{\cos x}$

By using first principle of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{\cos(x+h)} - \sqrt{\cos x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{\cos(x+h)} - \sqrt{\cos x}}{h} \times \frac{(\sqrt{\cos(x+h)} + \sqrt{\cos x})}{(\sqrt{\cos(x+h)} + \sqrt{\cos x})}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h(\sqrt{\cos(x+h)} + \sqrt{\cos x})}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h(\sqrt{\cos(x+h)} + \sqrt{\cos x})}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -\lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{(\sqrt{\cos(x+h)} + \sqrt{\cos x})} \times \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\frac{\sin\left(\frac{2x}{2}\right)}{\sqrt{\cos(x)} + \sqrt{\cos(x)}} \times 1 = -\frac{\sin x}{2\sqrt{\cos x}}$$

EXAMPLE |20| Find the derivative of $x\sin x$ from first principle. **[NCERT]**

Sol. We have, $f(x) = x\sin x$

By using first principle of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \; f'(x) = \lim_{h \to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)[\sin x \cdot \cos h + \cos x \cdot \sin h] - x\sin x}{h}$$

$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$$= \lim_{h\to 0} \frac{\begin{bmatrix} x\sin x \cdot \cos h + x \cdot \cos x \cdot \sin h + h\sin x \cdot \cos h \\ + h\cos x \cdot \sin h - x\sin x \end{bmatrix}}{h}$$

$$= \lim_{h\to 0} \frac{\begin{bmatrix} x\sin x(\cos h - 1) + x \cdot \cos x \cdot \sin h \\ + h(\sin x \cdot \cos h + \cos x \cdot \sin h) \end{bmatrix}}{h}$$

$$= \lim_{h\to 0} \frac{x\sin x(\cos h - 1)}{h} + \lim_{h\to 0} x \cdot \cos x \cdot \frac{\sin h}{h} + \lim_{h\to 0} \frac{h(\sin x \cdot \cos h + \cos x \cdot \sin h)}{h}$$

$$= x\sin x \lim_{h\to 0}\left[\frac{-(1-\cos h)}{h}\right] + x \cdot \cos x(1) + \sin x$$

$$= -x\sin x \cdot \lim_{\frac{h}{2}\to 0} \frac{2\sin^2 \frac{h}{2}}{h \times \frac{h}{4}} \times \frac{h}{4} + x \cos x + \sin x$$

$$= -x \cdot \sin x \cdot \frac{2}{4} \lim_{\frac{h}{2}\to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 \times h + x \cos x + \sin x$$

$$= -x\sin x \cdot \frac{1}{2}(1) \times 0 + x \cdot \cos x + \sin x \qquad \left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$= x\cos x + \sin x$$

EXAMPLE |21| Differentiate $e^{\sqrt{\tan x}}$ from first principles.

Sol. Let $f(x) = e^{\sqrt{\tan x}}$

Then, $f(x+h) = e^{\sqrt{\tan(x+h)}}$

$$\because \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{\sqrt{\tan(x+h)}} - e^{\sqrt{\tan x}}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} e^{\sqrt{\tan x}} \left\{\frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{h}\right\}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \lim_{h\to 0} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}} \times \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} \right\}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\tan x} \times 1 \times \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h\to 0} \frac{\sin h}{h\cos(x+h)\cos x} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \frac{1}{\cos^2 x} \times \frac{1}{2\sqrt{\tan x}}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \frac{e^{\sqrt{\tan x}}}{2\sqrt{\tan x}} \sec^2 x$$

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 Let f is a real valued function and a is a point in its domain of definition, then the derivative of f at a is defined by

(a) $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ (b) $\lim_{h\to 0} \frac{f(a) - f(a+h)}{h}$

(c) $\lim_{h\to 0} h\ f(a+h) - f(a)$ (d) $\lim_{h\to 0} \frac{h}{f(a+h) - f(a)}$

2 If $y = f(x)$ is the function, then derivative of f at any x is denoted by

(a) $f'(x)$ (b) $\frac{dy}{dx}$

(c) $D(f(x))$ (d) All of these

3 The derivative of $f(x) = 3$ at $x = 0$ and at $x = 3$ are

(a) negative (b) zero

(c) different (d) not defined

SHORT ANSWER Type I Questions

4 Find the derivative of $f(x) = ax^2 + bx + c$, where a, b and c are non-zero constant, by first principle. **[NCERT Exemplar]**

5 Find the derivative of $x^2 - 2$ at $x = 10$ from first principle.

6 By using first principle, find the derivative of $99x$ at $x = 100$. **[NCERT]**

SHORT ANSWER Type II Questions

Directions (Q. Nos. 7-36) *Find the derivative of the following functions from first principle.*

7 $x^3 - 27$ **8** $\frac{1}{x^3}$

9 $\sqrt{4-x}$ **10** $x^{2/3}$

11 $(x^2+1)(x-5)$

12 $\dfrac{ax+b}{cx+d}$

13 e^{2x}

14 a^{2x}

15 $\log x^2$

16 $\log(x+k)$

17 $\cos x$

18 $\operatorname{cosec} x$

19 $\cot x$

20 $\sin(x+1)$

21 $\cos\left(x-\dfrac{\pi}{8}\right)$

22 $\sin(2x-3)$

23 $\cos\sqrt{x}$

24 $\sec\sqrt{x}$

25 $\tan\sqrt{x}$

26 $\sin(x^2+1)$

27 $\tan(x^2+1)$

28 $\dfrac{\sin x}{x}$

29 $\cos^2 x$

30 $x\cos x$

31 $x\tan x$

32 $x\sec x$

33 xe^x

34 $e^{\sqrt{\cot x}}$

35 $e^{\sqrt{\sec x}}$

36 $(\sin x - \cos x)$

HINTS & ANSWERS

1. (a) If f is a real valued function and a is a point in its domain of definition, the derivative of f at a is defined by $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ provided this limit exists.

2. (d) There are different notations to denote the derivative of a function. Sometimes $f'(x)$ is denoted by $\dfrac{d}{dx}(f(x))$ or if $y=f(x)$, it is denoted by $\dfrac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by $D(f(x))$.

3. (b) $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{3-3}{h}=0$

Similarly, $f'(3)=\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$

$$=\lim_{h\to 0}\frac{3-3}{h}=0$$

4. $f'(x)=\lim\limits_{h\to 0}\dfrac{a(x+h)^2+b(x+h)+c-ax^2-bx-c}{h}$

Ans. $b+2ax$

5. $f'(10)=\lim\limits_{h\to 0}\dfrac{[(10+h)^2-2]-[(10)^2-2]}{h}$ **Ans.** 20

6. $f'(x)=\lim\limits_{h\to 0}\dfrac{99(x+h)-99x}{h}$ **Ans.** 99

7. $f'(x)=\lim\limits_{h\to 0}\dfrac{[(x+h)^3-27]-(x^3-27)}{h}$

$[\because f(x)=x^3-27]$

$$=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{h^3+3xh(x+h)}{h}$$

$$=\lim_{h\to 0}\frac{h[h^2+3x(x+h)]}{h}$$

Ans. $3x^2$

8. $f'(x)=\lim\limits_{h\to 0}\dfrac{\dfrac{1}{(x+h)^3}-\dfrac{1}{x^3}}{h}=\lim\limits_{h\to 0}\dfrac{x^3-(x+h)^3}{(x+h)^3x^3h}$

$$=\lim_{h\to 0}\frac{x^3-[x^3+h^3+3xh(x+h)]}{(x+h)^3x^3h}$$

$$=\lim_{h\to 0}\frac{-h[h^2+3x(x+h)]}{(x+h)^3x^3h}$$ **Ans.** $\dfrac{-3}{x^4}$

9. $f'(x)=\lim\limits_{h\to 0}\dfrac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h}$

$$\Rightarrow \frac{d}{dx}f(x)=\lim_{h\to 0}\frac{\{\sqrt{4-(x+h)}-\sqrt{4-x}\}\{\sqrt{4-(x+h)}+\sqrt{4-x}\}}{h\{\sqrt{4-(x+h)}+\sqrt{4-x}\}}$$

Ans. $\dfrac{-1}{2\sqrt{4-x}}$

10. $f'(x)=\lim\limits_{h\to 0}\left[\dfrac{(x+h)^{2/3}-x^{2/3}}{h}\right]$

$$=\lim_{(x+h)\to x}\left[\frac{(x+h)^{2/3}-x^{2/3}}{(x+h)-x}\right]$$ **Ans.** $\dfrac{2}{3}x^{-1/3}$

11. $f'(x)=\lim\limits_{h\to 0}\dfrac{((x+h)^2+1)(x+h-5)-(x^2+1)(x-5)}{h}$

Ans. $3x^2-10x+1$

12. $f'(x)=\lim\limits_{h\to 0}\dfrac{1}{h}\left[\dfrac{a(x+h)+b}{c(x+h)+d}-\dfrac{ax+b}{cx+d}\right]$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(ax+ah+b)(cx+d)-(ax+b)(cx+ch+d)}{\{c(x+h)+d\}(cx+d)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{adh-bch}{\{c(x+h)+d\}(cx+d)}\right]$$ **Ans.** $\dfrac{ad-bc}{(cx+d)^2}$

13. $f'(x)=\lim\limits_{h\to 0}\dfrac{e^{2x+2h}-e^{2x}}{h}$

$$=\lim_{h\to 0}\frac{e^{2x}(e^{2h}-1)}{2h}\times 2$$

Ans. $2e^{2x}$

14. $f'(x)=\lim\limits_{h\to 0}\dfrac{a^{2(x+h)}-a^{2x}}{h}$

$$=\lim_{h\to 0}\frac{a^{2x}(a^{2h}-1)}{2h}\times 2$$ **Ans.** $2a^{2x}\log a$

15. $f'(x) = \lim_{h\to 0}\frac{\log(x+h)^2 - \log x^2}{h}$

$$= \lim_{h\to 0}\frac{2\log(x+h) - 2\log x}{h}$$

$$= \lim_{h\to 0}\frac{2\log\left[1+\frac{h}{x}\right]}{\frac{h}{x}} \times \frac{1}{x}$$ **Ans.** $\frac{2}{x}$

16. $f'(x) = \lim_{h\to 0}\frac{\log(x+k+h) - \log(x+k)}{h}$

$$= \lim_{h\to 0}\frac{\log\left(1+\frac{h}{x+k}\right)}{\frac{h}{(x+k)}\cdot(x+k)}$$ **Ans.** $\frac{1}{x+k}$

17. $f'(x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h\to 0}\left\{-\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\}$$ **Ans.** $-\sin x$

18. $f'(x) = \lim_{h\to 0}\frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$

$$= \lim_{h\to 0}\frac{\sin x - \sin(x+h)}{h\sin x\sin(x+h)} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin x\sin(x+h)h}$$

Ans. $-\operatorname{cosec} x\cot x$

19. $f'(x) = \lim_{h\to 0}\frac{\cot(x+h) - \cot x}{h}$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{\sin x\cos(x+h) - \sin(x+h)\cos x}{h\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\frac{\sin(-h)}{h\sin(x+h)\sin x}$$

Ans. $-\operatorname{cosec}^2 x$

20. $f'(x) = \lim_{h\to 0}\frac{\sin(x+h+1) - \sin(x+1)}{h}$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\frac{2x+h+2}{2}\sin\frac{h}{2}}{2\times\frac{h}{2}}$$ **Ans.** $\cos(x+1)$

21. $f'(x) = \lim_{h\to 0}\frac{\cos\left(x+h-\frac{\pi}{8}\right) - \cos\left(x-\frac{\pi}{8}\right)}{h}$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{x+h-\frac{\pi}{8}+x-\frac{\pi}{8}}{2}\right)\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{2x-2\left(\frac{\pi}{8}\right)+h}{2}\right)\sin\frac{h}{2}}{2\times\frac{h}{2}}$$ **Ans.** $-\sin\left(x-\frac{\pi}{8}\right)$

22. $2\cos(2x-3)$

23. $f'(x) = \lim_{h\to 0}\frac{\cos\sqrt{x+h} - \cos\sqrt{x}}{h}$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{(x+h)-x}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)}{(\sqrt{x+h}+\sqrt{x})} \times \lim_{h\to 0}\frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{2\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}$$

Ans. $-\frac{1}{2}\frac{\sin\sqrt{x}}{\sqrt{x}}$.

24. $f'(x) = \lim_{h\to 0}\frac{\sec\sqrt{x+h} - \sec\sqrt{x}}{h}$

$$= \lim_{h\to 0}\frac{\cos\sqrt{x} - \cos\sqrt{x+h}}{h\cos\sqrt{x}\cos\sqrt{x+h}}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{\sqrt{x}+\sqrt{x+h}}{2}\right)\sin\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}{\{(x+h)-x\}\cos\sqrt{x}\cos\sqrt{x+h}}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\frac{\sqrt{x}+\sqrt{x+h}}{2}\right)}{(\sqrt{x+h}+\sqrt{x})\cos\sqrt{x}\cos\sqrt{x+h}}$$

$$\times \lim_{h\to 0}\frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}{(\sqrt{x+h}-\sqrt{x})}$$

Ans. $\frac{\sec x\tan x}{2\sqrt{x}}$

25. $f'(x) = \lim_{h\to 0}\frac{\frac{\sin\sqrt{x+h}}{\cos\sqrt{x+h}} - \frac{\sin\sqrt{x}}{\cos\sqrt{x}}}{h}$

$$= \lim_{h\to 0}\frac{\sin\sqrt{x+h}\cdot\cos\sqrt{x} - \sin\sqrt{x}\cdot\cos\sqrt{x+h}}{h\cdot\cos\sqrt{x+h}\cos\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\sin(\sqrt{x+h}-\sqrt{x})}{h\cos\sqrt{x}\cdot\cos\sqrt{x+h}} \times \frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}-\sqrt{\sqrt{x}}}$$

$$= \lim_{h\to 0} \frac{1}{\cos\sqrt{x}\cdot\cos\sqrt{x+h}} \times \lim_{h\to 0} \frac{\sin(\sqrt{x+h}-\sqrt{x})}{\sqrt{x+h}-\sqrt{x}} \times \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Ans. $\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

26. $f'(x) = \lim_{h\to 0} \frac{\sin((x+h)^2+1)-\sin(x^2+1)}{h}$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{(x+h)^2+x^2+2}{2}\right)\sin\left(\frac{(x+h)^2-x^2}{2}\right)}{h}$$

$$= \lim_{h\to 0} 2\cos\left(\frac{2x^2+h^2+2hx+2}{2}\right) \times \lim_{h\to 0} \frac{\sin\left(\frac{h(h+2x)}{2}\right)}{h(h+2x)} \times \lim_{h\to 0}(h+2x)$$

Ans. $2x\cos(x^2+1)$

27. $f'(x) = \lim_{h\to 0} \frac{\tan((x+h)^2+1)-\tan(x^2+1)}{h}$

$$= \lim_{h\to 0} \frac{\left[\sin((x+h)^2+1)\cos(x^2+1) - \cos((x+h)^2+1)\sin(x^2+1)\right]}{h\cdot\cos((x+h)^2+1)\cos(x^2+1)}$$

$$= \lim_{h\to 0} \frac{\sin((x+h)^2+1-x^2-1)}{h\cos((x+h)^2+1)\cos(x^2+1)}$$ **Ans.** $2x\sec^2(x^2+1)$

28. $f'(x) = \lim_{h\to 0} \frac{\frac{\sin(x+h)}{x+h} - \frac{\sin x}{x}}{h}$

$$= \lim_{h\to 0} \frac{x[\sin(x+h)-\sin x] - h\sin x}{h\cdot x(x+h)}$$

$$= \lim_{h\to 0} \frac{x\left[2\cdot\cos\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)\right] - h\sin x}{h\cdot x(x+h)}$$

$$= \lim_{h\to 0} \frac{x\left[2\cdot\sin\frac{h}{2}\cdot\cos\left(x+\frac{h}{2}\right)\right] - h\sin x}{h\cdot x(x+h)}$$

$$= \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)}{(x+h)} - \lim_{h\to 0} \frac{\sin x}{x(x+h)}$$

Ans. $\frac{\cos x}{x} - \frac{\sin x}{x^2}$

29. $f'(x) = \lim_{h\to 0} \frac{\cos^2(x+h)-\cos^2 x}{h}$

$$= \lim_{h\to 0}(\cos(x+h)+\cos x)\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$= 2\cos x \lim_{h\to 0} - \frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

Ans. $-\sin 2x$

30. $f'(x) = \lim_{h\to 0}\frac{1}{h}[(x+h)\cos(x+h) - x\cos x]$

$$= \lim_{h\to 0}\frac{1}{h}[x\{\cos(x+h)-\cos x\} + h\cos(x+h)]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[x\left\{-2\sin\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right\} + h\cos(x+h)\right]$$

$$= \lim_{h\to 0}\left[-2x\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h} + \cos(x+h)\right]$$

$$= -2x\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\frac{1}{2} + \lim_{h\to 0}\cos(x+h)$$

Ans. $\cos x - x\sin x$

31. $f'(x) = \lim_{h\to 0}\frac{[(x+h)\tan(x+h) - x\tan x]}{h}$

$$= \lim_{h\to 0}\frac{x[\tan(x+h)-\tan x] + h\tan(x+h)}{h}$$

$$= \lim_{h\to 0}\frac{x[\sin(x+h)\cos x - \cos(x+h)\sin x]}{h\cos(x+h)\cos x} + \lim_{h\to 0}\tan(x+h)$$

$$= x\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x} \times \lim_{h\to 0}\frac{\sin h}{h} + \tan x$$

Ans. $x\sec^2 x + \tan x$

32. $\sec x + x\sec x\tan x$

33. $\lim_{h\to 0}\frac{(xe^{x+h} - xe^x) + he^{x+h}}{h}$

$$= xe^x\lim_{h\to 0}\left(\frac{e^h-1}{h}\right) + \lim_{h\to 0}e^{x+h}$$

Ans. $(x+1)e^x$

34. $\frac{-e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}\cdot\operatorname{cosec}^2 x$

35. $\frac{e^{\sqrt{\sec x}}}{2\sqrt{\sec x}}\cdot\sec x\tan x$

36. $\cos x + \sin x.$

|TOPIC 6|
Algebra of Derivative of Functions

Let f and g be two functions such that their derivatives are defined in a common domain. Then,

(i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

(ii) Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

(iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

This is also known as Leibnitz product rule of derivative.

(iv) Derivative of quotient of two functions is given by the following quotient rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}, g(x)\neq 0$$

Note

$\frac{d}{dx}[c\cdot f(x)]=c\frac{d}{dx}f(x)$

Some Important Theorems

Theorem 1 *Derivative of* $f(x)=x^n$ *is* nx^{n-1} *for any positive integer* n.

Note

The above theorem is true for all powers of x i.e. n can be any real number.

Theorem 2 Derivative of Polynomial Functions

Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$

be a polynomial function, where a_i's are all real numbers and $a_n\neq 0$.

Then, the derivative function is given by

$$\frac{d}{dx}[f(x)]=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots+2a_2x+a_1$$

Proof By definition of first principle of derivative, we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h} \quad \ldots(i)$$

By binomial theorem, we have

$$(x+h)^n={}^nC_0x^n+{}^nC_1x^{n-1}h+{}^nC_2x^{n-2}h^2+\ldots+{}^nC_nh^n$$

$$\Rightarrow \quad (x+h)^n=x^n+nhx^{n-1}+\frac{n(n-1)}{2}h^2x^{n-2}+\ldots+h^n$$

$$\Rightarrow (x+h)^n-x^n=nhx^{n-1}+\frac{n(n-1)}{2}h^2x^{n-2}+\ldots+h^n$$

On putting this value in Eq. (i), we get

$$f'(x)=\lim_{h\to 0}\frac{nhx^{n-1}+\frac{n(n-1)}{2}h^2x^{n-2}+\ldots+h^n}{h}$$

$$=\lim_{h\to 0}\frac{h\left[nx^{n-1}+\frac{n(n-1)}{2}hx^{n-2}+\ldots+h^{n-1}\right]}{h}$$

$$=\lim_{h\to 0} nx^{n-1}+\frac{n(n-1)}{2}hx^{n-2}+\ldots+h^{n-1}$$

$$=nx^{n-1}$$

Hence, $f'(x)$ or $\frac{d}{dx}f(x)=nx^{n-1}$.

EXAMPLE |1| Differentiate $2x^3-4x^2+6x+8$ w.r.t. x.

Sol. Let $y=2x^3-4x^2+6x+8$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(2x^3-4x^2+6x+8)$$

$$=2\frac{d}{dx}(x^3)-4\frac{d}{dx}(x^2)+6\frac{d}{dx}(x)+\frac{d}{dx}(8)$$

$$=2(3x^2)-4(2x)+6(1)+0 \quad \left[\because \frac{d}{dx}(x^n)=nx^{n-1}\right]$$

$$=6x^2-8x+6$$

EXAMPLE |2| Find the derivative of $f(x)=1+x+x^2+x^3+\ldots+x^{50}$ at $x=1$. **[NCERT]**

Sol. Given, $f(x)=1+x+x^2+x^3+\ldots+x^{50}$

On differentiating both sides w.r.t. x, we get

$$f'(x)=0+1+2x+3x^2+\ldots+50x^{49}$$

At $x=1$,

$$f'(1)=1+2(1)+3(1)^2+\ldots+50(1)^{49}$$

$$=1+2+3+\ldots+50$$

$$=\frac{(50)(51)}{2}=1275 \quad \left[\because \Sigma n=\frac{n(n+1)}{2}\right]$$

EXAMPLE |3| If $u = 7t^4 - 2t^3 - 8t - 5$, then find $\frac{du}{dt}$ at $t = 2$.

Sol. We have, $u = 7t^4 - 2t^3 - 8t - 5$

On differentiating both sides w.r.t. t, we get

$$\frac{du}{dt} = \frac{d}{dt}[7t^4 - 2t^3 - 8t - 5]$$

$$= 7(4t^3) - 2(3t^2) - 8(1) - 0 \quad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

$$= 28t^3 - 6t^2 - 8$$

Now, $\left(\frac{du}{dt}\right)_{t=2} = 28(2)^3 - 6(2)^2 - 8$

$$= 224 - 24 - 8 = 192$$

EXAMPLE |4| For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1.$$

Prove that $f'(1) = 100 f'(0)$. **[NCERT]**

Sol. Given, $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$

$$\Rightarrow \quad f'(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0$$

[if $f(x) = x^n$, then $f'(x) = nx^{n-1}$]

$$\Rightarrow \quad f'(x) = x^{99} + x^{98} + \dots + x + 1 \quad \text{...(i)}$$

On putting $x = 1$, we get

$$f'(1) = (1)^{99} + (1)^{98} + \dots + 1 + 1 = \underbrace{1 + 1 + 1 \dots + 1 + 1}_{100 \text{ times}}$$

$$\Rightarrow \quad f'(1) = 100 \quad \text{...(ii)}$$

Again, putting $x = 0$, we get

$$f'(0) = 0 + 0 + \dots + 0 + 1$$

$$\Rightarrow \quad f'(0) = 1 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), $f'(1) = 100 f'(0)$ **Hence proved.**

EXAMPLE |5| Differentiate the following functions w.r.t. x.

(i) $\left(x + \frac{1}{x}\right)^3$

(ii) $(ax + b)(cx + d)^2$ (iii) $\frac{px^2 + qx + r}{(ax + b)}$ **[NCERT]**

Sol. (i) Let $y = \left(x + \frac{1}{x}\right)^3 = x^3 + 3x \times \frac{1}{x^2} + 3x^2 \times \frac{1}{x} + \frac{1}{x^3}$

$[\because (A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3]$

$$= x^3 + \frac{3}{x} + 3x + \frac{1}{x^3}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}\left(\frac{3}{x}\right) + \frac{d}{dx}(3x) + \frac{d}{dx}\left(\frac{1}{x^3}\right)$$

$$= 3x^2 + 3\frac{d}{dx}(x^{-1}) + 3 \times 1 + \frac{d}{dx}(x^{-3})$$

$$= 3x^2 + \frac{-3}{x^2} + 3 - \frac{3}{x^4}$$

(ii) Let $y = (ax + b)(cx + d)^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = (ax + b)\frac{d}{dx}(cx + d)^2 + (cx + d)^2 \frac{d}{dx}(ax + b)$$

[using product rule of derivative]

$$= (ax + b)\frac{d}{dx}(c^2x^2 + d^2 + 2cxd) + (cx + d)^2 (a \times 1 + 0)$$

$[\because (A + B)^2 = A^2 + 2AB + B^2]$

$$= (ax + b)(c^2(2x) + 0 + 2c \times 1 \times d) + (cx + d)^2 \times a$$

$$= (ax + b)(2c^2x + 2cd) + a(cx + d)^2$$

$$= (ax + b)\,2c\,(cx + d) + a(cx + d)^2$$

$$= (cx + d)[2c(ax + b) + a(cx + d)]$$

$$= (cx + d)(2acx + 2bc + acx + ad)$$

$$= (cx + d)(3acx + 2bc + ad)$$

(iii) Let $y = \frac{px^2 + qx + r}{(ax + b)}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left[(ax + b)\frac{d}{dx}(px^2 + qx + r) - (px^2 + qx + r)\frac{d}{dx}(ax + b)\right]}{(ax + b)^2}$$

[by quotient rule]

$$= \frac{(ax + b)(2px + q + 0) - (px^2 + qx + r)(a \times 1 + 0)}{(ax + b)^2}$$

$$= \frac{(ax + b)(2px + q) - (px^2 + qx + r)\,a}{(ax + b)^2}$$

$$= \frac{(2apx^2 + aqx + 2bpx + bq) - (apx^2 + aqx + ra)}{(ax + b)^2}$$

$$= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - aqx - ra}{(ax + b)^2}$$

$$\frac{dy}{dx} = \frac{apx^2 + 2bpx + bq - ra}{(ax + b)^2}$$

3 If $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}$, then the value of $\frac{dy}{dx}-y+\frac{x^n}{n!}$ is

(a) 0 (b) n
(c) $2n$ (d) None of these

4 If a, b are fixed non-zero constants, then the derivative of $\frac{a}{x^4}-\frac{b}{x^2}$ is $ma+nb$, where

(a) $m=4x^3, n=\frac{-2}{x^3}$

(b) $m=\frac{-4}{x^5}, n=\frac{2}{x^3}$

(c) $m=\frac{-4}{x^5}, n=\frac{-2}{x^3}$

(d) $m=4x^3, n=\frac{2}{x^3}$

VERY SHORT ANSWER Type Questions

5 Find the derivative of $2x^4+x$. **[NCERT Exemplar]**

6 Find $f'(x)$, if $f(x)=(x-2)^2(2x-3)$.

7 For some constants a and b, find the derivative of the following functions.
(Each part carries 1 mark)

(i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$

(iii) $\frac{x-a}{x-b}$ (iv) $\frac{1}{ax^2+bx+c}$

8 Find the derivative of the following functions.
(Each part carries 1 mark)

(i) $\frac{x^4+x^3+x^2+1}{x}$ (ii) $\frac{x^2+x+1}{\sqrt{x}}$

(iii) $(5x^3+3x-1)(x-1)$ (iv) $x^{-4}(3-4x^{-5})$

(v) $\frac{2}{x+1}-\frac{x^2}{3x-1}$ **[NCERT]**

(vi) $\frac{3x+4}{5x^2-7x+9}$ **[NCERT Exemplar]**

SHORT ANSWER Type Questions

9 If $y=\sqrt{x}+\frac{1}{\sqrt{x}}$, then find $\frac{dy}{dx}$ at $x=1$.
[NCERT Exemplar]

10 If $y=\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, then find $\frac{dy}{dx}$.
[NCERT Exemplar]

LONG ANSWER Type Questions

11 Find the derivative of $\frac{(x-1)(x-2)}{(x-3)(x-4)}$.

12 If $f(x)=1-x+x^2-x^3+\ldots-x^{99}+x^{100}$, then find $f'(1)$.

13 If $f(x)=x^{100}+x^{99}+\ldots+x+1$, then find $f'(1)$.
[NCERT Exemplar]

HINTS & ANSWERS

1. (b) We have, $f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots\ldots+a_1x+a_0$ be a polynomial function, where a_i's are real numbers and $a_n\neq 0$.
Then, the derivative function is given by

$$\frac{df(x)}{dx}=na_n x^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots\ldots+2a_2x+a_1$$

2. (b) Let $y=\frac{x^n-a^n}{x-a}$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx}=\frac{(x-a)\frac{d}{dx}(x^n-a^n)-(x^n-a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$=\frac{(x-a)[nx^{n-1}-0]-(x^n-a^n)(1-0)}{(x-a)^2}$$

$$=\frac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$$

3. (a) We have,

$$y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}$$

$$\therefore \frac{dy}{dx}=0+\frac{1}{1!}+\frac{2x}{2!}+\frac{3x^2}{3!}+\ldots+\frac{nx^{n-1}}{n!}=y-\frac{x^n}{n!}$$

$$\Rightarrow \frac{dy}{dx}-y+\frac{x^n}{n!}=0$$

4. (c) Let $y=\frac{a}{x^4}-\frac{b}{x^2}$

$$\Rightarrow \quad y=a\,x^{-4}-bx^{-2}$$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx}=a\frac{d}{dx}(x^{-4})-b\frac{d}{dx}(x^{-2})$$

$$=-\frac{4a}{x^5}+\frac{2b}{x^3} \qquad \left[\because \frac{d}{dx}(x^n)=nx^{n-1}\right]$$

5. $8x^3+1$

6. $f(x)=2x^3-11x^2+20x-12$

Ans. $6x^2-22x+20$

7. (i) Let $f(x)=(x-a)(x-b)$

$= x^2-(a+b)x+ab$

Ans. $2x-a-b$

(ii) Let $f(x)=(ax^2+b)^2=a^2x^4+b^2+2abx^2$

Ans. $4ax(ax^2+b)$

(iii) $\frac{a-b}{(x-b)^2}$ (iv) $\frac{-(2ax+b)}{(ax^2+bx+c)^2}$

8. (i) $\frac{d}{dx}\left(\frac{x^4+x^3+x^2+1}{x}\right)=\frac{d}{dx}\left(x^3+x^2+x+\frac{1}{x}\right)$

Ans. $\frac{3x^4+2x^3+x^2-1}{x^2}$

(ii) $\frac{d}{dx}\left(\frac{x^2+x+1}{\sqrt{x}}\right)=x^{3/2}+x^{1/2}+x^{-1/2}$

Ans. $\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$

(iii) $20x^3-15x^2+6x-4$

(iv) $\frac{d}{dx}(x^{-4}(3-4x^{-5}))=\frac{d}{dx}(3x^{-4}-4x^{-9})$

Ans. $-12x^{-5}+36x^{-10}$

(v) $\frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$

(vi) Let $y=\frac{3x+4}{5x^2-7x+9}$

$$\frac{dy}{dx}=\frac{(5x^2-7x+9)\frac{d}{dx}(3x+4)-(3x+4)\frac{d}{dx}(5x^2-7x+9)}{(5x^2-7x+9)^2}$$

Ans. $\frac{55-15x^2-40x}{(5x^2-7x+9)^2}$

9. $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}-\frac{1}{2x^{3/2}}$ **Ans.** 0

10. $y=\frac{x^2+1}{x^2-1}$ **Ans.** $-\frac{4x}{(x^2-1)^2}$

11. Let $y=\frac{(x-1)(x-2)}{(x-3)(x-4)}=\frac{x^2-3x+2}{x^2-7x+12}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\left[(x^2-7x+12)\frac{d}{dx}(x^2-3x+20)-(x^2-3x+2)\frac{d}{dx}(x^2-7x+12)\right]}{(x^2-7x+12)^2}$$

Ans. $\frac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$

12. $f'(x)=0-1+2x-3x^2+\ldots-99x^{98}+100x^{99}$

$=-1+2x-3x^2+\ldots-99x^{98}+100x^{99}$

$\therefore f'(1)=-1+2-3+\ldots-99+100$

$=(-1-3-5-\ldots-99)+(2+4+\ldots+100)$

$=-\frac{50}{2}[2\times1+(50-1)2]+\frac{50}{2}[2\times2+(50-1)2]$

$=50$

13. $f'(x)=100x^{99}+99x^{98}+\ldots+1+0$

$=100x^{99}+99x^{98}+\ldots+1$

Now, $f'(1)=100+99+\ldots+1$

$=\frac{100}{2}[2\times100+(100-1)(-1)]$

$=5050$

|TOPIC 7|
Derivative of Trigonometric Functions

To find the derivative of trigonometric functions, we use the algebra of derivative and the following formulae

(i) $\frac{d}{dx}(\sin x) = \cos x$ (ii) $\frac{d}{dx}(\cos x) = -\sin x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$ (iv) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

EXAMPLE |1| Differentiate the following functions w.r.t. x.

(i) $\cos(x+a)$ (ii) $\frac{\sin(x+a)}{\cos x}$

Sol. (i) Let $y = \cos(x+a)$

$y = \cos x \cos a - \sin x \sin a$

$[\because \cos(A+B) = \cos A \cos B - \sin A \sin B]$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = \cos a \frac{d}{dx}(\cos x) - \sin a \frac{d}{dx}(\sin x)$

$= \cos a(-\sin x) - \sin a(\cos x)$

$= -\cos a \sin x - \sin a \cos x = -\sin(x+a)$

$[\because \sin A \cos B + \cos A \sin B = \sin(A+B)]$

(ii) Let $y = \frac{\sin(x+a)}{\cos x} = \frac{\sin x \cos a + \cos x \sin a}{\cos x}$

$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$

$= \frac{\sin x \cos a}{\cos x} + \frac{\cos x \sin a}{\cos x} = \cos a \tan x + \sin a$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = \cos a \frac{d}{dx}(\tan x) + \frac{d}{dx}(\sin a)$

$\frac{dy}{dx} = \cos a \frac{d}{dx}(\tan x) + \frac{d}{dx}(\sin a)$

$= \cos a \sec^2 x + 0$ $\left[\because \frac{d}{dx}(\text{constant}) = 0\right]$

$= \frac{\cos a}{\cos^2 x}$ $\left[\because \sec x = \frac{1}{\cos x}\right]$

EXAMPLE |2| Find the derivative of the following functions. **[NCERT]**

(i) $f(x) = \sin^2 x$ (ii) $f(x) = \sin 2x$

(iii) $f(x) = (\cos^2 x - \sin^2 x)$

Sol. (i) We have, $f(x) = \sin^2 x$

On differentiating both sides w.r.t. x, we get

$f'(x) = \frac{d}{dx}(\sin x \sin x)$

$= (\sin x)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(\sin x)$

[using product rule of derivative]

$= (\cos x)\sin x + \sin x(\cos x)$

$= 2\sin x \cos x = \sin 2x$

(ii) We have, $f(x) = \sin 2x$

$f(x) = 2\sin x \cos x$

On differentiating both sides w.r.t. x, we get

$f'(x) = 2\left[\sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)\right]$

[using product rule of derivative]

$= 2[\sin x \cdot (-\sin x) + \cos x \cdot \cos x]$

$= 2(-\sin^2 x + \cos^2 x)$

$= 2(\cos^2 x - \sin^2 x) = 2\cos 2x$

(iii) We have, $f(x) = (\cos^2 x - \sin^2 x)$

$f(x) = (\cos x - \sin x)(\cos x + \sin x)$

On differentiating both sides w.r.t. x, we get

$f'(x) = (\cos x - \sin x)\frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x)$
$\frac{d}{dx}(\cos x - \sin x)$

[using product rule of derivative]

$= (\cos x - \sin x)(-\sin x + \cos x) + (\cos x + \sin x)$
$(-\sin x - \cos x)$

$= (\cos x - \sin x)(\cos x - \sin x) - (\cos x + \sin x)$
$(\cos x + \sin x)$

$= (\cos x - \sin x)^2 - (\cos x + \sin x)^2$

$= (\cos^2 x + \sin^2 x - 2\sin x \cos x) - (\cos^2 x + \sin^2 x$
$+ 2\sin x \cos x)$

$= -2\sin x \cos x - 2\sin x \cos x$

$= -\sin 2x - \sin 2x = -2\sin 2x$

$= -2\sin x \cos x - 2\sin x \cos x$

$= -\sin 2x - \sin 2x = -2\sin 2x$

EXAMPLE |3| Find the derivative of following functions.

(i) $x^2 \cos x$ (ii) $x^2 \sec x$ (iii) $x^2 \tan x$

First, consider the given function as y. Then, use product rule of derivative to get the result.

Sol. (i) Let $y = x^2 \cos x$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cos x) = x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2)$

[using product rule of derivative]

$= x^2(-\sin x) + \cos x(2x) = 2x\cos x - x^2 \sin x$

(ii) Let $y = x^2 \sec x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(x^2)$$

[using product rule of derivative]

$$= x^2 \cdot \sec x \tan x + \sec x(2x)$$

(iii) Let $y = x^2 \tan x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(x^2)$$

[using product rule of derivative]

$$= x^2 \sec^2 x + \tan x\,(2x)$$

EXAMPLE |4| Find the derivative of $\frac{4x + 5\sin x}{3x + 7\cos x}$.

Sol. Let $f(x) = \frac{4x + 5\sin x}{3x + 7\cos x}$ **[NCERT]**

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{\left[(3x + 7\cos x)\frac{d}{dx}(4x + 5\sin x) - (4x + 5\sin x) \times \frac{d}{dx}(3x + 7\cos x)\right]}{(3x + 7\cos x)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}\right]$$

$$= \frac{(3x + 7\cos x)(4 + 5\cos x) - (4x + 5\sin x)[3 + 7(-\sin x)]}{(3x + 7\cos x)^2}$$

$$= \frac{\left[(12x + 15x \cdot \cos x + 28 \cdot \cos x + 35\cos^2 x) - (12x - 28x \cdot \sin x + 15\sin x - 35\sin^2 x)\right]}{(3x + 7\cos x)^2}$$

$$= \frac{\left[35(\sin^2 x + \cos^2 x) + 28x \cdot \sin x + 15x \cdot \cos x + 28\cos x - 15\sin x\right]}{(3x + 7\cos x)^2}$$

$$= \frac{35 + 28x\sin x + 15x \cdot \cos x - 15\sin x + 28\cos x}{(3x + \cos x)^2}$$

EXAMPLE |5| Differentiate $(x + \sec x) \cdot (x - \tan x)$ w.r.t. x. **[NCERT]**

Sol. We have, $y = (x + \sec x)(x - \tan x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(x + \sec x)(x - \tan x)]$$

$$= (x + \sec x) \cdot \frac{d}{dx}(x - \tan x) + (x - \tan x)\frac{d}{dx}(x + \sec x)$$

[using product rule of derivative]

$$= (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \cdot \tan x)$$

EXAMPLE |6| Differentiate the following functions w.r.t. x.

(i) $1 - 2\sin^2 \frac{x}{2}$ (ii) $2\cos^2 \frac{x}{2} - 1$

(ii) $3\sin\frac{x}{3} - 4\sin^3\frac{x}{3}$ (iv) $e^{\log_e \sin x}$ (v) $\log_e e^{\cos x}$

Sol. (i) Let $y = 1 - 2\sin^2 \frac{x}{2}$

$\Rightarrow$ $y = \cos x$ $[\because \cos 2\theta = 1 - 2\sin^2\theta]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x) = -\sin x$$

(ii) Let $y = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow$ $y = \cos x$ $[\because \cos 2\theta = 2\cos^2\theta - 1]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x) = -\sin x$$

(iii) Let $y = 3\sin\frac{x}{3} - 4\sin^3\frac{x}{3} = \sin 3\left(\frac{x}{3}\right)$

$[\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$

$\Rightarrow$ $y = \sin x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

(iv) Let $y = e^{\log_e \sin x}$

$\Rightarrow$ $y = \sin x$ $[\because e^{\log_e f(x)} = f(x)]$

On differentiating both side w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

(v) Let $y = \log_e e^{\cos x}$

$\Rightarrow$ $y = \cos x$ $\left[\begin{array}{l}\because \log m^n = n\log m \text{ and } \log_e e = 1 \\ \because \log_e e^{\cos x} = \cos x \log_e e = \cos x\end{array}\right]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE |7| Differentiate $(ax^2 + \sin x)(p + q\cos x)$ w.r.t. x.

Sol Let $y = (ax^2 + \sin x)(p + q\cos x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$

[using product rule of derivative]

$$= (ax^2 + \sin x)(0 - q\sin x) + (p + q\cos x)(2ax + \cos x)$$

$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

EXAMPLE |8| If $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$, $x \in \left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right)$, then find $\dfrac{dy}{dx}$.

Sol. We have, $y = \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x}$

$$\left[\because \cos 2\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \text{ and } \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow 2\cos^2\theta = 1 + \cos 2\theta\right]$$

$$\Rightarrow \quad y = |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

Now, $$y = \begin{cases} \tan x, & x \in \left(0, \dfrac{\pi}{2}\right) \\ -\tan x, & x \in \left(\dfrac{\pi}{2}, \pi\right) \end{cases}$$

$$\therefore \quad \frac{dy}{dx} = \begin{cases} \sec^2 x, & \text{if } x \in \left(0, \dfrac{\pi}{2}\right) \\ -\sec^2 x, & \text{if } x \in \left(\dfrac{\pi}{2}, \pi\right) \end{cases}$$

EXAMPLE |9| Find the derivative of the following functions.

(i) $\dfrac{\sin x + \cos x}{\sin x - \cos x}$ (ii) $\dfrac{\sec x + \tan x}{\sec x - \tan x}$

[NCERT; NCERT Exemplar]

First, consider the given function as $f(x)$. Then, use quotient rule of derivative to find the required derivative.

Sol. Let $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{\left[(\sin x - \cos x)\dfrac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\dfrac{d}{dx}(\sin x - \cos x)\right]}{(\sin x - \cos x)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2}\right]$$

$$= \frac{\left[(\sin x - \cos x)\left[\dfrac{d}{dx}(\sin x) + \dfrac{d}{dx}(\cos x)\right] - (\sin x + \cos x)\left[\dfrac{d}{dx}(\sin x) - \dfrac{d}{dx}(\cos x)\right]\right]}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$\left[\because \frac{d}{dx}(\sin x) = \cos x \text{ and } \frac{d}{dx}(\cos x) = -\sin x\right]$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= -\left[\frac{(\sin x - \cos x)^2 + (\sin x + \cos x)^2}{(\sin x - \cos x)^2}\right]$$

$$= \frac{\left[-(\sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x)\right]}{(\sin x - \cos x)^2}$$

$$= \frac{-2(\sin^2 x + \cos^2 x)}{(\sin x - \cos x)^2} = \frac{-2}{(\sin x - \cos x)^2}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

(ii) Let $y = \dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{\dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}} \Rightarrow y = \dfrac{1+\sin x}{1-\sin x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sec x + \tan x}{\sec x - \tan x}\right) = \frac{d}{dx}\left(\frac{1+\sin x}{1-\sin x}\right)$$

$$= \frac{(1-\sin x)\dfrac{d}{dx}(1+\sin x) - (1+\sin x)\dfrac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

[using quotient rule of derivative]

$$= \frac{(1-\sin x)(0+\cos x) - (1+\sin x)(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{2\cos x}{(1-\sin x)^2}$$

EXAMPLE |10| If $y = \left\{\dfrac{1-\tan x}{1+\tan x}\right\}$, then show that $\dfrac{dy}{dx} = \dfrac{-2}{1+\sin 2x}$.

Sol. We have, $y = \dfrac{1-\tan x}{1+\tan x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\tan x)\cdot\dfrac{d}{dx}(1-\tan x) - (1-\tan x)\cdot\dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

[using quotient rule of derivative]

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} = \frac{-2}{(\cos^2 x)(1+\tan^2 x + 2\tan x)}$$

$$= \frac{-2}{(\cos^2 x)\left\{1 + \dfrac{\sin^2 x}{\cos^2 x} + \dfrac{2\sin x}{\cos x}\right\}} = \frac{-2}{(1+\sin 2x)}$$

Hence proved.

TOPIC PRACTICE 7

OBJECTIVE TYPE QUESTIONS

1 The derivative of $\frac{\sec x - 1}{\sec x + 1}$ is $\frac{p}{(\sec x + 1)^2}$, where p is equal to

(a) $2\sec x \tan x$ (b) $2\sin x \tan x$
(c) $2\operatorname{cosec} x \cot x$ (d) $2\cos x \cot x$

2 If a, p and q are fixed non-zero constants, then the derivative of $(ax^2 + \sin x)(p + q\cos x)$ is $A(ax^2 + \sin x) + B(p + q\cos x)$, where A and B respectively are

(a) $-q\sin x, 2ax + \cos x$ (b) $2ax + \cos x, -q\sin x$
(c) $2ax - \cos x, q\sin x$ (d) $q\sin x, 2ax - \cos x$

3 If the derivative of $(x + \cos x)(x - \tan x)$ is $A(x + \cos x) + B(x - \tan x)$, then A and B respectively are

(a) $\tan^2 x, 1 - \sin x$ (b) $-\tan^2 x, 1 - \sin x$
(c) $\tan^2 x, \sin x - 1$ (d) $-\tan^2 x, \sin x - 1$

4 The derivative of $\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$ is

(a) $\frac{x\sin\left(\frac{\pi}{4}\right)[2\sin x + x\cos x]}{\sin^2 x}$

(b) $\frac{x\cos\left(\frac{\pi}{4}\right)[2\sin x - x\cos x]}{\sin^2 x}$

(c) $\frac{x\sin\left(\frac{\pi}{4}\right)[2\sin x - x\cos x]}{\cos^2 x}$

(d) $\frac{x\cos\left(\frac{\pi}{4}\right)[2\sin x + x\cos x]}{\cos^2 x}$

5 If $f(x) = x\sin x$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(a) 0 (b) 1 (c) -1 (d) 1/2

VERY SHORT ANSWER Type Questions

6 Find the derivative of the following functions. (Each part carries 1 mark)

(i) $\sin x \cos x$
(ii) $5\sec x + 4\cos x$
(iii) $3\cot x + 5\operatorname{cosec} x$
(iv) $5\sin x + 6\cos x + 7$
(v) $2\tan x - 7\sec x$ **[NCERT]**
(vi) $\operatorname{cosec} x \cot x$ (vii) $(\sec x - 1)(\sec x + 1)$
(viii) $(ax^2 + \cot x)(p + q\cos x)$ **[NCERT Exemplar]**
(ix) $(x^2 + 1)\cos x$ (x) $\sin(x + a)$

SHORT ANSWER Type I Questions

7 Find the derivative of the following functions. (Each part carries 2 marks)

(i) $(x + \cos x)(x - \tan x)$ (ii) $x^4(5\sin x - 3\cos x)$

(iii) $\frac{x}{1+\tan x}$ (iv) $(3x+5)(1+\tan x)$

(v) $\frac{x^5 - \cos x}{\sin x}$ (vi) $\frac{a + b\sin x}{c + d\cos x}$

(vii) $\frac{\cos x}{1+\sin x}$ (viii) $\frac{\cos(x-a)}{\cos x}$

8 If $y = \frac{\sin(x+9)}{\cos x}$, then find $\frac{dy}{dx}$ at $x=0$.

9 Find the derivative of $\frac{1-\cos x}{1+\cos x}$.

SHORT ANSWER Type II Questions

10 If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then find $\frac{dy}{dx}$ at $x=0$. **[NCERT Exemplar]**

11 Find the derivative of $x^2 \sin x + \cos 2x$.

12 Find the derivative of the following functions.

(i) $\frac{x\tan x}{(\sec x + \tan x)}$ (ii) $\frac{\sin x - x\cos x}{x\sin x + \cos x}$

HINTS & ANSWERS

1. (a) Let $y = \frac{\sec x - 1}{\sec x + 1}$

Then $\frac{dy}{dx} = \frac{(\sec x + 1)\frac{d}{dx}(\sec x - 1) - (\sec x - 1)\frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$

$= \frac{2\sec x \tan x}{(\sec x + 1)^2}$

2. (a) Let $y = (ax^2 + \sin x)(p + q\cos x)$, then by product rule

$\frac{dy}{dx} = -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$

3. (b) Let $y = (x + \cos x)(x - \tan x)$, then

$\frac{dy}{dx} = (x + \cos x)\frac{d}{dx}(x - \tan x) + (x - \tan x)\frac{d}{dx}(x + \cos x)$

$= -(x + \cos x)\tan^2 x + (x - \tan x)(1 - \sin x)$

$[\because \sec^2 x - \tan^2 x = 1]$

4. (b) Let $y = \frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

$\Rightarrow y = \left(\cos\frac{\pi}{4}\right)\frac{x^2}{\sin x}$

$$\frac{dy}{dx} = \cos\frac{\pi}{4}\cdot\frac{d}{dx}\left(\frac{x^2}{\sin x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\cos\frac{\pi}{4}\right)\frac{\sin x\frac{d}{dx}(x^2) - x^2\left(\frac{d}{dx}\sin x\right)}{\sin^2 x}$$

(by quotient rule)

$$= \frac{x\cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

5. (b) Clearly $f'(x) = x\cos x + \sin x$

$$\therefore \quad f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$$

6. (i) $\cos 2x$ (ii) $5\sec x\tan x - 4\sin x$

(iii) $-3\text{cosec}^2 x - 5\,\text{cosec}\,x\cot x$ (iv) $5\cos x + 6\sin x$

(v) $2\sec^2 x - 7\sec x\tan x$ (vi) $-\text{cosec}^3 x - \cot^2 x\,\text{cosec}\,x$

(vii) $(\sec x - 1)(\sec x + 1) = \sec^2 x - 1 = \tan^2 x$

$= \tan x\cdot\tan x$ **Ans.** $2\tan x\sec^2 x$

(viii) Let $y = (ax^2 + \cot x)(p + q\cos x)$.

Then, $\frac{dy}{dx} = (ax^2+\cot x)\frac{d}{dx}(p+q\cos x)$

$+(p+q\cos x)\frac{d}{dx}(ax^2+\cot x)$

$= -q\sin x(ax^2+\cot x) + (p+q\cos x)(2ax - \text{cosec}^2 x)$

(ix) $-x^2\sin x - \sin x + 2x\cos x$

(x) $\sin(x+a) = \sin x\cos a + \cos x\sin a$ **Ans.** $\cos(x+a)$

7. (i) $\frac{d}{dx}\{(x + \text{cosec}x)(x - \tan x)^2\}$

$= (x+\cos x)\cdot\frac{d}{dx}(x-\tan x) + (x-\tan x)\cdot\frac{d}{dx}(x+\cos x)$

Ans. $(x+\cos x)(-\tan^2 x) + (x-\tan x)(1-\sin x)$

(ii) $x^3[5x\cos x + 3x\sin x + 20\sin x - 12\cos x]$

(iii) Let $y = \frac{x}{1+\tan x}$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\tan x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$$

(iv) Let $y = (3x+5)(1+\tan x)$.

Then, $\frac{dy}{dx} = \frac{d}{dx}[(3x+5)(1+\tan x)]$

$= (3x+5)\frac{d}{dx}(1+\tan x) + (1+\tan x)\frac{d}{dx}(3x+5)$

$= 3x\sec^2 x + 5\sec^2 x + 3\tan x + 3$

(v) $\frac{d}{dx}\left(\frac{x^5 - \cos x}{\sin x}\right)$

$$= \frac{\sin x\frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x(5x^4 + \sin x) - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{5x^4\sin x - x^5\cos x + 1}{\sin^2 x}$$

(vi) Let $y = \frac{a + b\sin x}{c + d\cos x}$. Then,

$$\frac{dy}{dx} = \frac{\left[(c+d\cos x)\frac{d}{dx}(a+b\sin x) - (a+b\sin x)\frac{d}{dx}(c+d\cos x)\right]}{(c+d\cos x)^2}$$

$$= \frac{(c+d\cos x)(b\cos x) - (a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$= \frac{bc\cos x + ad\sin x + bd}{(c+d\cos x)^2}$$

(vii) $\frac{-1}{1+\sin x}$

(viii) $y = \frac{\cos x\cos a + \sin x\sin a}{\cos x} = \cos a + \tan x\sin a$

Ans. $\sec^2 x\cdot\sin a$

8. Given, $y = \frac{\sin(x+9)}{\cos x} = \frac{\sin x\cos 9 + \cos x\sin 9}{\cos x}$

$= \tan x\cos 9 + \sin 9$

$\therefore \frac{dy}{dx} = \sec^2 x\cos 9$ **Ans.** $\cos 9$

9. Use quotient rule of derivative.

Ans. $\frac{2\sin x}{(1+\cos x)^2}$

10. Given, $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

$$\therefore \quad \frac{dy}{dx} = \frac{\left[(\sin x - \cos x)(\cos x - \sin x) - (\sin + \cos x)(\cos x + \sin x)\right]}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = -2$$

11. $\frac{d}{dx}(x^2\sin x + \cos 2x) = \frac{d}{dx}(x^2\sin x) + \frac{d}{dx}(\cos 2x)$

$= \frac{d}{dx}(x^2\sin x) + \frac{d}{dx}(2\cos^2 x - 1)$

$= \frac{d}{dx}(x^2\sin x) + \frac{d}{dx}(2\cos x\cos x - 1)$

$= x^2\cdot\cos x + \sin x\,2x$

$+\left[2\left(\cos x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\cos x)\right) - \frac{d}{dx}(1)\right]$

$= x^2\cos x + 2x\sin x - 2\sin 2x$

12. (i) $\frac{x\sec x(\sec x - \tan x) + \tan x}{(\sec x + \tan x)}$ (ii) $\frac{x^2}{(x\sin x + \cos x)^2}$

SUMMARY

- If $x \to a$, then $f(x) \to l$, where l is a real number, then l is the limit of the function $f(x)$, i.e. $\lim\limits_{x \to 0} f(x) = l$
- The value of $f(x)$, which is dictated by the values of $f(x)$, when x tends to a from the right is called right hand limit of function f. Thus, LHL $= \lim\limits_{x \to a^-} f(x)$
- The value of $f(x)$ which is dictated by the values of $f(x)$, when x tends to a from the left is called left hand limit of function f. Thus, RHL $= \lim\limits_{x \to a^+} f(x)$
- If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$ i.e. LHL = RHL. Then, limit exists and denoted by $\lim\limits_{x \to a} f(x)$.
- **Algebra of Limits** Let f and g be two real functions with common domain 'D' such that $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then,

  (i) $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$. (ii) $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x)$, where c is a constant.

  (iii) $\lim\limits_{x \to a} [f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$. (iv) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, where $\lim\limits_{x \to a} g(x) \neq 0$ for $x \in D$.

- **Limits of Rational Functions** $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{f(a)}{g(a)}$

  (i) If $g(a) = 0$ and $f(a) \neq 0$, then limit does not exist.(ii) If $g(a) = 0$ and $f(a) = 0$, then we can find limit.

- $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ where, n is any positive integer.
- **Limits of Trigonometric Functions**

  (i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x \to 0} \dfrac{\tan x}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

- **Limits of Exponential Function** $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1.$
- **Limits of Logarithmic Function** $\lim\limits_{x \to 0} \dfrac{\log_e (1 + x)}{x} = 1$
- **Derivative at a Point** Suppose f is real valued function and a is a point in its domain. Then, derivative of f at a is defined by $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$, provided this limit exists.

- **First Principle of Derivative**

  If f is a real valued function, then $\dfrac{d}{dx} f(x) = f'(x) = \lim\limits_{x \to 0} \dfrac{f(x + h) - f(x)}{h}$, provided this limit exists.

- **Algebra of Derivative of Function** Let $f(x)$ and $g(x)$ be two functions, then

  (i) $\dfrac{d}{dx}[f(x) \pm g(x)] = \dfrac{d}{dx} f(x) \pm \dfrac{d}{dx} g(x)$ (ii) $\dfrac{d}{dx}[f(x) \cdot g(x)] = f(x) \dfrac{d}{dx} g(x) + g(x) \dfrac{d}{dx} f(x)$.

  (iii) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x) \dfrac{d}{dx} f(x) - f(x) \dfrac{d}{dx} g(x)}{[g(x)]^2}$

- **Derivative of Polynomial Function** If $f(x) = x^n$, then $\dfrac{d}{dx} f(x) = nx^{n-1}$.

- **Derivative of Trigonometric Functions**

  (i) $\dfrac{d}{dx}(\sin x) = \cos x$ (ii) $\dfrac{d}{dx}(\cos x) = -\sin x$ (iii) $\dfrac{d}{dx}(\tan x) = \sec^2 x$ (iv) $\dfrac{d}{dx}(\cot x) = -\text{cosec}^2 x$

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. Let $f(x)=\begin{cases}\dfrac{k\cos\pi}{\pi-2x}, & x\neq\dfrac{\pi}{2}\\ 3, & x=\dfrac{\pi}{2}\end{cases}$ and if $\lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$, then the value of k is

(a) 3 (b) 4 (c) 5 (d) 6

2. If $f(x)=\begin{cases}x^2-1, & x\le 1\\ -x^2-1, & x>1\end{cases}$, then

(a) LHL ≠ RHL , at $x=1$
(b) LHL = RHL = − 2 , at $x=1$
(c) LHL = RHL = −1, at $x=1$
(d) None of the above

3. $\lim_{n\to\infty}\dfrac{1+2+3+\dots\dots+n}{n^2}$, $n\in N$ is equal to **[NCERT Exemplar]**

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

4. $\lim_{x\to 0}\dfrac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$ is equal to

(a) 2 (b) 0 (c) 1 (d) −1

5. $\lim_{x\to 0}\dfrac{10^x-2^x-5^x+1}{x\tan x}$ is equal to

(a) log 10
(b) log 5 + log 2 − log10
(c) (log 5)(log 2)
(d) (log 10) (log 5) (log 2)

6. The derivative of the function $f(x)=3x$ at $x=2$ is

(a) 0 (b) 1 (c) 2 (d) 3

7. If $\dfrac{d}{dx}\left\{\dfrac{1}{f(x)}\right\}=\dfrac{p}{q}$, where p and q respectively are

(a) $f'(x), \{f(x)\}^2$
(b) $-f'(x), \{f(x)\}^2$
(c) $-f'(x), f(x)$
(d) $f(x), f'(x)$

8. If $y=\dfrac{1+1/x^2}{1-1/x^2}$, then $\dfrac{dy}{dx}$ is **[NCERT Exemplar]**

(a) $\dfrac{-4x}{(x^2-1)^2}$
(b) $\dfrac{-4x}{x^2-1}$
(c) $\dfrac{1-x^2}{4x}$
(d) $\dfrac{4x}{x^2-1}$

9. Which of the following is/are true?

I. The derivative of $f(x)=\sin 2x$ is $2(\cos^2 x-\sin^2 x)$.

II. The derivative of $g(x)=\cot x$ is $-\operatorname{cosec}^2 x$.

(a) Both I and II are true
(b) Only I is true
(c) Only II is true
(d) Both I and II are false

10. If $y=\left\{\dfrac{1-\tan x}{1+\tan x}\right\}$ and $\dfrac{dy}{dx}=\dfrac{k}{1+\sin 2x}$, then the value of k is

(a) 1 (b) 2
(c) −1 (d) − 2

VERY SHORT ANSWER Type Questions

11. Evaluate the following.
(Each part carries 1 mark)

(i) $\lim_{x\to 1}\dfrac{x^2+3x+2}{x^2+1}$

(ii) $\lim_{x\to 4}\dfrac{x^3-2x^2-9x+4}{x^2-2x-8}$

(iii) $\lim_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(iv) $\lim_{x\to 0}\dfrac{(1+x)^5-1}{3x+5x^2}$

(v) $\lim_{x\to 2}\dfrac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}}$

(vi) $\lim_{x\to -3}\dfrac{x^3+27}{x^5+243}$

12. Find the positive integer n so that $\lim_{x\to 3}\dfrac{x^n-3^n}{x-3}=108$.

SHORT ANSWER Type I Questions

13. Differentiate $\sin^3 x$ w.r.t. x from first principle.

14. For the function f, given by $f(x)=x^2-6x+8$, prove that $f'(5)-3f'(2)=f'(8)$.

15. Evaluate the following.
(Each part carries 2 marks)

(i) $\lim_{x\to 0}\frac{x\,2^x - x}{1-\cos x}$

(ii) $\lim_{x\to 0}\frac{e^x - x - 1}{x - 5}$

(iii) $\lim_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$

(iv) $\lim_{x\to 0}\frac{3\sin x - \sin 3x}{x(\cos 2x - \cos 4x)}$

16. Evaluate the following.
(Each part carries 2 marks)

(i) $\lim_{x\to \alpha}\frac{x\sin\alpha - \alpha\sin x}{x-\alpha}$

(ii) $\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{\sin^2 2x}$

SHORT ANSWER Type II Questions

17. If $y=\frac{2\sin^2 x + 3\cos x - 1}{\sin x}$, then find $\frac{dy}{dx}$.

18. If $y = a\sin x + b\cos x$, show that

$$y^2+\left(\frac{dy}{dx}\right)^2 = a^2+b^2.$$

HINTS & ANSWERS

1. (d) We have,

$$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-}\frac{k\cos x}{\pi-2x}=3$$

$$\Rightarrow \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}=3$$

$$\Rightarrow \lim_{h\to 0}\frac{k\sin h}{2h}=3$$

$$\Rightarrow k=6$$

2. (a) Given, $f(x)=\begin{cases} x^2-1, & x\le 1\\ -x^2-1, & x>1\end{cases}$

At $x=1$,

RHL $= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} -(1+h)^2-1$ (put $x=1+h$)

$=-2$

LHL $= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$ (put $x = 1-h$)

$= \lim_{h\to 0}(1-h)^2-1$

$=0$

$\Rightarrow$ RHL $\ne$ LHL

3. (c) Clearly, $\lim_{n\to\infty}\frac{1+2+3+\ldots+n}{n^2}$

$$=\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=\frac{1}{2}$$

4. (c) Given, $\lim_{x\to 0}\frac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$

$$=\lim_{x\to 0}\frac{\sin x(\sqrt{x+1}+\sqrt{1-x})}{(x+1)-(1-x)}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}(\sqrt{x+1}+\sqrt{1-x})$$

$$=\frac{1}{2}\cdot 1\cdot 2 = 1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

5. (c) $\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}$

$$=\lim_{x\to 0}\frac{5^x(2^x-1)-(2^x-1)}{x\tan x}$$

$$=\lim_{x\to 0}\frac{5^x-1}{x}\times\frac{2^x-1}{x}\times\frac{x}{\tan x}$$

$$=(\log 5)(\log 2)$$

6. (d) We have,

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{3(2+h)-3(2)}{h}$$

$$=\lim_{h\to 0}\frac{3h}{h}=3$$

7. (b) Let $\phi(x)=\frac{1}{f(x)}$. Then, $\phi(x+h)=\frac{1}{f(x+h)}$

$$\therefore \frac{d}{dx}\{\phi(x)\}=\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}$$

$$\Rightarrow \frac{d}{dx}\{\phi(x)\}=\lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$$

$$\Rightarrow \frac{d}{dx}\{\phi(x)\}=-\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\times\lim_{h\to 0}\frac{1}{f(x)f(x+h)}$$

$$\Rightarrow \frac{d}{dx}\{\phi(x)\} = -f'(x) \times \frac{1}{f(x)f(x)}$$

$$\Rightarrow \frac{d}{dx}\{\phi(x)\} = \frac{-f'(x)}{\{f(x)\}^2}$$

8. (a) We have, $y = \frac{x^2+1}{x^2-1}$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)\cdot 2x - (x^2+1)\cdot 2x}{(x^2-1)^2}$$

$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

9. (a) I. Recall the trigonometric rule $\sin 2x = 2\sin x \cos x$.

Thus, $\frac{df(x)}{dx} = \frac{d}{dx}(2\sin x \cos x) = 2\frac{d}{dx}(\sin x \cos x)$

$$= 2(\cos^2 x - \sin^2 x)$$

II. $g(x) = \cot x = \frac{\cos x}{\sin x}$

$$\Rightarrow \quad \frac{d}{dx}(g(x)) = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

10. (d) We have, $y = \frac{1-\tan x}{1+\tan x}$

$$\therefore \frac{dy}{dx} = \frac{(1+\tan x)\frac{d}{dx}(1-\tan x) - (1-\tan x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} = \frac{-2}{\cos^2 x(1+\tan^2 x + 2\tan x)}$$

$$= \frac{-2}{\cos^2 x + \sin^2 x + 2\sin x \cos x}$$

$$= \frac{-2}{1+\sin 2x}$$

$\therefore k = -2$

11. (i) By $\lim_{x\to 1} \frac{x^2+3x+2}{x^2+1} = \frac{1^2+3\times 1+2}{1^2+1} = \frac{6}{2}$

Ans. 3

(ii) $\lim_{x\to 4} \frac{x^3-2x^2-9x+4}{x^2-2x-8}$

$$= \lim_{x\to 4} \frac{(x-4)(x^2+2x-1)}{(x-4)(x+2)} \quad \textbf{Ans. } \frac{23}{6}$$

(iii) $\lim_{x\to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(\sqrt{x})^4-\sqrt{x}}{(\sqrt{x}-1)}$

$$= \lim_{x\to 1} \frac{\sqrt{x}\,[(\sqrt{x})^3-1]}{(\sqrt{x}-1)} = \lim_{x\to 1} \frac{\sqrt{x}\,(\sqrt{x}-1)(x+1+x)}{(\sqrt{x}-1)}$$

$$= \lim_{\sqrt{x}\to 1} \sqrt{x}\,(2x+1)$$

[as $x \to 1$, so $\sqrt{x} \to \sqrt{1}$ i.e. $\sqrt{x} \to 1$] **Ans.** 3

(iv) $\lim_{x\to 0} \frac{(1+x)^5-1}{3x+5x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$$= \frac{\lim_{x\to 0} \frac{(1+x)^5-1}{x}}{\lim_{x\to 0} \frac{3x+5x^2}{x}} = \frac{\lim_{(1+x)\to 1} \frac{(1+x)^5-1}{(1+x)-1}}{\lim_{x\to 0}(3x+5x)} \quad \textbf{Ans. } \frac{5}{3}$$

(v) $\lim_{x\to 2} \frac{x-2}{\sqrt{x^2-4}+\sqrt{x-2}} = \lim_{x\to 2} \frac{x-2}{\sqrt{x-2}\,[\sqrt{x+2}+1]}$

$$= \lim_{x\to 2} \frac{\sqrt{x-2}}{(\sqrt{x+2}+1)} \quad \textbf{Ans. } 0$$

(vi) $\lim_{x\to -3} \frac{x^3+27}{x^5+243} = \lim_{x\to -3} \frac{x^3-(-3)^3}{x^5-(-3)^5}$

$$= \frac{\lim_{x\to -3} \frac{x^3-(-3)^3}{(x-3)}}{\lim_{x\to -3} \frac{x^5-(-3)^5}{(x-3)}}$$

[multiplying and dividing by $(x-3)$]

$$= \frac{3(-3)^{3-1}}{5(-3)^{5-1}} = \frac{3\times 9}{5\times 81}\left[\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right] \textbf{Ans. } \frac{1}{15}$$

12. Given, $\lim_{x\to 3} \frac{x^n-3^n}{x-3} = 108$

$$\Rightarrow \quad n\cdot 3^{n-1} = 108 \quad \left[\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$\Rightarrow \quad n\cdot 3^{n-1} = 4\times 3^{4-1} \quad \textbf{Ans. } n = 4$$

13. Let $f(x) = \sin^3 x$ ...(i)

By definition of first principle of derivative, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin^3(x+h)-\sin^3 x}{h} \quad \text{[from Eq. (i)]}$$

$$= \lim_{h\to 0} \frac{\left[\{\sin(x+h)-\sin x\}\{\sin^2(x+h) + \sin^2 x\} + \sin(x+h)\sin x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)(\sin^2(x+h) + \sin^2 x + \sin(x+h)\sin x)\right]}{h}$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \cdot \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \cdot$$

$$\lim_{h \to 0} [\sin^2(x+h) + \sin^2 x + \sin(x+h)\sin x]$$

$$= 1 \cdot \cos x \cdot (\sin^2 x + \sin^2 x + \sin^2 x)$$

Ans. $3\sin^2 x \cos x$

14. Given, $f(x) = x^2 - 6x + 8$

On differentiating both sides w.r.t., x, we get

$$f'(x) = 2x - 6 \quad \text{...(i)}$$

Now, $f'(5) - 3f'(2) = 2 \times 5 - 6 - 3(2 \times 2 - 6)$ [from Eq. (i)]

$$= 10 - 6 - 12 + 18 = 10 \quad \text{...(ii)}$$

and $f'(8) = 2 \times 8 - 6$ [from Eq. (ii)]

$$= 10 \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get $f'(5) - 3f'(2) = f'(8)$

15. (i) $\lim_{x \to 0} \frac{x 2^x - x}{1 - \cos x} = \lim_{x \to 0} \frac{x(2^x - 1)}{1 - \cos x}$

$$= \lim_{x \to 0} \frac{x^2\left(\frac{2^x - 1}{x}\right)}{1 - \cos x} = \lim_{x \to 0} \frac{x^2\left(\frac{2^x - 1}{x}\right)}{2\sin^2 x/2}$$

$$= 2\lim_{x \to 0} \frac{1}{\left(\frac{\sin x/2}{x/2}\right)^2}\left(\frac{2^x - 1}{x}\right) = 2\log 2$$ **Ans.** $\log 4$

(ii) Use direct substitution method. **Ans.** 0

16. (i) Let $x = \alpha + h$, then as $x \to \alpha$, $h \to 0$

Now, $\lim_{x \to \alpha} \frac{x \sin\alpha - \alpha \sin x}{x - \alpha}$

$$= \lim_{h \to 0} \frac{(\alpha + h)\sin\alpha - \alpha\sin(\alpha + h)}{\alpha + h - \alpha}$$

$$= \lim_{h \to 0} \frac{\alpha\sin\alpha + h\sin\alpha - \alpha\sin(\alpha + h)}{h}$$

$$= \lim_{h \to 0} \frac{\alpha \cdot 2\cos\left(\frac{2\alpha + h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h} + \frac{h\sin\alpha}{h}$$

$$= \lim_{h \to 0} \frac{\alpha \cdot 2\cos\left(\frac{2\alpha + h}{2}\right) \cdot \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{-h}{2}\right)} \cdot \left(\frac{-h}{2}\right)}{h} + \sin\alpha$$

Ans. $\sin\alpha - \alpha\cos\alpha$

(ii) Let $\cos x \cos 2x \cos 3x = \frac{1}{2}(2\cos x \cos 3x \cos 2x)$

$$= \frac{1}{2}[(\cos 2x + \cos 4x)\cos 2x]$$

$$= \frac{1}{4}(2\cos^2 2x + 2\cos 2x \cos 4x)$$

$$= \frac{1}{4}(1 + \cos 4x + \cos 2x + \cos 6x)$$

Now, $\lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{4}(1 + \cos 4x + \cos 2x + \cos 6x)}{\sin^2 2x}$$ **Ans.** $\frac{7}{4}$

(iii) $\lim_{x \to 0} \frac{8}{x^8}\left(1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$

$$= \lim_{x \to 0} \frac{8}{x^8}\left[\left(1 - \cos\frac{x^2}{2}\right) - \cos\frac{x^2}{4}\left(1 - \cos\frac{x^2}{2}\right)\right]$$

$$= \lim_{x \to 0} \frac{8}{x^8}\left(1 - \cos\frac{x^2}{2}\right)\left(1 - \cos\frac{x^2}{4}\right)$$

$$= \lim_{x \to 0} \frac{8}{x^8} 2\sin^2\frac{x^2}{4} \cdot 2\sin^2\frac{x^2}{8}$$ **Ans.** $\frac{1}{32}$

(iv) $\lim_{x \to 0} \frac{3\sin x - \sin 3x}{x(\cos 2x - \cos 4x)}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{3\sin x - 3\sin x + 4\sin^3 x}{x\left[2\sin\left(\frac{4x + 2x}{2}\right)\sin\left(\frac{4x - 2x}{2}\right)\right]}$$

$$= \lim_{x \to 0} \frac{2\sin^3 x}{x\sin 3x \sin x} = \frac{2}{3}\lim_{x \to 0} \frac{\left(\frac{\sin x}{x}\right)^2}{\frac{\sin 3x}{3x}}$$ **Ans.** $\frac{2}{3}$

17. Given, $y = \frac{2\sin^2 x + 3\cos x - 1}{\sin x}$

$$= \frac{2\sin^2 x}{\sin x} + 3\frac{\cos x}{\sin x} - \frac{1}{\sin x}$$

$\Rightarrow$ $y = 2\sin x + 3\cot x - \operatorname{cosec} x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\cos x + 3(-\operatorname{cosec}^2 x) - (-\operatorname{cosec} x \cot x)$$

Ans. $2\cos x - 3\operatorname{cosec}^2 x + \operatorname{cosec} x \cdot \cot x$

18. We have, $y = a\sin x + b\cos x$

$$\Rightarrow \quad \frac{dy}{dx} = a\cos x - b\sin x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = (a\cos x - b\sin x)^2$$

$$\Rightarrow \quad y^2 + \left(\frac{dy}{dx}\right)^2 = (a\sin x + b\cos x)^2 + (a\cos x - b\sin x)^2$$

$$= a^2 + b^2$$