04

A number that determine whether the system of linear equations, like $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$ has a unique solution or not, is called determinant. In the previous chapter, we study about matrices and algebra of matrices. Here, we will study determinants, its properties and various applications.

DETERMINANTS

|TOPIC 1|

Determinant

To every square matrix $A=[a_{ij}]$ of order n, a unique number (real or complex) can be associated, which is called determinant of the square matrix. In other words, we can say that if M is the set of square matrices, K is the set of numbers (real or complex) and $f: M \to K$ is defined by $f(A)=k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A. It is denoted by Δ (read as delta) or det (A) or $|A|$

i.e. if $A=[a_{ij}]_{n\times n}$, then $|A|=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Note

(i) For matrix A, $|A|$ is read as determinant of A and not read as modulus of A.
(ii) Only square matrix has determinant.
(iii) In determinant we can determine the value but in matrix we cannot determine the value.
(iv) Here, we will study determinants upto order three only with real entries.

CHAPTER CHECKLIST

- Determinant
- Use of Determinants in Coordinate Geometry
- Minors and Cofactors
- Adjoint and Inverse of a Matrix
- Applications of Determinants and Matrices

Determinants of Matrix of Different Orders and Their Expansions

DETERMINANT OF A MATRIX OF ORDER 1

Let $A=[a]$ be a square matrix of order 1, then $|A|=|a|=a$, i.e. element itself a determinant. e.g. $|3|=3$

DETERMINANT OF A MATRIX OF ORDER 2

Let $A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ be a square matrix of order 2×2.

Then, det (A) or $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$

i.e. determinant of order 2 is equal to the product of diagonal elements minus the product of non-diagonal elements. e.g. $\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}=2\times(-1)-0\times 3=-2$

DETERMINANT OF MATRIX OF ORDER 3

(EXPANSION ALONG ROW OR COLUMN STEP-BY-STEP)

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants.

This is known as expansion of a determinant along a row or a column. There are six ways of expanding a determinant of order 3, corresponding to each of three rows R_1, R_2, R_3 and three columns C_1, C_2, C_3 giving the same value. Let $A=[a_{ij}]_{3\times 3}$ be a square matrix of order 3, then

EXAMPLE |1| Evaluate the following determinant.

(i) $|-12|$ (ii) $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$

(iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$ (iv) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Sol. (i) Clearly, $|-12| = -12$ $[\because |-a| = -a]$

(ii) We have, $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 3 \times 4 - (-2) \times 5$

$\left[\because \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\right]$

$= 12 + 10 = 22$

(iii) We have, $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix} = (x-1)(x^2+x+1) - x^3$

$\left[\because \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\right]$

$= (x^3 - 1) - x^3$ $[\because (a-b)(a^2+ab+b^2) = a^3 - b^3]$

$= -1$

(iv) We have,

$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta \times \cos\theta - (\sin\theta)(-\sin\theta)$

$\left[\because \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}\,a_{12}\right]$

$= \cos^2\theta + \sin^2\theta = 1$ $[\because \cos^2 x + \sin^2 x = 1]$

EXAMPLE |2| Evaluate x if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$.

[Delhi 2016C]

Sol. Given, $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow$ $2 \times 1 - 5 \times 4 = 2x \times x - 6 \times 4$

$\Rightarrow$ $2 - 20 = 2x^2 - 24$

$\Rightarrow$ $-18 = 2x^2 - 24$

$\Rightarrow$ $2x^2 = 6 \Rightarrow x^2 = 3$

$\therefore$ $x = \pm\sqrt{3}$ [taking square root]

EXAMPLE |3| Evaluate the determinant of matrix

$$\begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}.$$

Sol. Let $A = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along First Row (R_1) For expanding the determinant of A along first row, we use the following steps

I. Multiply first element a_{11} of R_1 by $(-1)^{(1+1)}$ [i.e. $(-1)^{\text{sum of suffixes in } a_{11}}$] with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of $|A|$ as a_{11} lies in R_1 and C_1, i.e. $(-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

II. Multiply second element a_{12} of R_1 by $(-1)^{(1+2)}$ [i.e. $(-1)^{\text{sum of suffixes in } a_{12}}$] with the second order determinant obtained by deleting the elements of first row (R_1) and second column (C_2) of $|A|$ as a_{12} lies in R_1 and C_2, i.e. $(-1)^{1+2} a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

III. Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ [i.e. $(-1)^{\text{sum of suffixes in } a_{13}}$] with the second order determinant obtained by deleting the elements of first row (R_1) and third column (C_3) as a_{13} lies in R_1 and C_3, i.e. $(-1)^{1+3} a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

IV. Now, the expansion of determinant of A written as sum of all three terms obtained in steps I, II and III.

$$\det(A) = |A| = (-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $|A| = a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}(a_{21}\,a_{32} - a_{31}\,a_{22})$

$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}\,a_{31}$ ...(i)

Note We can find the determinant of order 3 by using step IV directly.

EXAMPLE |4| Evaluate the determinant

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}.$$

Sol. Given, $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

$= 0 \cdot [0 - (-\sin\beta)(\sin\beta)] - (-\sin\alpha)(\sin\alpha \cdot 0 - \cos\alpha\sin\beta) + \cos\alpha(\sin\alpha\sin\beta - 0)$ [expanding along C_1]

$= 0 + \sin\alpha(-\cos\alpha\sin\beta) + \cos\alpha\sin\alpha\sin\beta$

$$= (-1)^{1+1}(3)\begin{vmatrix}1 & -2\\3 & 1\end{vmatrix} + (-1)^{1+2}(-4)\begin{vmatrix}1 & -2\\2 & 1\end{vmatrix} + (-1)^{1+3}(5)\begin{vmatrix}1 & 1\\2 & 3\end{vmatrix}$$

[expanding along R_1]

$$= (-1)^2 3\,[1\times1 - 3\times(-2)] + (-1)^3(-4)[1\times1 - 2\times(-2)] + (-1)^4 5(3-2)$$

$$= 3(1+6) + 4(1+4) + 5(1) = 21 + 20 + 5 = 46$$

Expansion along First Column (C_1) On expanding the determinant of A along first column (C_1), we get

$$|A| = (-1)^{1+1} a_{11}\begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix} + (-1)^{2+1} a_{21}\begin{vmatrix}a_{12} & a_{13}\\a_{32} & a_{33}\end{vmatrix} + (-1)^{3+1} a_{31}\begin{vmatrix}a_{12} & a_{13}\\a_{22} & a_{23}\end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \quad \ldots(i)$$

or $|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$

or $|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \quad \ldots(ii)$

Clearly, the values of $|A|$ from Eqs. (i) and (ii) are equal.

Hence, expanding a determinant along any row or column gives same value.

Similarly, we can expand along R_2, R_3, C_2 and C_3 step-by-step.

Note

(i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeroes.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or −1 according to $(i+j)$ is even or odd.

Or

While expanding the determinant, we can attach the following '+' and '−' signs.

$$\begin{vmatrix}+ & - & +\\- & + & -\\+ & - & +\end{vmatrix}$$

9 If $\begin{vmatrix}2x & x+3\\2(x+1) & x+1\end{vmatrix} = \begin{vmatrix}1 & 5\\3 & 3\end{vmatrix}$, then find the value of x. **[Delhi 2013C]**

10 If $A = \begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 3\\-1 & 1\end{bmatrix}$, then write the value of $|AB|$. **[Delhi 2015C]**

SHORT ANSWER Type Questions

11 If there are two values of a which makes determinant, $\Delta = \begin{vmatrix}1 & -2 & 5\\2 & a & -1\\0 & 4 & 2a\end{vmatrix} = 86$, then find the sum of these numbers. **[NCERT Exemplar]**

12 Evaluate $\begin{vmatrix}\cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha\\-\sin\beta & \cos\beta & 0\\\sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha\end{vmatrix}$.

$$= -\sin\alpha\cos\alpha\sin\beta + \sin\alpha\cos\alpha\sin\beta = 0$$

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 If $\begin{vmatrix}x & 2\\18 & x\end{vmatrix} = \begin{vmatrix}6 & 2\\18 & 6\end{vmatrix}$, then x is equal to **[NCERT]**

(a) 6 (b) ± 6
(c) −6 (d) zero

2 If $f(x) = \begin{vmatrix}0 & x-a & x-b\\x+a & 0 & x-c\\x+b & x+c & 0\end{vmatrix}$, then **[NCERT Exemplar]**

(a) $f(a) = 0$ (b) $f(b) = 0$
(c) $f(0) = 0$ (d) $f(1) = 0$

3 The determinant $\begin{vmatrix}x & \sin\theta & \cos\theta\\-\sin\theta & -x & 1\\\cos\theta & 1 & x\end{vmatrix}$ is

(a) independent of θ only
(b) independent of x only
(c) independent of both θ and x
(d) None of the above

VERY SHORT ANSWER Type Questions

4 Write value of the determinant $\begin{vmatrix}p & p+1\\p-1 & p\end{vmatrix}$. **[Delhi 2014C]**

5 Evaluate $\begin{vmatrix}\sec\theta & -\tan\theta\\-\tan\theta & \sec\theta\end{vmatrix}$ **[NCERT]**

6 Evaluate $\begin{vmatrix}\cos 15^\circ & \sin 15^\circ\\\sin 75^\circ & \cos 75^\circ\end{vmatrix}$ **[All India 2011]**

7 Evaluate $\begin{vmatrix}\log_3 256 & \log_4 3\\\log_3 8 & \log_4 9\end{vmatrix}$.

8 If $\begin{vmatrix}2x & 5\\8 & x\end{vmatrix} = \begin{vmatrix}6 & -2\\7 & 3\end{vmatrix}$, then find the value of x. **[Delhi 2014]**

4. Let $\Delta = \begin{vmatrix}p & p+1\\p-1 & p\end{vmatrix}$

$= p^2 - (p-1)(p+1)$

$= p^2 - (p^2 - 1)$ $\quad [\because (a+b)(a-b) = a^2 - b^2]$

$= p^2 - p^2 + 1 = 1$

5. We have, $\begin{vmatrix}\sec\theta & -\tan\theta\\-\tan\theta & \sec\theta\end{vmatrix}$

$= \sec\theta\times\sec\theta - (-\tan\theta)\times(-\tan\theta)$

$= \sec^2\theta - \tan^2\theta = 1$ $\quad [\because \sec^2\theta - \tan^2\theta = 1]$

6. We have, $\begin{vmatrix}\cos 15^\circ & \sin 15^\circ\\\sin 75^\circ & \cos 75^\circ\end{vmatrix}$

$= \cos 15^\circ\cos 75^\circ - \sin 15^\circ\sin 75^\circ$

$= \cos(15^\circ + 75^\circ)$

$[\because \cos A\cos B - \sin A\sin B = \cos(A+B)]$

$= \cos 90^\circ = 0$

HINTS & SOLUTIONS

1. (b) Given, $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix} \Rightarrow x^2 - 36 = 36 - 36$

$\Rightarrow \quad x^2 = 36 \Rightarrow x = \pm 6$

2. (c) Clearly, $f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix}$

$= [(a-b)\{2a \cdot (a+c)\}] \neq 0$

$\therefore \quad f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix}$

$= -(b-a)[2b(b-c)]$

$= -2b(b-a)(b-c) \neq 0$

$\therefore \quad f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$

$= a(bc) - b(ac) = abc - abc = 0$

3. (a) Let $\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$

$= x(-x^2 - 1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$

$= -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$

$= -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$= -x^3$, which is independent of θ.

7. We have, $\begin{vmatrix} \log_3 256 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix}$

$= \log_3 256 \times \log_4 9 - \log_3 8 \times \log_4 3$

$= \log_3 2^8 \times \log_4 3^2 - \log_3 2^3 \times \log_4 3$

$= 8\log_3 2 \times 2\log_4 3 - 3\log_3 2 \times \log_4 3$

$[\because \log m^n = n \log m]$

$= 16\log_3 2 \times \frac{1}{2}\log_2 3 - 3\log_3 2 \times \frac{1}{2}\log_2 3$

$\left[\because \log_{a^m} b = \frac{1}{m}\log_a b\right]$

$= 8 - \frac{3}{2} \qquad [\because \log_3 2 \times \log_2 3 = 1]$

$= \frac{13}{2}$

8. Similar as Example 2. [**Ans.** $x = \pm 6$]

9. We have, $\begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$

$\Rightarrow \quad 2x(x+1) - 2(x+1)(x+3) = 3 - 15$

$\Rightarrow \quad 2x^2 + 2x - 2(x^2 + 3x + x + 3) = -12$

$\Rightarrow \quad 2x^2 + 2x - 2x^2 - 6x - 2x - 6 = -12$

$\Rightarrow \quad -6x - 6 = -12$

$\Rightarrow \quad -6x = -12 + 6 = -6$

$\Rightarrow \quad x = 1$

10. We have, $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-2 & 3+2 \\ 3+1 & 9-1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 4 & 8 \end{bmatrix}$

$\Rightarrow \quad |AB| = \begin{vmatrix} -1 & 5 \\ 4 & 8 \end{vmatrix}$

$= -8 - 20 = -28$

11. We have, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$

$\Rightarrow \quad 1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$

[expanding along first column]

$\Rightarrow \quad 2a^2 + 4 + 8a + 40 = 86$

$\Rightarrow \quad 2a^2 + 8a + 44 - 86 = 0$

$\Rightarrow \quad a^2 + 4a - 21 = 0$

$\Rightarrow \quad a^2 + 7a - 3a - 21 = 0$

$\Rightarrow \quad (a+7)(a-3) = 0$

$\Rightarrow \quad a = -7$ and 3

$\therefore$ Required sum $= -7 + 3 = -4$

12. Let $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along R_1, we get

$\Delta = \cos\alpha\cos\beta(\cos\alpha\cos\beta - 0) - \cos\alpha\sin\beta$

$(-\cos\alpha\sin\beta - 0) - \sin\alpha(-\sin^2\beta\sin\alpha - \cos^2\beta\sin\alpha)$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha(\sin^2\beta + \cos^2\beta)$

$= \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha(\sin^2\beta + \cos^2\beta)$

$= (\cos^2\alpha)(1) + (\sin^2\alpha)(1) \qquad [\because \cos^2\theta + \sin^2\theta = 1]$

$= \cos^2\alpha + \sin^2\alpha$

$= 1$

|TOPIC 2|

Use of Determinants in Coordinate Geometry

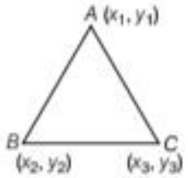
We can use determinants in geometry also. Here, we will use determinant to find the area of triangle and to find

$= \frac{1}{2}\left[3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 8\begin{vmatrix} -4 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} -4 & 2 \\ 5 & 1 \end{vmatrix}\right]$

equation of line.

AREA OF A TRIANGLE

Let the vertices of a ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.



Then, in the form of determinant,

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Note

(i) Area is a positive quantity, so we take the absolute value of determinant.

(ii) If area is given, then take both negative and positive values of determinant for calculation.

EXAMPLE |1| Find the area of the triangle, whose vertices are (3, 8), (− 4, 2) and (5, 1).

Sol. Given vertices of a triangle are (3, 8), (− 4, 2) and (5, 1).

Let $(x_1, y_1) = (3, 8)$, $(x_2, y_2) = (-4, 2)$ and $(x_3, y_3) = (5, 1)$

$$\text{Then, area of triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

EXAMPLE |3| If the points (2, − 3), (λ, − 1) and (0, 4) are collinear, then find the value of λ.

Sol. Given points (2, − 3), (λ, − 1) and (0, 4) are collinear.

So, area of triangle formed by these three points will be zero.

$$\therefore \quad \begin{vmatrix} 2 & -3 & 1 \\ \lambda & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-1-4) + 3(\lambda - 0) + 1(4\lambda - 0) = 0$ [expanding along R_1]

$\Rightarrow \quad 2(-5) + 3\lambda + 4\lambda = 0$

$\Rightarrow \quad 7\lambda - 10 = 0 \Rightarrow \lambda = \frac{10}{7}$

Hence, required value of λ is $\frac{10}{7}$.

EQUATION OF A LINE THROUGH TWO POINTS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points, then the line joining A and B is given by considering any point $P(x, y)$ on the line, so that the points P, A and B are

$$= \frac{1}{2}[3(2-1) - 8(-4-5) + 1(-4-10)]$$

$$= \frac{1}{2}[3 + 72 - 14] = \frac{61}{2} \text{ sq units}$$

EXAMPLE |2| If the area of a triangle with vertices (−3, 0), (3, 0) and (0, k) is 9 sq units, then find the value of k.

Sol. Given area of a triangle with vertices (−3, 0), (3, 0) and (0, k) is 9 sq units.

$$\text{We have,} \quad \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 9$$

$$\Rightarrow \quad \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 18$$

$\Rightarrow \quad -3(0-k) - 0 + 1(3k - 0) = \pm 18$

$\Rightarrow \quad 3k + 3k = \pm 18 \Rightarrow 6k = \pm 18$

$\Rightarrow \quad k = \pm 3$

Condition of Collinearity for Three Points

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if the area of triangle formed by these three points is zero

$$\text{i.e.} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2x - 6y + 20 = 0$

$\Rightarrow \quad x - 3y + 10 = 0$ [dividing by 2]

which is the required equation of line joining points P and Q.

Now, according to the question,

Area of $\Delta PQR = 9 \text{ m}^2$

$$\therefore \quad \frac{1}{2}\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & k & 1 \end{vmatrix} = \pm 9$$

$$\left[\because \text{area of a triangle with vertices } (x_1, y_1), (x_2, y_2) \text{ and } (x_3, y_3) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right]$$

$\Rightarrow \quad \frac{1}{2}[11(5-k) - 7(5+1) + 1(5k+5)] = \pm 9$ [expanding along R_1]

$\Rightarrow \quad 55 - 11k - 42 + 5k + 5 = \pm 18$

$\Rightarrow \quad (-6k + 18) = \pm 18 \Rightarrow -6k = -18 \pm 18$

For positive sign, $k = \frac{-18 + 18}{-6} = 0$

collinear.

Thus, we have $\frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ or $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

which gives the required equation of line.

EXAMPLE |4| Find the equation of line joining (2, 3) and (−1, 2) using determinants.

Sol. Equation of line is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Consider, $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-1, 2)$

$\therefore \quad \begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow x(3-2) - y(2+1) + 1(4+3) = 0$ [expanding along R_1]

$\Rightarrow \quad x - 3y + 7 = 0$

EXAMPLE |5| Find the equation of line joining $P(11, 7)$ and $Q(5, 5)$ using determinants. Also, find the value of k, if $R(-1, k)$ is the point such that area of ΔPQR is 9 m^2.

Sol. Let $A(x, y)$ be any point on line PQ. Then, the points P, Q and A are collinear.

So, $\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ x & y & 1 \end{vmatrix} = 0 \Rightarrow 11(5-y) - 7(5-x) + 1(5y-5x) = 0$

[expanding along R_1]

$\Rightarrow \quad 55 - 11y - 35 + 7x + 5y - 5x = 0$

and for negative sign, $k = \frac{-18-18}{-6} = 6$

Hence, the required values of k are 0 and 6.

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 The area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

(a) $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ (b) $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ y_1 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(c) $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ (d) None of these

2 Area of the triangle whose vertices are $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$, is

(a) 2 sq units (b) 3 sq units
(c) 0 sq unit (d) None of these

3 If area of a triangle is 35 sq units with vertices (2, − 6), (5, 4) and (k, 4), then k is **[NCERT]**

(a) 12 (b) −2 (c) −12, −2 (d) 12, −2

4 The area of the triangle formed by 3 collinear points is

(a) one (b) two
(c) zero (d) four

VERY SHORT ANSWER Type Questions

5 Find the area of the triangle whose vertices are (−2, − 3), (3, 2) and (−1, −8). **[NCERT]**

6 If the points (a_1, b_1), (a_2, b_2) and $(a_1 + a_2, b_1 + b_2)$ are collinear, then show that $a_1b_2 = a_2b_1$.

7 Find the equation of the line joining (1, 2) and (3, 6) using determinants. **[NCERT]**

SHORT ANSWER Type I Questions

8 If area of a triangle is 35 sq units with vertices (2, − 6), (5, 4) and (k, 4), then find the values of k. **[NCERT]**

9 Using determinants, find the area of the triangle whose vertices are (1, 4), (2, 3) and (− 5, − 3). Are the given points collinear? **[NCERT]**

10 Find the value of k, if the points $(k+1, 1)$, $(2k+1, 3)$ and $(2k+2, 2k)$ are collinear.

5. The area of triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Consider, $(x_1, y_1) = (-2, -3)$, $(x_2, y_2) = (3, 2)$ and $(x_3, y_3) = (-1, -8)$

$$\therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-2(2+8) + 3(3+1) + 1(-24+2)]$$

[expanding along R_1]

$$= \frac{1}{2}[-20 + 12 - 22] = -15$$

Since, area is always positive, so we neglect the negative sign.
Hence, the area of triangle is 15 sq units.

6. If the given points are collinear, then

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_1 + a_2 & b_1 + b_2 & 1 \end{vmatrix} = 0$$

SHORT ANSWER Type II Question

11 Find the equation of the line joining $A(1, 3)$ and $B(0, 0)$ using determinants and find k, if $D(k, 0)$ is a point such that area of ΔABD is 3 sq units. **[NCERT]**

| HINTS & SOLUTIONS |

1. (a) By formula of area of triangle.

2. (c) Area of triangle,

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}[a\{(c+a)-(a+b)\}-(b+c)(b-c) + 1\{b(a+b)-c(c+a)\}]$$
[expanding along R_1]

$$= \frac{1}{2}[a(c-b)-(b^2-c^2)+(ab+b^2-c^2-ac)]$$

$$= \frac{1}{2}[ac-ab-b^2+c^2+ab+b^2-c^2-ac]$$

$$= \frac{1}{2}\times 0 = 0$$

3. (d) **Hint** $\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35$

4. (c) By definition of collinearity.

$$\Rightarrow a_1(b_2-b_1-b_2)-b_1(a_2-a_1-a_2) + 1\{a_2(b_1+b_2)-b_2(a_1+a_2)\} = 0$$
[expanding along R_1]

$$\Rightarrow a_1(-b_1)-b_1(-a_1)+a_2b_1+a_2b_2-b_2a_1-b_2a_2 = 0$$

$$\Rightarrow -a_1b_1+b_1a_1+a_2b_1+a_2b_2-b_2a_1-b_2a_2 = 0$$

$$\Rightarrow -a_1b_2+a_2b_1 = 0$$

$$\Rightarrow a_1b_2 = a_2b_1$$ **Hence proved.**

7. Similar as Example 4. [**Ans.** $y = 2x$]

8. Given, $\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35$

$$\Rightarrow 2(4-4)+6(5-k)+1(20-4k) = \pm 70$$

$$\Rightarrow 0+30-6k+20-4k = \pm 70$$

On taking positive sign, we get

$$-10k+50 = 70$$
$$\Rightarrow -10k = 20$$
$$\Rightarrow k = -2$$

On taking negative sign, we get

$$-10k+50 = -70$$
$$\Rightarrow -10k = -120$$
$$\Rightarrow k = 12$$

Hence, the values of k are 12 and -2.

9. Solve as Question 5. $\left[\textbf{Ans. } \frac{13}{2}, \text{No}\right]$

10. Similar as Example 3.
$\left[\textbf{Ans. } k = 2 \text{ or } k = -\frac{1}{2}\right]$

11. Similar as Example 5. [**Ans.** $y = 3x$ and $k = \pm 2$]

|TOPIC 3|
Minors and Cofactors

Here, we will study how to write the expansion of a determinant in compact form using minors and cofactors.

MINOR

Minor of an element a_{ij} of a determinant is the determinant obtained by deleting ith row and jth column. It is denoted by M_{ij}.

e.g. Let $\Delta = \begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 4 \\ 7 & 6 & 5 \end{vmatrix}$. Then, minor of a_{11}, i.e. minor of 2, is the determinant obtained by deleting first row and first column,

i.e. $M_{11} = \begin{vmatrix} -3 & 4 \\ 6 & 5 \end{vmatrix} = -15-24 = -39$

Note Minor of an element a_{ij} of a determinant of order $n\ (n \ge 2)$ has order $(n-1)$.

COFACTOR

If elements of a row or column are multiplied with cofactors of any other row or column, then their sum is zero.

e.g. $a_{11}C_{21}+a_{12}C_{22}+a_{13}C_{23}$

$$= a_{11}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{11}(a_{12}a_{33}-a_{13}a_{32})+a_{12}(a_{11}a_{33}-a_{31}a_{13}) - a_{13}(a_{11}a_{32}-a_{31}a_{12})$$

$$= -a_{11}a_{12}a_{33}+a_{11}a_{13}a_{32}+a_{12}a_{11}a_{33} - a_{12}a_{31}a_{13}-a_{13}a_{11}a_{32}+a_{13}a_{31}a_{12} = 0$$

EXAMPLE |1| If $\Delta = \begin{vmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$, then write the cofactor of the element a_{21}.

If M_{ij} is the minor of an element a_{ij}, then the cofactor of a_{ij} is denoted by C_{ij} or A_{ij} and defined as

$$C_{ij} \text{ or } A_{ij} = (-1)^{i+j} M_{ij}$$

e.g. Let $\Delta = \begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 4 \\ 7 & 6 & 5 \end{vmatrix}$.

Then, cofactor of a_{11}, i.e. cofactor of 2

$= (-1)^{1+1}$ minor of a_{11}

i.e. $C_{11} = (-1)^{1+1}(-39) = -39$

Expansion of Determinant in Terms of Cofactors

The determinant, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ can be written in terms of cofactors as $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

or $a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

or $a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$.

i.e. Δ = Sum of product of elements of a row with their corresponding cofactors.

We can also write, Δ = Sum of product of elements of a column with their corresponding cofactors.

EXAMPLE |3| Find the minors and cofactors of the elements of first row of determinant $\begin{vmatrix} 1 & 2 & 0 \\ 3 & 5 & -1 \\ 4 & 7 & 8 \end{vmatrix}$.

Sol. The minors and cofactors of the elements of first row of the determinant $\begin{vmatrix} 1 & 2 & 0 \\ 3 & 5 & -1 \\ 4 & 7 & 8 \end{vmatrix}$ are given below

$M_{11} = \begin{vmatrix} 5 & -1 \\ 7 & 8 \end{vmatrix}$ [minor of an element a_{11}]

$= 40 + 7 = 47$

$M_{12} = \begin{vmatrix} 3 & -1 \\ 4 & 8 \end{vmatrix}$ [minor of an element a_{12}]

$= 24 + 4 = 28$

$M_{13} = \begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix}$ [minor of an element a_{13}]

$= 21 - 20 = 1$

$C_{11} = (-1)^{1+1} M_{11}$ [cofactor of an element a_{11}]

$= (-1)^2 \times 47 = 47$

$C_{12} = (-1)^{1+2} M_{12}$ [cofactor of an element a_{12}]

$= (-1)^3 \times 28 = -28$

$C_{13} = (-1)^{1+3} M_{13}$ [cofactor of an element a_{13}]

$= (-1)^4 \times 1 = 1$

Sol. Given, $\Delta = \begin{vmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$

Now, cofactors of $a_{21} = (-1)^{2+1}\begin{vmatrix} 6 & -3 \\ -7 & 3 \end{vmatrix} = -(18-21) = 3$

EXAMPLE |2| Find minors and cofactors of all the elements of the determinant $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$.

Sol. Minors of elements of determinant $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ are

$M_{11} = 5$ [minor of an element a_{11}]

$M_{12} = 3$ [minor of an element a_{12}]

$M_{21} = -1$ [minor of an element a_{21}]

and $M_{22} = 2$ [minor of an element a_{22}]

Cofactors of elements of determinant $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$ are

$C_{11} = (-1)^{1+1} M_{11}$ [cofactor of element a_{11}]

$= (-1)^2 \times 5 = 5$

$C_{12} = (-1)^{1+2} M_{12}$ [cofactor of element a_{12}]

$= (-1)^3 \times 3 = -3$

$C_{21} = (-1)^{2+1} M_{21}$ [cofactor of element a_{21}]

$= (-1)^3 \times (-1) = 1$

$C_{22} = (-1)^{2+2} M_{22}$ [cofactor of element a_{22}]

$= (-1)^4 \times 2 = 2$

4 If $M_{11} = -40$, $M_{12} = -10$ and $M_{13} = 35$ of the determinant $\Delta = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$, then the value of Δ is

(a) -80 (b) 60

(c) 70 (d) 100

5 If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is cofactor of a_{ij}, then value of Δ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

VERY SHORT ANSWER Type Questions

6 Find the minor of the element of second row and third column in the determinant $\begin{vmatrix} 3 & -2 & 4 \\ 5 & 2 & 1 \\ 1 & 6 & -5 \end{vmatrix}$.

7 If A is a matrix of order 3×3, then find the number of minors in determinant A. **[NCERT Exemplar]**

8 Write minors and cofactors of elements of following determinants. (Each part carries 1 Mark)

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 Minor of an element of a determinant of order $n (n \geq 2)$ is a determinant of order

(a) n (b) $n-1$

(c) $n-2$ (d) $n+1$

2 If $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$, then the minor M_{31} is

(a) $-c(a^2-b^2)$ (b) $c(b^2-a^2)$

(c) $c(a^2+b^2)$ (d) $c(a^2-b^2)$

3 If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, then the cofactor A_{21} is

(a) $-(hc+fg)$ (b) $fg-hc$

(c) $fg+hc$ (d) $hc-fg$

SHORT ANSWER Type II Questions

13 Using cofactor of third row, evaluate

$$\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}.$$

14 Find minors and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$. [NCERT]

HINTS & SOLUTIONS

1. (b) By defination of minor.

2. (d) $M_{31} = \begin{vmatrix} a & bc \\ b & ca \end{vmatrix}$

$= ca^2 - b^2c = c(a^2-b^2)$

3. (b) $A_{21} = (-1)^{2+1} M_{21} = -M_{21} = -\begin{vmatrix} h & g \\ f & c \end{vmatrix}$

$= -(hc-fg) = fg-hc$

4. (a) $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

$= 1\cdot(-40) - 3(-10) + (-2)(35)$

$= -40 + 30 - 70$

$= -80$

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ [NCERT]

SHORT ANSWER Type I Questions

9 Find the minors of the diagonal elements of the determinant $\begin{vmatrix} 1 & i & -i \\ -i & 1 & i \\ 1 & -i & i \end{vmatrix}$.

10 Using cofactors of the elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$. [NCERT]

11 Using cofactors of elements of third row, evaluate $\Delta = \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$.

12 Using cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$ [NCERT]

9. The minors of the diagonal elements of the determinant $\begin{vmatrix} 1 & i & -i \\ -i & 1 & i \\ 1 & -i & i \end{vmatrix}$ are given below

$M_{11} = \begin{vmatrix} 1 & i \\ -i & i \end{vmatrix} = i + i^2 = i - 1$ $[\because i^2 = -1]$

$M_{22} = \begin{vmatrix} 1 & -i \\ 1 & i \end{vmatrix} = i + i = 2i$

and $M_{33} = \begin{vmatrix} 1 & i \\ -i & 1 \end{vmatrix} = 1 + i^2 = 1 - 1 = 0$

10. Given, $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Using cofactors of elements of second row, we have

$\Delta = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$= 2\cdot(-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} + 0\cdot(-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} + 1\cdot(-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= -2(9-16) + (-1)(10-3)$

$= (-2)(-7) + (-1)(7)$

$= 14 - 7 = 7$

11. Given, $\Delta = \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$

Using cofactors of elements of third row, we have

$\Delta = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$= (-1)^{3+1}\cdot 1\begin{vmatrix} x & y+z \\ y & z+x \end{vmatrix}$

5. (d) $\Delta =$ Sum of product of elements of any row (or column) with their corresponding cofactors.

6. Clearly, $M_{23} =$ Determinant obtained after deleting second row and third column

$$= \begin{vmatrix} 3 & -2 \\ 1 & 6 \end{vmatrix} = 18 - (-2)$$

$$= 18 + 2 = 20$$

7. Since, the matrix have 9 elements, therefore its corresponding determinant have 9 elements. Hence, there are 9 minors in determinant of A.

8. (i) Similar as Example 2.

[**Ans.** $M_{11} = 3$, $M_{12} = 0$, $M_{21} = -4$ and $M_{22} = 2$, $C_{11} = 3$, $C_{12} = 0$, $C_{21} = 4$ and $C_{22} = 2$]

(ii) Given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

$\therefore$ Minors, $M_{11} = d$, $M_{12} = b$, $M_{21} = c$ and $M_{22} = a$

Also, cofactors are $A_{11} = (-1)^{1+1} M_{11} = 1 \times d = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1) \times b = -b$

$A_{21} = (-1)^{2+1} M_{21} = (-1) \times c = -c$

$A_{22} = (-1)^{2+2} M_{22} = 1 \times a = a$

$$= (3+\sqrt{23}\sqrt{5})(\sqrt{5}\sqrt{10} - 5\sqrt{5})$$
$$-\sqrt{15}(\sqrt{10}\sqrt{23} + \sqrt{10}\sqrt{3} - \sqrt{5}\sqrt{15} - \sqrt{5}\sqrt{46})$$
$$+5(5\sqrt{23} + 5\sqrt{3} - \sqrt{5}\sqrt{15} - \sqrt{5}\sqrt{46})$$
$$= (3+\sqrt{23}\sqrt{5})(5\sqrt{2} - 5\sqrt{5})$$
$$-\sqrt{15}(\sqrt{10}\sqrt{23} + \sqrt{10}\sqrt{3} - 5\sqrt{3} - \sqrt{5}\sqrt{46})$$
$$+5(5\sqrt{23} + 5\sqrt{3} - 5\sqrt{3} - \sqrt{5}\sqrt{46})$$
$$= 15\sqrt{2} - 15\sqrt{5} + 5\sqrt{23}\sqrt{10} - 25\sqrt{23}$$
$$-\sqrt{15}\sqrt{10}\sqrt{23} - \sqrt{15}\sqrt{10}\sqrt{3} + \sqrt{15}\,5\sqrt{3}$$
$$+\sqrt{15}\sqrt{5}\sqrt{46} + 25\sqrt{23} - 5\sqrt{5}\sqrt{46}$$
$$= 15\sqrt{2} - 15\sqrt{5} + 5\sqrt{23}\sqrt{10} - \sqrt{15}\sqrt{5}\sqrt{46} - 15\sqrt{2}$$
$$+15\sqrt{5} + \sqrt{15}\sqrt{5}\sqrt{46} - 5\sqrt{10}\sqrt{23}$$
$$= 0$$

14. Given, determinant $= \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

We have, $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$;

$C_{11} = (-1)^{1+1} M_{11}$; $\quad$ [$\because$ cofactor $C_{ij} = (-1)^{i+j} M_{ij}$]

$= (-20) = -20$

$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46$;

$C_{12} = (-1)^{1+2} M_{12} = -(-46) = 46$;

$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30$;

$$+ (-1)^{3+2} \cdot z \begin{vmatrix} 1 & y+z \\ 1 & z+x \end{vmatrix} + (-1)^{3+3} \cdot (x+y) \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}$$

[$\because$ cofactor $A_{ij} = (-1)^{i+j} M_{ij}$]

$$= 1(zx + x^2 - y^2 - yz) - z(z + x - y - z) + (x+y)(y-x)$$
$$= zx + x^2 - y^2 - yz - zx + yz + y^2 - x^2 = 0$$

12. **Hint** $\Delta = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

[**Ans.** $(x-y)(y-z)(z-x)$]

13. Given, $\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$

Using cofactors of elements of third row, we have

$\Delta = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

$$= (3+\sqrt{115})(-1)^{3+1} \begin{vmatrix} \sqrt{5} & \sqrt{5} \\ 5 & \sqrt{10} \end{vmatrix}$$
$$+ \sqrt{15}(-1)^{3+2} \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & \sqrt{10} \end{vmatrix}$$
$$+ 5(-1)^{3+3} \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 \end{vmatrix}$$

$C_{13} = (-1)^{1+3} M_{13} = (30) = 30$;

$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4$;

$C_{21} = (-1)^{2+1} M_{21} = -(-4) = 4$;

$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19$;

$C_{22} = (-1)^{2+2} M_{22} = (-19) = -19$

$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13$;

$C_{23} = (-1)^{2+3} M_{23} = -(13) = -13$;

$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12$;

$C_{31} = (-1)^{3+1} M_{31} = (-12) = -12$;

$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22$;

$C_{32} = (-1)^{3+2} M_{32} = -(-22) = 22$;

$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18$;

$C_{33} = (-1)^{3+3} M_{33} = (18) = 18$

We have, $a_{11} = 2$, $a_{12} = -3$, $a_{13} = 5$, $C_{31} = -12$, $C_{32} = 22$, $C_{33} = 18$

$\therefore$ $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$

$$= (2)(-12) + (-3)(22) + (5)(18)$$
$$= -24 - 66 + 90$$
$$= -90 + 90 = 0$$

|TOPIC 4|
Adjoint and Inverse of a Matrix

SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be a singular matrix, if $|A|=0$ and if $|A| \neq 0$, then matrix A is said to be non-singular matrix.

e.g. Let $A=\begin{bmatrix}2 & 1\\ 6 & 3\end{bmatrix}$. Then, $|A|=6-6=0$

So, the matrix A is a singular matrix.

EXAMPLE |1| Check whether the matrix $\begin{bmatrix}2 & 1 & 3\\ 4 & -1 & 0\\ -7 & 2 & 1\end{bmatrix}$ is singular or not.

Sol. Let $A=\begin{bmatrix}2 & 1 & 3\\ 4 & -1 & 0\\ -7 & 2 & 1\end{bmatrix}$.

Now, $|A|=\begin{vmatrix}2 & 1 & 3\\ 4 & -1 & 0\\ -7 & 2 & 1\end{vmatrix}$

$= 2(-1-0)-1(4-0)+3(8-7)$

$=-2-4+3$

$=-6+3$

$=-3 \neq 0$

Hence, A is a non-singular matrix.

ADJOINT OF A MATRIX

The adjoint of a square matrix $A=[a_{ij}]_{n\times n}$ is defined as the transpose of the matrix formed by cofactors of the elements a_{ij} and it is denoted by adj (A).

e.g. If $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$.

Then, adj (A) = Transpose of $\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}$

or adj $(A)=\begin{bmatrix}C_{11} & C_{21} & C_{31}\\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33}\end{bmatrix}$

where, C_{ij} is the cofactor of element a_{ij}.

Note For a square matrix of order 2 given by $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$, adj (A) can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}.

i.e. adj $(A)=\begin{bmatrix}a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{bmatrix}$.

EXAMPLE |2| Find the adjoint of the matrix

(i) $\begin{bmatrix}2 & -1\\ 4 & 3\end{bmatrix}$ **[Delhi 2020]**

(ii) $\begin{bmatrix}1 & -1 & 2\\ 2 & 3 & 5\\ -2 & 0 & 1\end{bmatrix}$. **[NCERT]**

Sol. (i) Let $A=\begin{bmatrix}2 & -1\\ 4 & 3\end{bmatrix}$, then $|A|=\begin{vmatrix}2 & -1\\ 4 & 3\end{vmatrix}$

$A_{12}=-\begin{vmatrix}2 & 5\\ -2 & 1\end{vmatrix}=-(2+10)=-12$

and $A_{13}=\begin{vmatrix}2 & 3\\ -2 & 0\end{vmatrix}=0-(-6)=6$

Cofactors of elements of second row are

$A_{21}=-\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-(-1-0)=1,$

$A_{22}=\begin{vmatrix}1 & 2\\ -2 & 1\end{vmatrix}=1+4=5$

and $A_{23}=-\begin{vmatrix}1 & -1\\ -2 & 0\end{vmatrix}=-(0-2)=2$

Cofactors of elements of third row are

$A_{31}=\begin{vmatrix}-1 & 2\\ 3 & 5\end{vmatrix}=-5-6=-11,$

$A_{32}=-\begin{vmatrix}1 & 2\\ 2 & 5\end{vmatrix}=-(5-4)=-1$

and $A_{33}=\begin{vmatrix}1 & -1\\ 2 & 3\end{vmatrix}=3+2=5$

Hence, adj $(A)=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^T$

$=\begin{bmatrix}3 & -12 & 6\\ 1 & 5 & 2\\ -11 & -1 & 5\end{bmatrix}^T$

$=\begin{bmatrix}3 & 1 & -11\\ -12 & 5 & -1\\ 6 & 2 & 5\end{bmatrix}$

Now, cofactors of elements of $|A|$ are

$C_{11} = (-1)^{1+1}(3) = 3 \qquad [\because C_{ij} = (-1)^{i+j} m_{ij}]$

$C_{12} = (-1)^{1+2}(4) = -4$

$C_{21} = (-1)^{2+1}(-1) = 1$

and $C_{22} = (-1)^{2+2}(2) = 2$

Now, $\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$= \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}^T$

$= \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}$

(ii) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Cofactors of elements of first row of $|A|$ are

$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3,$

Sol. We have, $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

Let C_{ij} be the cofactor of the element a_{ij} of $|A|$.
Now, cofactors of $|A|$ are

$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4 - 2 = -6$

$C_{21} = (-1)^{2+1} \begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -(-2-4) = 6$

$C_{22} = (-1)^{2+2} \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3$

$C_{23} = (-1)^{2+3} \begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$

$C_{31} = (-1)^{3+1} \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} = 4 + 2 = 6$

$C_{32} = (-1)^{3+2} \begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$

$C_{33} = (-1)^{3+3} \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3$

Now, the adjoint of the matrix A is given by

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$

$= -1(1-4) + 2(2+4) - 2(-4-2)$

$= -1(-3) + 2(6) - 2(-6) = 3 + 12 + 12 = 27$

and $A \cdot (\text{adj } A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

Some Important Theorems

Theorem 1 If A is any given square matrix of order n, then $A(\text{adj } A) = (\text{adj } A)A = |A| \, I_n$, where I is the identity matrix of order n.

Theorem 2 If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

Theorem 3 The determinant of the product of matrices is equal to the product of their respective determinants,

i.e. $\quad |AB| = |A||B|$

where, A and B are square matrices of the same order.

EXAMPLE |3| Find the adjoint of the matrix

$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that

$A(\text{adj } A) = |A| I_3$. **[All India 2015]**

(iii) $\text{adj}(AB) = [\text{adj}(B)][\text{adj}(A)]$

(iv) $|\text{adj } A| = |A|^{n-1}$

(v) $|\text{adj}[\text{adj}(A)]| = |A|^{(n-1)^2}$

(vi) $\text{adj}(\text{adj } A) = |A|^{n-2} \cdot A$

EXAMPLE |4| If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then verify the following results.

(i) $|\text{adj } A| = |A|$ (ii) $\text{adj}(A^T) = (\text{adj } A)^T$

(iii) $|\text{adj}(\text{adj } A)| = |A|$

Sol. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Now, cofactors of elements of $|A|$ are

$A_{11} = 4, A_{12} = -3, A_{21} = -2$ and $A_{22} = 1$

$\therefore \text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ ...(i)

(i) Now, $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$ ...(ii)

and $|\text{adj}(A)| = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = 4 - 6 = -2$ ...(iii)

$\therefore \quad |\text{adj}(A)| = |A|$ [from Eqs. (ii) and (iii)]

(ii) We have, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\therefore \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $|A^T| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

Now, cofactors of elements of $|A^T|$ are $C_{11} = 4$, $C_{12} = -2$, $C_{21} = -3$, $C_{22} = 1$.

$\therefore \text{adj}(A^T) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$

$= \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ ...(iv)

From Eq. (i), we get

$(\text{adj } A)^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3+12+12 & -6-6+12 & -6+12-6 \\ -6-6+12 & 12+3+12 & 12-6-6 \\ -6+12-6 & 12-6-6 & 12+12+3 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix} = 27 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 27 I_3 = |A| I_3$$

PROPERTIES OF ADJOINT OF A MATRIX

Suppose A and B are two non-singular matrices of same order n. Then,

(i) $\text{adj}\,(A^T) = [\text{adj}\,(A)]^T$

(ii) $\text{adj}\,(kA) = k^{n-1}\,(\text{adj}\,A), k \in R$

INVERSE OF A MATRIX

Suppose A is a non-zero square matrix of order n and there exists matrix B of same order n such that $AB = BA = I_n$, then such matrix B is called an inverse of matrix A. It is denoted by A^{-1}.

> **Theorem 4** A square matrix A is invertible if and only if A is non-singular matrix.

Proof Let A be an invertible matrix of order n and I be an identity matrix of same order n. Then, there exists a square matrix B of same order n such that

$$AB = BA = I$$

Now, $\quad AB = I$

$\therefore \quad |AB| = |I|$

$\Rightarrow \quad |AB| = 1 \qquad [\because |I| = 1]$

$\Rightarrow \quad |A||B| = 1$

$\Rightarrow \quad |A| \neq 0$

Hence, A is non-singular matrix.

Now, conversely let A be non-singular matrix, then $|A| \neq 0$. By theorem 1, we have

$$A(\text{adj}\ A) = (\text{adj}\ A)A = |A|\,I$$

$$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj}\ A\right) = \left(\frac{1}{|A|}\text{adj}\ A\right)A = I$$

$$\Rightarrow \quad AB = BA = I$$

where, $B = \dfrac{1}{|A|}\text{adj}\,(A)$

Thus, A is invertible and $A^{-1} = \dfrac{1}{|A|}\text{adj}\,(A)$.

Properties of Inverse of a Matrix

Let A and B be two square invertible matrices of same order. Then,

(i) $(A^{-1})^{-1} = A$ $\qquad$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow \quad (\text{adj}\ A)^T = \text{adj}\,(A^T) \qquad$ [from Eq. (iv)]

(iii) We have, $(\text{adj}\ A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$\therefore \quad |\text{adj}\ A| = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix}$

Now, cofactors of elements of $|\text{adj}\ A|$ are

$C_{11} = 1, C_{12} = 3, C_{21} = 2$ and $C_{22} = 4$.

$\therefore \quad \text{adj}\,(\text{adj}\ A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow \quad |\text{adj}(\text{adj}\ A)| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$= |A|$ [from Eq. (ii)] **Hence verified.**

Method of Finding the Inverse of a Matrix

Suppose a square matrix A of order n is given. Sometimes, we have to find inverse of A by using formula and sometimes an algebraic equation in A is given to us and we have to find inverse of A by using this equation. For these cases, we use the following steps

|TYPE I|

INVERSE BY USING THE FORMULA

I. Let the given matrix be A. Then, find $|A|$ and check whether it is singular or non-singular matrix. If A is singular, i.e. $|A| = 0$, then inverse of A does not exists and if A is non-singular, i.e. $|A| \neq 0$, then inverse of A exists, so go to next step.

II. Find the cofactors corresponding to each element of $|A|$.

III. Find adj (A), i.e. write the matrix of cofactors and then find transpose of this cofactors matrix, to get adj (A).

IV. Now, put the value of $|A|$ and adj (A) in the formula $A^{-1} = \dfrac{1}{|A|}\text{adj}\,(A)$ to get required inverse of matrix A.

EXAMPLE |5| Find the inverse of following matrices.

(i) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ $\qquad$ (ii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **[NCERT]**

Sol. (i) Let $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$, then $|A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = -2 + 15 = 13 \neq 0$

So, A is non-singular matrix and therefore A^{-1} exists.

Now, cofactors of each element of $|A|$ are

$A_{11} = (-1)^{1+1}(2) = 2,$

$A_{12} = (-1)^{1+2}(-3) = 3,$

$A_{21} = (-1)^{2+1}(5) = -5$

(iii) $(A^T)^{-1} = (A^{-1})^T$ (iv) $|A^{-1}| = |A|^{-1}$

(v) $AA^{-1} = A^{-1}A = I$ (vi) $(kA)^{-1} = \frac{1}{k}A^{-1}, k \neq 0$

Note

(i) If A, B and C are invertible matrices of the same order, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

(ii) Only square matrices have adjoint or inverse.

(ii) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Then, $|A| = 1\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$

[expanding along C_1]

$= 1(0) - 2(0) + 1(0+1) = 1 \neq 0$

Thus, A is a non-singular matrix, so A^{-1} exists.

Now, cofactors corresponding to each element of determinant A are

$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix}$ [$\because C_{ij} = (-1)^{i+j} M_{ij}$]

$= 1(-0-0) = 0$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = -1(0-0) = 0$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1(0+1) = 1$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = -1(0-0) = 0$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1(0-1) = -1$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1(0+1) = -1$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1(0+1) = 1$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -1(0-2) = 2$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1(-1+2) = 1$

Thus, matrix of cofactors

$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

and $\text{adj}(A) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

[interchange rows and columns]

Now, $A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

which is the required inverse of given matrix A.

EXAMPLE |6| If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $AA^{-1} = I$

and $A_{22} = (-1)^{2+2}(-1) = -1$

Thus, matrix of cofactors $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

and $\text{adj}(A) = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

Now, $A^{-1} = \frac{1}{|A|}\cdot\text{adj}A = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

which is the required inverse of given matrix A.

Sol. (i) Given, $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then

$AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-3 & -4+9 \\ 1+4 & -2-12 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -8-3 = -11 \neq 0$

$|B| = \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} = 3-2 = 1 \neq 0$

and $|AB| = \begin{vmatrix} -1 & 5 \\ 5 & -14 \end{vmatrix} = 14 - 25 = -11 \neq 0$

Thus A, B and AB are non-singular matrices, so their inverse exists.

Now, $\text{adj}(A) = \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$, $\left[\because \text{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

$\text{adj}(B) = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $\text{adj}(AB) = \begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$

$\therefore \quad (AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = \frac{1}{-11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$ ...(i)

$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{-11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

and $\quad B^{-1} = \frac{1}{|B|}\text{adj}(B) = \frac{1}{1}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

Now, $B^{-1}A^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 12+2 & 9-4 \\ 4+1 & 3-2 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$

$= (AB)^{-1}$ [from Eq. (i)]

$\therefore \quad (AB)^{-1} = B^{-1}A^{-1}$ **Hence verified.**

(ii) Now, $AA^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 8+3 & 6-6 \\ 4-4 & 3+8 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$

(iii) $|A^{-1}| = |A|^{-1}$. **[All India 2015C]**

(iii) Now, $|A^{-1}| = \left(\frac{1}{11}\right)^2 \begin{vmatrix} 4 & 3 \\ 1 & -2 \end{vmatrix}$

[if A is a matrix of order 2×2, then $|kA| = k^2|A|$]

$$= \frac{1}{(11)^2}[-8-3] = \frac{-11}{(11)^2} = -\frac{1}{11}$$

and $|A|^{-1} = (-11)^{-1} = -\frac{1}{11}$ **Hence verified.**

|TYPE II|

INVERSE BY USING ALGEBRAIC EQUATION

Sometimes, a matrix (say A) and an algebraic equation in matrix A is given to us and we have to show that A satisfies this algebraic equation and then find inverse of A by using this equation.

For this, we use the following steps

I. Firstly, we show that matrix A satisfies given algebraic equation in A and for this

(i) Find the value of all terms having power of A by multiplying A itself (as $A^2 = A \cdot A$, $A^3 = A \cdot A^2$).

(ii) Put the values of I, A, A^2, A^3, etc., in given algebraic equation and show that LHS = RHS.

II. Now, pre-multiply or post-multiply both sides in given algebraic equation by A^{-1}.

III. Simplify the equation obtained in Step II by using properties of inverse, as $AA^{-1} = I, A^2A^{-1} = A$, $IA^{-1} = A^{-1}$, etc., and then take A^{-1} in LHS and other terms in RHS.

IV. Now, put the values of I, A, A^2, A^3, etc., in RHS of equation obtained in Step III and find required value of inverse of A.

Note If a matrix A and an algebraic equation in variable x is given to us and we have to show that matrix A satisfies this equation in x, then we replace x by A in the given equation and then solve by using above method.

EXAMPLE |7| If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and I is the identity matrix of order 2, then show that $A^2 = 4A - 3I$. Hence, find A^{-1}. **[Foreign 2015]**

Sol. Here, $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Now, $A^2 = A \cdot A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix}$$

[multiplying row by column]

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$ **Hence verified.**

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad \text{...(i)}$$

Also, $4A - 3I = 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$A^2 = 4A - 3I \quad \text{...(iii)}$$

Now, $|A| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$

$\therefore$ A^{-1} exists.

Now, pre-multiplying both sides of Eq. (iii) by A^{-1}, we get

$$A^{-1} \cdot A^2 = A^{-1} \cdot (4A - 3I)$$

$\Rightarrow (A^{-1} \cdot A) \cdot A = 4A^{-1} \cdot A - 3A^{-1} \cdot I$

$\Rightarrow IA = 4I - 3A^{-1}$

[$\because A \cdot A^{-1} = I = A^{-1}A$ and $A^{-1} \cdot I = A^{-1}$]

$\Rightarrow A = 4I - 3A^{-1}$ [$\because IA = A = AI$]

$\Rightarrow 3A^{-1} = 4I - A$

$\therefore A^{-1} = \frac{1}{3}\left(4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)$

$$= \frac{1}{3}\left(\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

|TOPIC PRACTICE 4|

OBJECTIVE TYPE QUESTIONS

1 If A is a square matrix of order 3, such that $A(\text{adj } A) = 10\,I$, then $|\text{adj } A|$ is equal to **[All India 2020]**

(a) 1 (b) 10 (c) 100 (d) 101

2 If $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$, then which of the following is true?

(a) $A(\text{adj } A) \neq |A|\,I$

(b) $A(\text{adj } A) \neq (\text{adj } A)\,A$

(c) $A(\text{adj } A) = (\text{adj } A)A = |A|\,I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) None of the above

3 If $A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$, then A^{-1} exists, if **[NCERT Exemplar]**

(a) $\lambda = 2$ (b) $\lambda \neq 2$

(c) $\lambda \neq -2$ (d) None of these

4 If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to [NCERT]

(a) $\det(A)$ (b) $\frac{1}{\det(A)}$ (c) 1 (d) 0

5 If A and B are invertible matrices, then which of the following is not correct?

(a) $\text{adj } A = |A| \cdot A^{-1}$ (b) $\det(A)^{-1} = [\det(A)]^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$ (d) $(A+B)^{-1} = B^{-1} + A^{-1}$

VERY SHORT ANSWER Type Questions

6 For what value of λ, the matrix $\begin{bmatrix} 1 & \lambda & 0 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ is singular?

7 In the interval $\frac{\pi}{2} < x < \pi$, find the value of x for which the matrix $\begin{bmatrix} 2\sin x & 3 \\ 1 & 2\sin x \end{bmatrix}$ is singular. [All India 2015C]

8 Find $|(\text{adj } A)|$, if $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$. [Delhi 2014]

9 If for any 2×2 square matrix A, $A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$. [All India 2017]

10 If A and B are matrix of order 3 and $|A| = 5$, $|B| = 3$, then find the value of $|3AB|$. [NCERT Exemplar]

11 If A is matrix of order 3 and $|A| = 4$, then find the value of $|\text{adj}(A)|$.

12 If A is a square matrix of order 3 such that $|\text{adj}(A)| = 64$, then find $|A|$. [Delhi 2013C]

13 Find the inverse of the matrix $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$. [NCERT]

14 If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then show that $A^{-1} = \frac{1}{19} A$.

15 If A is a matrix of order 3×3, then show that $(A^2)^{-1} = (A^{-1})^2$. [NCERT Exemplar]

SHORT ANSWER Type I Questions

16 Let A be the non-singular square matrix of order 3×3, then prove that $|\text{adj}(A)| = |A|^2$. [NCERT]

17 Given, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. [CBSE 2018]

18 Find the inverse of the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

19 If A is a matrix of order 2×2, then find the value of $(A^3)^{-1}$. [NCERT Exemplar]

20 If A and B are invertible matrices, then which of the following is incorrect?

(i) $\text{adj}(A) = |A| A^{-1}$

(ii) $(A+B)^{-1} = B^{-1} + A^{-1}$

(iii) $\det(A^{-1}) \cdot [\det(A)] = 1$ [NCERT]

SHORT ANSWER Type II Questions

21 Find the adjoint of the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. [NCERT]

22 Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and verify that $A[\text{adj}(A)] = |A| I$.

23 Compute the adjoint of the following matrices and verify that $A(\text{adj } A) = |A| I = (\text{adj } A) A$.

(i) $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

24 Given, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

25 If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Directions (Q. Nos. 21-22) *Find the inverse of the following matrices.*

26 $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

27 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

28 Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$. [NCERT]

29 If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, then show that $A^{-1}=A^2$.

30 Find A^{-1}, if $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$ and show that $A^{-1}=\dfrac{A^2-3I}{2}$. **[NCERT Exemplar]**

31 If $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$, then find $(A')^{-1}$. **[Delhi 2015]**

32 Compute $(AB)^{-1}$,
if $A=\begin{bmatrix}1 & 1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$ and $B^{-1}=\begin{bmatrix}1 & 2 & 0\\ 0 & 3 & -1\\ 1 & 0 & 2\end{bmatrix}$. **[NCERT]**

33 Suppose a matrix B of order 2×2 such that $B\begin{bmatrix}1 & -2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$. Find matrix B by using the inverse of matrix method.

34 Suppose a matrix A of order 2×2 such that $\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}A\begin{bmatrix}-3 & 2\\ 5 & -3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Find the matrix A by using the inverse of matrix method.

35 Show that the matrix $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$ satisfies the equation $A^2-5A+7I=O$. Hence, find A^{-1}. **[NCERT; Delhi 2010C]**

36 Show that $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ satisfies the equation $x^2-6x+17=0$. Hence, find A^{-1}.

37 For the matrix $A=\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$, find x and y, so that $A^2+xI=yA$. Hence, find A^{-1}.

38 For the matrix $A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, find the numbers a and b such that $A^2+aA+bI=O$. Hence, find A^{-1}.

LONG ANSWER II Type Questions

39 If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$, then prove that $A^2-4A-5I=O$.
Hence, find A^{-1}.

40 If $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$, then verify that $A^3-6A^2+9A-4I=O$ and hence find A^{-1}. **[NCERT]**

HINTS & SOLUTIONS

1. (c) We know that

$$A\,(\text{adj }A)=|A|I$$

Now, we have

$$A\,(\text{adj }A)=10I$$

$\therefore$ $|A|=10$

Again, $|\text{adj }A|=|A|^{n-1}$

$\therefore$ $|\text{adj }A|=|A|^{3-1}=|A|^2=(10)^2=100.$

2. (c) We know, if A is any square matrix of order n, then $A(\text{adj }A)=(\text{adj }A)\cdot A=|A|\cdot I$.

3. (d) Hint A^{-1} exist iff $|A|\neq 0$

4. (b) We know, $AA^{-1}=I$

$\therefore$ $|AA^{-1}|=|I|$

$\Rightarrow$ $|A||A^{-1}|=1$

$\Rightarrow$ $|A^{-1}|=\dfrac{1}{|A|}$

5. (d) Since, A and B are invertible matrices. So, we can say that

$$(AB)^{-1}=B^{-1}A^{-1} \qquad \text{...(i)}$$

We know that $A^{-1}=\dfrac{1}{|A|}(\text{adj }A)$

$\Rightarrow$ $\text{adj }A=|A|\cdot A^{-1}$ ...(ii)

Also, $\det(A)^{-1}=[\det(A)]^{-1}$

$\Rightarrow$ $\det(A)^{-1}=\dfrac{1}{[\det(A)]}$

$\Rightarrow$ $\det(A)\cdot\det(A)^{-1}=1$ ...(iii)

which is true.

6. Let $A=\begin{bmatrix}1 & \lambda & 0\\ 3 & -1 & 2\\ 4 & 1 & 5\end{bmatrix}$.

Since, the matrix is singular.

$\therefore$ $|A|=0$

$\Rightarrow$ $\begin{vmatrix}1 & \lambda & 0\\ 3 & -1 & 2\\ 4 & 1 & 5\end{vmatrix}=0$

$\Rightarrow$ $1(-5-2)-\lambda(15-8)+0=0$

$\Rightarrow$ $-7-7\lambda=0$

$\Rightarrow$ $-7\lambda=7$

$\therefore$ $\lambda=-1$

7. Let $A=\begin{bmatrix}2\sin x & 3\\ 1 & 2\sin x\end{bmatrix}$

$\because$ A is a singular matrix.

$\therefore$ $|A|=0\Rightarrow\begin{vmatrix}2\sin x & 3\\ 1 & 2\sin x\end{vmatrix}=0$

$\Rightarrow$ $4\sin^2 x-3=0 \Rightarrow \sin^2 x=\dfrac{3}{4}$

$\Rightarrow \quad \sin x = \frac{\sqrt{3}}{2} \quad \left[\because \frac{\pi}{2} < x < \pi\right]$

$\Rightarrow \quad x = \frac{2\pi}{3}$

8. Given, $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15 - 14 = 1$

We know that $|\text{adj}(A)| = |A|^{n-1}$, if A is a non-singular square matrix of order n.

$\therefore \ |\text{adj}(A)| = |1|^{2-1}$

$\Rightarrow |\text{adj}(A)| = 1$

9. **Hint** Use $A(\text{adj } A) = |A| I$ [**Ans.** $|A| = 8$]

10. Clearly, $|3AB| = 3^3 |A||B|$

[if matrix A is of order $n \times n$, then $|kA| = k^n |A|$ and $|AB| = |A||B|$]

$= 27 \times 5 \times 3 = 405$

11. Given, $|A| = 4$

Clearly, $|\text{adj}(A)| = |A|^{3-1} = |A|^2 = 4^2 = 16$

12. We know that for a non-singular square matrix of order n, $|\text{adj}(A)| = |A|^{n-1}$.

Here, $n = 3$

$\therefore \quad |\text{adj}(A)| = |A|^{3-1} = |A|^2$

Given, $|\text{adj } A| = 64$

$\Rightarrow \quad 64 = |A|^2$

$\Rightarrow \quad (8)^2 = |A|^2$

$\Rightarrow \quad |A| = \pm 8$ [taking square root]

13. Similar as Example 5 (i). $\left[\text{Ans. } A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}\right]$

14. Similar as Example 5 (i), then show that LHS = RHS.

15. $(A^2)^{-1} = (AA)^{-1} = (A^{-1}A^{-1})$ $\quad [\because (AB)^{-1} = B^{-1}A^{-1}]$

$= (A^{-1})^2$

16. We know that $[\text{adj }(A)] \cdot A = |A| I$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

On taking determinant both sides, we get

$$|(\text{adj } A) \cdot (A)| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$\Rightarrow \quad |(\text{adj } A)A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= |A|^3 |I| = |A|^3 \quad [\because |I| = 1]$

$\Rightarrow \quad |\text{adj }(A)||A| = |A|^3 \quad [\because |AB| = |A| \cdot |B|]$

$\therefore \quad |\text{adj }(A)| = |A|^2 \quad [\because |A| \neq 0]$

Hence proved.

17. We have, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$

$\therefore \quad A^{-1}$ exists.

Clearly, $\text{adj}(A) = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\left[\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then adj }(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$= \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ ...(i)

Now, consider RHS $= 9I - A$

$= 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$= 2A^{-1}$ [using Eq. (i)]

= LHS **Hence proved.**

18. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then

$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$

$\therefore \ A^{-1}$ exists.

Also, $\text{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\left[\because \text{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

Now, $A^{-1} = \frac{1}{|A|}\text{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

19. Solve as Question 15. [**Ans.** $(A^3)^{-1} = (A^{-1})^3$]

20. Since, A is an invertible matrix, therefore we have

(i) $A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$\Rightarrow \text{adj}(A) = |A| \cdot A^{-1}$,

which is correct.

(ii) $\because (A + B)^{-1} \neq B^{-1} + A^{-1}$, so given result is incorrect.

(iii) $\because \det(A^{-1}) = \{\det(A)\}^{-1} \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

$\Rightarrow \det(A) \cdot \det(A^{-1}) = 1$,

which is correct.

21. Similar as Example 2 (ii). $\left[\text{Ans.} \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}\right]$

22. We have, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, therefore $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Now, cofactors of elements of $|A|$ are

$C_{11} = (-1)^{1+1} \times 4 = 4;\ C_{12} = (-1)^{1+2} \times 3 = -3;$

$C_{21} = (-1)^{2+1} \times 2 = -2$ and $C_{22} = (-1)^{2+2} \times 1 = 1$

$\therefore\ \text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \qquad \text{...(i)}$

and $A\,[\text{adj}(A)] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4-6 & -2+2 \\ 12-12 & -6+4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$= -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|\, I \qquad$ [from Eq. (i)]

Hence, $A[\text{adj}(A)] = |A|\, I$.

23. (i) Similar as Example 3. $\left[\text{Ans. adj}(A) = \begin{bmatrix} -6 & 5 & 14 \\ 0 & 0 & 9 \\ 3 & -1 & -10 \end{bmatrix}\right]$

(ii) Similar as Example 3.

$\left[\text{Ans. adj}(A) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$

24. **Hint**

(i) Compute A^{-1}, $2A^{-1}$ and $9I - A$.

(ii) Prove, LHS = RHS $\left[\text{Ans. } A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}\right]$

25. Given, $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x \neq 0$

$\therefore\ A^{-1}$ exists.

Let C_{ij} be the cofactor of a_{ij} in $|A|$.

Then,

$C_{11} = (-1)^{1+1}\,1 = 1;\ C_{12} = (-1)^{1+2}(-\tan x) = \tan x;$

$C_{21} = (-1)^{2+1}\tan x = -\tan x$ and $C_{22} = (-1)^{2+2}\,1 = 1$

$\therefore \text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T$

$= \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$\because\quad A^{-1} = \frac{1}{|A|}\text{adj}(A)$

$\therefore\quad A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$

Now, $A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$

$\Rightarrow A^T A^{-1} = \begin{bmatrix} \frac{1}{1+\tan^2 x} - \frac{\tan^2 x}{1+\tan^2 x} & -\frac{\tan x}{1+\tan^2 x} - \frac{\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} + \frac{\tan x}{1+\tan^2 x} & -\frac{\tan^2 x}{1+\tan^2 x} + \frac{1}{1+\tan^2 x} \end{bmatrix}$

[multiplying rows by columns]

$\Rightarrow A^T A^{-1} = \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & -\frac{2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$

$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$ **Hence verified.**

26. Similar as Example 5 (ii). $\left[\text{Ans. } \frac{-1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}\right]$

27. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$.

Then, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix} = 1(-\cos^2\alpha - \sin^2\alpha)$

$= -(\cos^2\alpha + \sin^2\alpha) = -(1) = -1 \neq 0$

$\therefore\ A^{-1}$ exists.

Now, cofactors of elements of $|A|$ are

$C_{11} = (-1)^{1+1}\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix}$

$= (-\cos^2\alpha - \sin^2\alpha) = -1 \qquad [\because \cos^2\theta + \sin^2\theta = 1]$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = -(0) = 0$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 1(0) = 0$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = -1(0) = 0$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 0\\ 0 & -\cos\alpha\end{vmatrix} = (-\cos\alpha - 0) = -\cos\alpha$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & 0\\ 0 & \sin\alpha\end{vmatrix} = -(\sin\alpha) = -\sin\alpha$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}0 & 0\\ \cos\alpha & \sin\alpha\end{vmatrix} = 0$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}1 & 0\\ 0 & \sin\alpha\end{vmatrix} = -(\sin\alpha - 0) = -\sin\alpha$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & 0\\ 0 & \cos\alpha\end{vmatrix} = (\cos\alpha - 0) = \cos\alpha$$

$$\therefore \text{adj}(A) = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}^T$$

$$= \begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}^T$$

$$= \begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$$

and $A^{-1} = \dfrac{1}{|A|}[\text{adj}(A)]$

$$= \frac{1}{-1}\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$$

28. Similar as Example 6 (i).

29. **Hint** Compute A^{-1} and A^2 and prove the result.

30. We have, $A = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$

Now, cofactors of elements of $|A|$ are

$A_{11} = -1$, $A_{12} = 1$, $A_{13} = 1$, $A_{21} = 1$, $A_{22} = -1$,

$A_{23} = 1$, $A_{31} = 1$, $A_{32} = 1$ and $A_{33} = -1$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}^T = \begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

Also, $\quad |A| = -1(-1) + 1\cdot 1 = 2$

$$\text{Now, } A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix} \quad \text{...(i)}$$

$$\text{and} \quad A^2 = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix} \quad \text{...(ii)}$$

$$\therefore \frac{A^2 - 3I}{2} = \frac{1}{2}\left\{\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix} - \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix} = A^{-1} \quad \text{[using Eq. (i)]}$$

Hence proved.

31. We have, $A = \begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$.

$$\text{Now, } |A| = \begin{vmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{vmatrix} = 1(-1-8) + 2(0+8) + 3(0-2)$$

[expanding along R_1]

$$= -9 + 16 - 6 = 1 \neq 0$$

$\therefore A^{-1}$ exists.

Cofactors of elements of $|A|$ are

$$C_{11} = (-1)^{1+1}\begin{vmatrix}-1 & 4\\ 2 & 1\end{vmatrix} = (-1-8) = -9$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}0 & 4\\ -2 & 1\end{vmatrix} = -(0+8) = -8$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}0 & -1\\ -2 & 2\end{vmatrix} = (0-2) = -2$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix}-2 & 3\\ 2 & 1\end{vmatrix} = -(-2-6) = 8$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix} = (1+6) = 7$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}1 & -2\\ -2 & 2\end{vmatrix} = -(2-4) = 2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}-2 & 3\\ -1 & 4\end{vmatrix} = (-8+3) = -5$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}1 & 3\\ 0 & 4\end{vmatrix} = -(4-0) = -4$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & -2\\ 0 & -1\end{vmatrix} = (-1-0) = -1$$

$$\text{Thus, } \text{adj}(A) = \begin{bmatrix}C_{11} & C_{21} & C_{31}\\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33}\end{bmatrix} = \begin{bmatrix}-9 & 8 & -5\\ -8 & 7 & -4\\ -2 & 2 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\,\text{adj}(A) = \frac{1}{1}\begin{bmatrix}-9 & 8 & -5\\ -8 & 7 & -4\\ -2 & 2 & -1\end{bmatrix}$$

$$\text{Now, } (A')^{-1} = (A^{-1})' = \begin{bmatrix}-9 & 8 & -5\\ -8 & 7 & -4\\ -2 & 2 & -1\end{bmatrix}' = \begin{bmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{bmatrix}$$

32. We have, $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix} = 1(8-6) - 0 + 3(-3-4)$

[expanding along C_1]

$= 2 - 21 = -19 \neq 0$

$\therefore$ A^{-1} exists.

Now, cofactors of elements of $|A|$ are

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 8 - 6 = 2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0+9) = -9$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = 0 - 6 = -6$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = -(4+4) = -8$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -(-2-3) = 5$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -3 - 4 = -7$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -9 & -6 \\ -8 & -2 & 5 \\ -7 & 3 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix}$$

and $\quad A^{-1} = \dfrac{1}{|A|} \text{adj}(A) = \dfrac{-1}{19} \begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix}$

Now, $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix} \cdot \frac{-1}{19} \begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix}$$

[$\because (AB)^{-1} = B^{-1}A^{-1}$]

Now, pre-multiplying both sides of Eq. (i) by B^{-1}, we get

$B^{-1}(BAC) = B^{-1} \cdot I$ $\qquad$ [$\because B^{-1}I = B^{-1}$]

$\Rightarrow \quad (B^{-1}B)(AC) = B^{-1}$

$$= \frac{-1}{19} \begin{bmatrix} 2-18-0 & -8-4+0 & -7+6+0 \\ 0-27+6 & 0-6-5 & 0+9-2 \\ 2+0-12 & -8+0+10 & -7+0+4 \end{bmatrix}$$

[multiplying row by column]

$$= \frac{-1}{19} \begin{bmatrix} -16 & -12 & -1 \\ -21 & -11 & 7 \\ -10 & 2 & -3 \end{bmatrix} = \begin{bmatrix} \frac{16}{19} & \frac{12}{19} & \frac{1}{19} \\ \frac{21}{19} & \frac{11}{19} & \frac{-7}{19} \\ \frac{10}{19} & \frac{-2}{19} & \frac{3}{19} \end{bmatrix}$$

33. Given, $B \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ $\qquad$ ...(i)

Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

Then, Eq. (i) becomes $BA = C \Rightarrow (BA)A^{-1} = CA^{-1}$

[post-multiplying both sides by A^{-1}, as $|A| = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 4 + 2 = 6 \neq 0$]

$\Rightarrow B(AA^{-1}) = CA^{-1}$ $\qquad$ [$\because AA^{-1} = I$]

$\Rightarrow \quad B(I) = CA^{-1}$

$\Rightarrow \quad B = CA^{-1}$ $\qquad$ ...(ii)

Let A_{ij} be the cofactors of a_{ij} in $|A|$.

Then, $A_{11} = (-1)^{1+1}\, 4 = 4$, $A_{12} = (-1)^{1+2}\, 1 = -1$,

$A_{21} = (-1)^{2+1}(-2) = 2$ and $A_{22} = (-1)^{2+2}\, 1 = 1$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

and $\quad A^{-1} = \dfrac{1}{|A|} \text{adj}(A) = \dfrac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$\therefore$ From Eq. (ii), we get

$$B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$= \dfrac{1}{6} \begin{bmatrix} 24+0 & 12+0 \\ 0-6 & 0+6 \end{bmatrix}$ [multiplying row by column]

$$= \frac{1}{6} \begin{bmatrix} 24 & 12 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

34. Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$, then the given equation reduces to

$BAC = I$ $\qquad$ ...(i)

Now, as $|B| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0$

and $\quad |C| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = 9 - 10 = -1 \neq 0$

$\therefore$ Both B^{-1} and C^{-1} exist.

39. $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$\Rightarrow \quad (B^{-1}B)(AC) = B^{-1}$

$\Rightarrow \quad I(AC) = B^{-1} \quad [\because B^{-1}B = I]$

$\Rightarrow \quad AC = B^{-1} \quad \text{...(ii)}$

Now, post-multiplying both sides of Eq. (ii) by C^{-1}, we get

$(AC)C^{-1} = B^{-1}C^{-1}$

$\Rightarrow \quad A(CC^{-1}) = B^{-1}C^{-1}$

$\Rightarrow \quad A \cdot I = B^{-1}C^{-1} \quad [\because CC^{-1} = I]$

$\Rightarrow \quad A = B^{-1}C^{-1} \quad \text{...(iii)}$

Now, let us find B^{-1} and C^{-1}.

Clearly, $B^{-1} = \frac{1}{|B|}\text{adj}(B) = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$\left[\because \text{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

and $C^{-1} = \frac{1}{|C|}\text{adj}(C) = \frac{1}{-1}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$\therefore$ From Eq. (iii), we get

$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6-5 & 4-3 \\ -9+10 & -6+6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

35. Similar as Example 7. $\left[\textbf{Ans. } A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\right]$

36. **Hint** (i) To show $A^2 - 6A + 17I_2 = O$

(ii) Use above result to find A^{-1}.

$\left[\textbf{Ans. } A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}\right]$

37. **Hint**

(i) First, find A^2.

(ii) Then, put value of A^2 and A in the given equation $A^2 + xI = yA$.

Also, solve it to find x and y.

(iii) Now, use the above result to find A^{-1}, as in Example 7.

$\left[\textbf{Ans. } x = 8,\ y = 8,\ A^{-1} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}\right]$

38. Similar as Question 37.

$\left[\textbf{Ans. } a = -4,\ b = 1 \text{ and } A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\right]$

$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$

[multiplying rows by columns]

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

Now, $A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow A^2 - 4A - 5I = O$

Now, $|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-4) + 2(4-2)$

[expanding along R_1]

$= -3 + 4 + 4 = 5 \neq 0$

$\therefore$ A^{-1} exists.

$\because \quad A^2 - 4A - 5I = O$

$\therefore$ On pre-multiplying both sides by A^{-1}, we get

$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = A^{-1}O$

$\Rightarrow \quad IA - 4I - 5A^{-1} = O \quad [\because A^{-1}I = A^{-1} \text{ and } A \cdot A^{-1} = I]$

$\Rightarrow \quad 5A^{-1} = A - 4I$

$\Rightarrow \quad A^{-1} = \frac{1}{5}(A - 4I)$

$\Rightarrow \quad A^{-1} = \frac{1}{5}\left(\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$= \frac{1}{5}\begin{bmatrix} 1-4 & 2-0 & 2-0 \\ 2-0 & 1-4 & 2-0 \\ 2-0 & 2-0 & 1-4 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

40. Solve as Question 39.

$\left[\textbf{Ans. } A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\right]$

|TOPIC 5|

Applications of Determinants and Matrices

Determinants and matrices can be used for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

(i) If $(\text{adj } A)B \neq O$ (where O being zero matrix), then solution does not exist and the system of equations is inconsistent.

(ii) If $(\text{adj } A)\ B = O$, then system of equations may be for checking the consistency of the system of linear equations.

CONSISTENT AND INCONSISTENT SYSTEM

A system of equations is said to be consistent, if its solution (one or more) exists and a system of equations is said to be inconsistent, if its solution does not exist, i.e. no solution exist.

Note Here, we will study only those system of linear equations, which have unique solution only.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS USING INVERSE OF A MATRIX

Let the system of linear equations be

$$a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2$$

and $a_3x+b_3y+c_3z=d_3$.

Now, let $A=\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix}$.

Then, the matrix representation of above system of linear equations is

$$AX=B \quad \text{...(i)}$$

Case* I *If A is a non-singular matrix

Here, A is a non-singular matrix, i.e. $|A|\neq 0$, therefore inverse of A exists.

Now, pre-multiplying Eq. (i) by A^{-1}, we get

$$A^{-1}(AX)=A^{-1}B$$

$\Rightarrow$ $(A^{-1}A)X=A^{-1}B$ [by associative property]

$\Rightarrow$ $IX=A^{-1}B \Rightarrow X=A^{-1}B$

This method of solving system of equations is known as **matrix method**. As we know that inverse of a matrix is unique, so we get unique solution for the given system of equations.

Case* II *If A is a singular matrix

Here, A is a singular matrix, i.e. $|A|=0$, therefore its inverse does not exist.

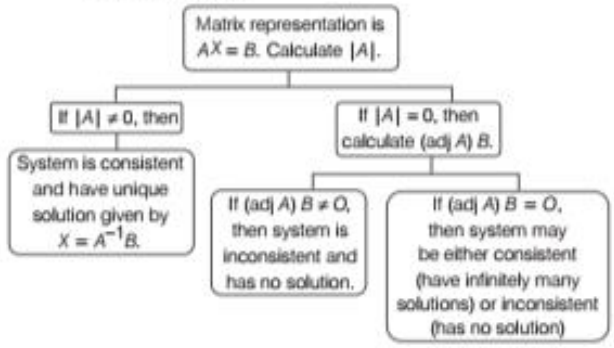
In this case, we calculate (adj A)B.

Then, we have two conditions

(i) If (adj A) $B\neq O$, then the system of equations is inconsistent.

(ii) If (adj A) $B=O$, then system of equations may be either consistent or inconsistent, according as system have either infinitely many solutions or no solution.

In the form of diagram, above cases can be represented as



Homogeneous System of Equations

If the system of linear equations, written in matrix form as $AX=O$, then the given system of equations is called homogeneous system of linear equations. In that case, if

(i) A is non-singular matrix, then A^{-1} exists. So, it has unique solution, i.e. $x_1=x_2=\dots=x_n=0$, which is also called as **trivial solution**.

(ii) A is singular matrix, i.e $|A|=0$, then the given system of equations is consistent and it has infinitely many solutions.

EXAMPLE |1| Examine the consistency of the system of equations $x+2y=2$ and $2x+3y=3$. **[NCERT]**

Sol. The given system of equations can be written in matrix form as $AX=B$,

where $A=\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$

and $B=\begin{bmatrix}2\\ 3\end{bmatrix}$

Now, $|A|=\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}=1(3)-2(2)=3-4=-1\neq 0$

Since, A is non-singular, so A^{-1} exists.

Hence, the given system of equations is consistent.

EXAMPLE |2| Test the consistency of the system of equations $3x-y=5$ and $6x-2y=3$.

Sol. Given system of equations are $3x-y=5$ ...(i)

and $6x-2y=3$ ...(ii)

Given equations can be written in matrix form as

$$AX=B,$$

EXAMPLE |3| Solve the system of linear equations by matrix method $4x-3y=3$ and $3x-5y=7$.

Sol. The given system of linear equations can be written in matrix form as $AX=B$,

where $A=\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}3\\ 7\end{bmatrix}$

where $A=\begin{bmatrix}3 & -1\\ 6 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}5\\ 3\end{bmatrix}$

Now, $|A|=\begin{vmatrix}3 & -1\\ 6 & -2\end{vmatrix}=-6+6=0$

$\therefore A^{-1}$ does not exist.

Now, $\text{adj}(A)=\begin{bmatrix}-2 & 1\\ -6 & 3\end{bmatrix}$ $\quad\left[\because \text{adj}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\right]$

$\therefore \quad \text{adj}(A)B=\begin{bmatrix}-2 & 1\\ -6 & 3\end{bmatrix}\begin{bmatrix}5\\ 3\end{bmatrix}=\begin{bmatrix}-10+3\\ -30+9\end{bmatrix}=\begin{bmatrix}-7\\ -21\end{bmatrix}$

$\Rightarrow \text{adj}(A)B \neq O$

Hence, equations are inconsistent and have no solution.

Note If (adj A) $B=O$, then first we check the solution of system. For this, put x(or y) $=k$ in both given equations and find the value of other variable y (or x) in terms of k. If we get same value of y (or x) from both equations, then equations have infinitely many solutions, since k can take any value, so consistent and if we get different values of y (or x) from both equations, then equations has no solution, so inconsistent.

Method to Solve System of Equations by Matrix Method

To solve the system of equations by matrix method, we use the following steps

I. First, write the given system of linear equations in matrix form as $AX=B$. ...(i)

If linear equations are not given, then first form them by using given conditions and then write in matrix form as Eq. (i).

II. Calculate $|A|$. If $|A|\neq 0$, then go to next step.

III. Find the inverse of A, i.e. A^{-1}. For this, firstly find adj (A) and then use formula $A^{-1}=\frac{1}{|A|}\text{adj}(A)$.

IV. The solution of Eq. (i) is given by

$$X=A^{-1}B \qquad \text{...(ii)}$$

So, put the values of A^{-1} and B in RHS of Eq. (ii) and simplify it.

V. Now, compare the LHS of Eq. (ii) with simplified RHS obtained in step IV and get the required values of x, y and z, etc. (variables in X).

Note Matrix method is applicable only, when the number of variables and number of linear equations are same.

$$C_{21}=(-1)^{2+1}\begin{vmatrix}1 & -1\\ 2 & 2\end{vmatrix}=-(2+2)=-4$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}5 & -1\\ 7 & 2\end{vmatrix}=(10+7)=17$$

$$C_{23}=(-1)^{2+3}\begin{vmatrix}5 & 1\\ 7 & 2\end{vmatrix}=-(10-7)=-3$$

Now, $|A|=\begin{vmatrix}4 & -3\\ 3 & -5\end{vmatrix}=-20+9=-11\neq 0$

So, A^{-1} exists and it is given by

$$A^{-1}=\frac{1}{|A|}\text{adj}(A)=\frac{1}{-11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}$$

$\left[\because \text{adj}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\right]$

$\because \quad X=A^{-1}B$

$\therefore \quad X=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}=\frac{1}{11}\begin{bmatrix}15-21\\ 9-28\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-6\\ -19\end{bmatrix}$

$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-6/11\\ -19/11\end{bmatrix}$

Now, comparing the corresponding elements, we get

$$x=\frac{-6}{11} \text{ and } y=\frac{-19}{11}.$$

EXAMPLE |4| Using matrix method, solve the following system of linear equations.

$$5x+y-z=7, 4x-2y-3z=5, 7x+2y+2z=7$$

Sol. The given system of equations is

$$5x+y-z=7, 4x-2y-3z=5 \text{ and } 7x+2y+2z=7$$

In matrix form, it may be represented as

$$AX=B \qquad \text{...(i)}$$

where, $A=\begin{bmatrix}5 & 1 & -1\\ 4 & -2 & -3\\ 7 & 2 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $B=\begin{bmatrix}7\\ 5\\ 7\end{bmatrix}$

Now, $|A|=\begin{vmatrix}5 & 1 & -1\\ 4 & -2 & -3\\ 7 & 2 & 2\end{vmatrix}$

$=5(-4+6)-4(2+2)+7(-3-2)$ [expanding along C_1]

$=10-16-35=-41\neq 0$

Thus, A is a non-singular matrix. So, A^{-1} exists.

Cofactors of elements of $|A|$ are

$$C_{11}=(-1)^{1+1}\begin{vmatrix}-2 & -3\\ 2 & 2\end{vmatrix}=(-4+6)=2$$

$$C_{12}=(-1)^{1+2}\begin{vmatrix}4 & -3\\ 7 & 2\end{vmatrix}=-(8+21)=-29$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}4 & -2\\ 7 & 2\end{vmatrix}=(8+14)=22$$

$$C_{21}=(-1)^{2+1}(3)=-3$$

$$C_{22}=(-1)^{2+2}(2)=2$$

Then, $\text{adj}(A)=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}^T=\begin{bmatrix}4 & 5\\ -3 & 2\end{bmatrix}^T=\begin{bmatrix}4 & -3\\ 5 & 2\end{bmatrix}$

Now, $A^{-1}=\frac{1}{|A|}\text{adj}(A)$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix} = (-3-2) = -5$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 5 & -1 \\ 4 & -3 \end{vmatrix} = -(-15+4) = 11$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 5 & 1 \\ 4 & -2 \end{vmatrix} = (-10-4) = -14$$

Then, $\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$$= \begin{bmatrix} 2 & -29 & 22 \\ -4 & 17 & -3 \\ -5 & 11 & -14 \end{bmatrix}^T = \begin{bmatrix} 2 & -4 & -5 \\ -29 & 17 & 11 \\ 22 & -3 & -14 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{1}{|A|}\text{adj}(A) = \dfrac{1}{-41}\begin{bmatrix} 2 & -4 & -5 \\ -29 & 17 & 11 \\ 22 & -3 & -14 \end{bmatrix}$

and the solution of Eq. (i) is given by $X = A^{-1}B$.

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{41}\begin{bmatrix} 2 & -4 & -5 \\ -29 & 17 & 11 \\ 22 & -3 & -14 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{41}\begin{bmatrix} 14-20-35 \\ -203+85+77 \\ 154-15-98 \end{bmatrix} = -\frac{1}{41}\begin{bmatrix} -41 \\ -41 \\ 41 \end{bmatrix}$$

[multiplying the row by column]

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{[dividing each element by } (-41)]$$

On comparing corresponding elements, we get

$x = 1$, $y = 1$ and $z = -1$

EXAMPLE |5| Given that $A = \begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix}$. Find A^{-1} and hence solve the simultaneous equations $2x + 3y + 4 = 0$ and $-5x + 4y + 13 = 0$.

Firstly, determine the inverse matrix of A, i.e. A^{-1}, then rewrite the two equations into matrix form and use A^{-1} to solve for x and y.

Sol. Given, $A = \begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix}$

Then, $|A| = 8 + 15 = 23 \neq 0$

Thus, A is a non-singular matrix. So, A^{-1} exists.

Now, cofactors of elements of determinant A are

$$C_{11} = (-1)^{1+1}(4) = 4$$

$$C_{12} = (-1)^{1+2}(-5) = 5$$

Now, cofactors corresponding to each element of $|A|$ are

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 1 + 6 = 7$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = -(0-3) = 3$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 0 - 1 = -1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = -(1+2) = -3$$

$$\Rightarrow \quad A^{-1} = \frac{1}{23}\begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$$

Given system of linear equations are

$$2x + 3y = -4$$

and $\quad -5x + 4y = -13$

These equations can be written in matrix form as

$$AX = B \qquad \text{...(i)}$$

where $\quad A = \begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ -13 \end{bmatrix}$

The solution of Eq. (i) is given by

$$X = A^{-1}B \qquad \text{...(ii)}$$

On putting the values of A^{-1} and B in RHS of Eq. (ii), we get

$$X = A^{-1}B = \frac{1}{23}\begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ -13 \end{bmatrix}$$

$$= \frac{1}{23}\begin{bmatrix} -16+39 \\ -20-26 \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 23 \\ -46 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\therefore \quad x = 1$ and $y = -2$

[comparing corresponding elements]

EXAMPLE |6| The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Sol. Let first, second and third numbers be denoted by x, y and z, respectively.

Then, according to the question, we get

$$x + y + z = 6, \; y + 3z = 11$$

and $\quad x + z = 2y \Rightarrow x - 2y + z = 0$

In matrix form, this system of equations can be written as

$$AX = B \qquad \text{...(i)}$$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(1+6) - 1(0-3) + 1(0-1)$

[expanding along R_1]

$$= 7 + 3 - 1 = 9 \neq 0$$

Since, $|A| \neq 0$, so the inverse of A exists.

Sol. Clearly, the system has a unique solution given by $X = A^{-1}B$.

Here, $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix}$

$$= 1(1-2) - 3(2-10) + 4(2-5)$$

[expanding along R_1]

$$= -1 + 24 - 12 = 11 \neq 0$$

Thus, A is invertible.

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -(-2-1) = 3$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3 - 1 = 2$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -(3-0) = -3$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

Then, $\text{adj}(A) = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

Thus, $A^{-1} = \dfrac{1}{|A|}\text{adj}(A) = \dfrac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

Now, the solution of Eq. (i) is given by

$X = A^{-1}B$ ...(ii)

On putting the values of A^{-1} and B in RHS of Eq. (ii), we get

$$X = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 7\times 6 - 3\times 11 + 2\times 0 \\ 3\times 6 + 0\times 11 - 3\times 0 \\ -1\times 6 + 3\times 11 + 1\times 0 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 - 0 \\ -6 + 33 + 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 9/9 \\ 18/9 \\ 27/9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

On comparing both sides, we get

$x = 1,\ y = 2$ and $z = 3$

which are the required numbers.

EXAMPLE |7| If $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, find A^{-1}.

Hence solve the system of equations

$$x + 3y + 4z = 8$$
$$2x + y + 2z = 5$$

and $5x + y + z = 7$ **[All India 2019]**

Now, the cofactors of $|A|$ are

$A_{11} = -1,\ A_{12} = 8,\ A_{13} = -3$
$A_{21} = 1,\ A_{22} = -19,\ A_{23} = 14$
$A_{31} = 2,\ A_{32} = 6,\ A_{33} = -5$

$$\therefore\ \text{adj}(A) = \begin{bmatrix} -1 & 8 & -3 \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}' = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

So, $A^{-1} = \dfrac{1}{|A|}\cdot\text{adj}(A)$

$$= \frac{1}{11}\cdot\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{11} & \frac{1}{11} & \frac{2}{11} \\ \frac{8}{11} & -\frac{19}{11} & \frac{6}{11} \\ -\frac{3}{11} & \frac{14}{11} & \frac{-5}{11} \end{bmatrix}$$

The given equations are

$x + 3y + 4z = 8$ ...(i)

$2x + y + 2z = 5$ ...(ii)

and $5x + y + z = 7$ ...(iii)

which can be written in matrix form as $AX' = B$,

where $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-1}{11} & \frac{1}{11} & \frac{2}{11} \\ \frac{8}{11} & \frac{-19}{11} & \frac{6}{11} \\ \frac{-3}{11} & \frac{14}{11} & \frac{-5}{11} \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{11} + \frac{5}{11} + \frac{14}{11} \\ \frac{64}{11} - \frac{95}{11} + \frac{42}{11} \\ -\frac{24}{11} + \frac{70}{11} - \frac{35}{11} \end{bmatrix} = \begin{bmatrix} \frac{11}{11} \\ \frac{11}{11} \\ \frac{11}{11} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore x = 1,\ y = 1$ and $z = 1$

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 If A is singular matrix and (adj A) $B \neq O$, then

(a) there is unique solution
(b) solution does not exist
(c) there are infinitely many solutions
(d) None of the above

2 For the system of equations

SHORT ANSWER Type I Questions

10 Examine the consistency of the system of equations $3x - y - 2z = 2,\ 2y - z = -1$ and $3x - 5y = 3$. **[NCERT]**

11 Solve the system of linear equations using matrix method.

(i) $2x - y = -2$ and $3x + 4y = 3$

(ii) $5x + 2y = 4$ and $7x + 3y = 5$

SHORT ANSWER Type II Questions

$5x + 2y = 4; 7x + 3y = 5$

the values of x and y are respectively

(a) $x = 2, y = -3$ (b) $x = 2, y = 3$

(c) $x = -2, y = -3$ (d) $x = -2, y = 3$

3 The simultaneous equations

$kx + 2y - z = 1, \ (k-1)y - 2z = 2, (k+2)z = 3$

have only one solution when

(a) $k = -2$ (b) $k = -1$

(c) $k = 0$ (d) $k = 1$

4 Given, $2x - y + 2z = 2, x - 2y + z = -4$ and $x + y + \lambda z = 4$, then the value of λ such that the given system of equation has no solution is

(a) 3 (b) 1 (c) 0 (d) -3

5 For what value of k, the following system of linear equations will have infinite solutions?

$$x - y + z = 3$$
$$2x + y - z = 2$$
$$-3x - 2ky + 6z = 3$$

(a) $k \neq 2$ (b) $k = 0$ (c) $k = 3$ (d) $k = -1$

VERY SHORT ANSWER Type Questions

Directions (Q. Nos. 6-8) *Examine the consistency of the system of equations.*

6 $2x - y = 5$ and $x + y = 4$. [NCERT]

7 $x + 3y = 5$ and $2x + 6y = 8$. [NCERT]

8 $x + y + z = 1, 2x + 3y + 2z = 2$ and $bx + by + 2bz = 4$. [NCERT]

9 For what values of k, the system of linear equations $x + y + z = 2, 2x + y - z = 3$ and $3x + 2y + kz = 4$ has a unique solution? [All India 2016]

Directions (Q. Nos. 12-14) *Solve the following systems of linear equations.*

12 $x - y + 2z = 7, 3x + 4y - 5z = -5$ and $2x - y + 3z = 12$. [Delhi 2012]

13 $x + 2y - 3z = -4, 2x + 3y + 2z = 2$ and $3x - 3y - 4z = 11$. [All India 2011]

14 $2x + y + z = 1, x - 2y - z = \frac{3}{2}$ and $3y - 5z = 9$. [NCERT]

LONG ANSWER Type Questions

15 Using matrix method, solve the following system of equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1 \text{ and } \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2,$$

where x, y and $z \neq 0$. [NCERT; Delhi 2011]

16 If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find the value of A^{-1}.

Hence, solve the following system of equations

$$2x - 3y + 5z = 11, \ 3x + 2y - 4z = -5$$

and $x + y - 2z = -3$ [Delhi 2019]

17 If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, then find A^{-1}. Using A^{-1}, solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$. [NCERT Exemplar]

18 If $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of equations $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2, \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5$ and $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4$. [Delhi 2017]

19 Determine the product of

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$ and then use to solve

the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$ and $2x + y + 3z = 1$. [All India 2017]

20 An amount of ₹ 5000 is put into three investments at the rate of interest of 6%, 7% and 8% per annum, respectively. The total annual income is ₹ 358. If the combined income from the first two investments is ₹ 70 more than the income from the third. Find the amount of each investment by matrix method.

21 An automobile company uses three types of steels S_1, S_2 and S_3 for producing three types of cars C_1, C_2 and C_3.

Steel requirements (in tonnes) for each type of

5. (c) The given system will have infinite solution, if

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -3 & -2k & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad 6k - 18 = 0 \Rightarrow k = 3$

Note There is no need to verify (adj A) $B = O$. For $k = 3$.

6. Similar as Example 1. [**Ans.** Consistent]

7. Similar as Example 2. [**Ans.** Inconsistent]

8. Given system of linear equations are

$x + y + z = 1, \ 2x + 3y + 2z = 2$ and $bx + by + 2bz = 4$

It can be written in matrix form as $AX = B$,

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ b & b & 2b \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ b & b & 2b \end{vmatrix} = 1[6b - 2b] - 1[4b - 2b] + 1[2b - 3b]$

$= 4b - 2b - b = b \neq 0$

cars are given below

Steel/Car	C_1	C_2	C_3
S_1	2	3	4
S_2	1	1	2
S_3	3	2	1

Find the number of cars of each type which can be produced using 29, 13 and 16 tonnes of steel of three types, respectively.

HINTS & SOLUTIONS

1. (b) If $|A| = 0$ and $(\text{adj } A)\, B \neq O$, then system of equations has no solution.

2. (a) From the option, we can see only option (a) satisfy both the equations.

3. (b) **Hint** Given system of equations has unique solution,

if $\begin{vmatrix} k & 2 & -1 \\ 0 & k-1 & -2 \\ 0 & 0 & k+2 \end{vmatrix} \neq 0$

$\Rightarrow \quad k \neq -2, 0, 1$

$\therefore$ $k = -1$ is the required value.

4. (b) The given system of equations will have no solution, if $|A| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow 2(-2\lambda - 1) + (\lambda - 1) + 2(1 + 2) = 0$

$\Rightarrow \quad -3\lambda + 3 = 0$

$\Rightarrow \quad \lambda = 1$

Hence, the system has no solution for $\lambda = 1$.

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} = -(-3+0) = 3$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = (6+0) = 6$$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}^T$$

$$= \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\text{Now, adj}(A)\cdot B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$$

Since, A is non-singular. So, A^{-1} exists.

Hence, the given system of equations is consistent.

9. **Hint** For unique solution, $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$ [**Ans.** $k \neq 0$]

10. Given system of equations can be written as

$3x - y - 2z = 2$, $0 \cdot x + 2y - z = -1$ and $3x - 5y + 0 \cdot z = 3$

and its matrix form is $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\text{Here, } |A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3(0-5) + 1(0+3) - 2(0-6)$$

$$= -15 + 3 + 12 = 0$$

$\therefore$ A^{-1} does not exist.

Consider, the cofactors of elements of $|A|$, i.e.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -5 & 0 \end{vmatrix} = (0-5) = -5$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -(0+3) = -3$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 3 & -5 \end{vmatrix} = (0-6) = -6$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} = -(0-10) = 10$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} = (0+6) = 6$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = -(-15+3) = 12$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} = (1+4) = 5$$

Since, $|A| \neq 0$, so unique solution exists.

Now, cofactors of elements of $|A|$ are

$$A_{11} = (-1)^2\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 1(120-45) = 75$$

$$A_{12} = (-1)^3\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -1(-80-30) = 110$$

$$A_{13} = (-1)^4\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 1(36+36) = 72$$

$$A_{21} = (-1)^3\begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = -1(-60-90) = 150$$

$$A_{22} = (-1)^4\begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = 1(-40-60) = -100$$

$$A_{23} = (-1)^5\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = -1(18-18) = 0$$

$$A_{31} = (-1)^4\begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 1(15+60) = 75$$

$$= \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \neq O$$

$\Rightarrow \quad \text{adj}(A) \cdot B \neq O$

Hence, the given system of linear equations is inconsistent.

11. Similar as Example 3. $\left[\textbf{Ans.} \text{ (i) } x = \dfrac{-5}{11} \text{ and } y = \dfrac{12}{11} \quad \text{(ii) } x = 2 \text{ and } y = -3\right]$

Solutions (Q. Nos. 12-14) *Similar as Example 4.*

12. **Ans.** $x = 2,\ y = 1$ and $z = 3$

13. **Ans.** $x = 3,\ y = -2$ and $z = 1$

14. **Ans.** $x = 1,\ y = \dfrac{1}{2}$ and $z = \dfrac{-3}{2}$

15. The given system of equations is

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1,$$

and $\quad \dfrac{6}{x} + \dfrac{9}{y} - \dfrac{20}{z} = 2;\ x, y, z \neq 0$

Let $\dfrac{1}{x} = u, \dfrac{1}{y} = v$ and $\dfrac{1}{z} = w$, then system of equations can be written as

$$\left.\begin{aligned} 2u + 3v + 10w &= 4 \\ 4u - 6v + 5w &= 1 \\ \text{and} \quad 6u + 9v - 20w &= 2 \end{aligned}\right\} \quad \text{...(i)}$$

Above system of Eq. (i) can be written in matrix form as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \text{ and } X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Its solution is given by

$$X = A^{-1}B \quad \text{...(ii)}$$

Here, $|A| = 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 2(75) - 3(-110) + 10(72)$

$= 150 + 330 + 720 = 1200$

$\Rightarrow \quad |A| = 1200$

$$A_{31} = (-1)^4 \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 1(15 + 60) = 75$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -1(10 - 40) = 30$$

$$A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = 1(-12 - 12) = -24$$

$$\therefore \text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

and $\quad A^{-1} = \dfrac{\text{adj}(A)}{|A|} = \dfrac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

On putting the values X, A^{-1} and B in Eq. (ii), we get

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300 + 150 + 150 \\ 440 - 100 + 60 \\ 288 + 0 - 48 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

On comparing corresponding elements, we get

$$u = \frac{600}{1200},\ v = \frac{400}{1200} \text{ and } w = \frac{240}{1200}$$

$\therefore \quad u = \dfrac{1}{2}, v = \dfrac{1}{3}$ and $w = \dfrac{1}{5}$

But $\quad \dfrac{1}{x} = u, \dfrac{1}{y} = v$ and $\dfrac{1}{z} = w$

$\Rightarrow \quad \dfrac{1}{x} = \dfrac{1}{2}, \dfrac{1}{y} = \dfrac{1}{3}$ and $\dfrac{1}{z} = \dfrac{1}{5}$

$\therefore \quad x = 2,\ y = 3$ and $z = 5$

16. **Hint** (i) Find A^{-1}

(ii) Write the given system of equation in matrix form as $AX = B$.

$\left[\textbf{Ans. } A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}; x = 1,\ y = 2 \text{ and } z = 3\right]$

17. **Hint** (i) Find A^{-1}.

(ii) Given system of equations can be written in matrix

form as $\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

i.e. $A^T \cdot X = B$,

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

$$\Rightarrow \quad X = (A^T)^{-1} \cdot B \quad \text{...(i)}$$

(iii) We know, $(A^T)^{-1} = (A^{-1})^T$, so find transpose of A^{-1}

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8I$$

$\Rightarrow BA = 8I \Rightarrow B(AA^{-1}) = 8IA^{-1}$

[post-multiplying both sides by A^{-1}]

$\Rightarrow \quad B = 8A^{-1} \quad [\because AA^{-1} = I]$

$$\therefore A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

Given system of equations can be written in matrix form as

$$AX = C \Rightarrow X = A^{-1}C,$$

where, $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and substitute in Eq. (i), to get the required values of x, y and z.

$$\left[\textbf{Ans. } A^{-1} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix};\ x = 0,\ y = -5 \text{ and } z = -3\right]$$

18. **Hint** (i) Find A^{-1}.

(ii) Write the given system of equations in matrix form as

$$AX = B, \text{ where } X = \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}.$$

(iii) Required solution is given by $X = A^{-1}B$.

$$\left[\text{Ans. } A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix};\ x = 2,\ y = -3,\ z = 5\right]$$

19. Let $B = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$\therefore BA = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\therefore X = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16 + 36 + 4 \\ -28 + 9 + 3 \\ 20 - 27 - 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

On comparing corresponding elements, we get

$x = 3$, $y = -2$ and $z = -1$

20. **Hint** The system of equations is $x + y + z = 5000$,

$$\frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100} = 358 \text{ and } \frac{6x}{100} + \frac{7y}{100} = 70 + \frac{8z}{100}$$

Then, similar as Example 4.

[**Ans.** ₹1000, ₹2200 and ₹1800]

21. **Hint** Let x be the number of C_1 cars produced,

y be the number of C_2 cars produced

and z be the number of C_3 cars produced.

Now, according to the given conditions, we have

$2x + 3y + 4z = 29$, $x + y + 2z = 13$ and $3x + 2y + z = 16$

Then, similar as Example 4.

[**Ans.** $x = 2$, $y = 3$ and $z = 4$]

SUMMARY

- **Determinants** To every square matrix $A = [a_{ij}]$ of order n, a unique number (real or complex) can be associated which is called determinant of the square matrix.
- **Determinant of Matrix of Order 1** Let $A = [a]$ be a square matrix of order 1, then $|A| = |a|_{1\times 1} = a$, i.e. element itself is determinant.
- **Determinant of Matrix of Order 2** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2. Then, det (A) or $|A| = a_{11}a_{22} - a_{21}a_{12}$.
- **Determinant of Matrix of Order 3** Let $A = [a_{ij}]_{3\times 3}$ be a square matrix of order 3, then

  $$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \text{ [expanding along } R_1]$$
- **Area of a Triangle** The area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), is given by $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
- **Condition of Collinearity for Three Points** The area of a triangle formed by three collinear points is zero.
- **Minor of an Element a_{ij}** of a determinant is the determinant obtained by deleting ith row and jth column in which element a_{ij} lies. It is denoted by M_{ij}.
- **Cofactor** If M_{ij} is the minor of an element a_{ij}, then the cofactor of a_{ij} is denoted by C_{ij} or A_{ij} and is defined as C_{ij} or $A_{ij} = (-1)^{i+j}M_{ij}$
- **Singular and Non-singular Matrices** A square matrix A is said to be singular matrix, if $|A| = 0$ and if $|A| \neq 0$, then matrix A is said to be non-singular matrix.

(iii) adj $(AB) = [\text{adj}(B)][\text{adj}(A)]$ (iv) $|\text{adj}[\text{adj}(A)]| = |A|^{(n-1)^2}$

(v) $|\text{adj } A| = |A|^{(n-1)}$ (vi) adj (adj A) $= |A|^{n-2}.A$

- **Inverse of a Matrix** Suppose A is a non-zero square matrix of order n and there exists matrix B of same order n such that $AB = BA = I_n$, then such matrix B is called an inverse of matrix A. It is denoted by A^{-1} and is given by $A^{-1} = \frac{1}{|A|}[\text{adj}(A)]$.
- **Properties of Inverse of a Matrix** Let A and B be two square invertible matrices of same order, then

  (i) $(A^{-1})^{-1} = A$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

  (iii) $(A^T)^{-1} = (A^{-1})^T$ (iv) $|A^{-1}| = |A|^{-1}$

  (v) $AA^{-1} = A^{-1}A = I$ (vi) $(kA)^{-1} = \frac{1}{k}A^{-1}$, $k \neq 0$
- **Consistent and Inconsistent System** A system of equations is said to be consistent, if its solution (one or more) exist and a system of equations is said to be inconsistent, if its solution does not exist, i.e. no solution exists.
- **Solution of System of Linear Equations** Let the system of linear equations be $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

  Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. Then, the matrix representation of system of linear equations is $AX = B$. Here, if A is a non-singular matrix, i.e. $|A| \neq 0$, then $X = A^{-1}B$ gives the unique solution for the given system.

  If A is singular matrix, i.e. $|A| = 0$, then two cases arise

- **Adjoint of a Matrix** The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix formed by cofactors of the elements a_{ij}. It is denoted by adj (A). Suppose A and B are two non-singular square matrices of same order n. Then,

  (i) $\text{adj}(A^T) = [\text{adj}(A)]^T$ (ii) $\text{adj}(kA) = k^{n-1}(\text{adj } A), k \in R$

(i) If $(\text{adj } A)B \neq O$ (where, O being zero matrix), then solution does not exist and the system of equations is inconsistent.

(ii) If $(\text{adj } A)B = O$, then system of equations may be either consistent or inconsistent, according as system has either infinitely many solutions or no solution.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 Which of the following is correct? **[NCERT]**

(a) Determinant is a square matrix

(b) Determinant is a number associated to a matrix

(c) Determinant is a number associated to a square matrix

(d) None of the above

2 If there are two values of a which makes determinant,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86,$$ then the sum of these number is **[NCERT Exemplar]**

(a) 4 (b) 5

(c) -4 (d) 9

3 If $\begin{vmatrix} 2 & 3 & 2 \\ x & x & x \\ 4 & 9 & 1 \end{vmatrix} + 3 = 0$, then the value of x is **[Delhi 2020]**

(a) 3 (b) 0

(c) -1 (d) 1

4 Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq 2\pi$, then **[NCERT]**

(a) $\det A = 0$ (b) $\det A \in (2, \infty)$

(c) $\det A \in (2, 4)$ (d) $\det A \in [2, 4]$

5 If $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$, then $\lim_{t \to 0} \frac{f(t)}{t^2}$ is equal to **[NCERT Exemplar]**

(a) 0 (b) -1

(c) 2 (d) 3

6 A square matrix A is said to be non-singular, if

(a) $|A| = 0$ (b) $|A| \neq 0$

(c) $|A| = -1$ (d) $|A| = 1$

7 If A and B are square matrices of same order, then

(a) $|AB| = |A| \cdot |B|$ (b) $|AB| \neq |A| \cdot |B|$

(c) $|AB| = \frac{|A|}{|B|}, |B| \neq 0$ (d) $|AB| = \frac{|B|}{|A|}, |A| \neq 0$

8 The adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is

(a) $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

VERY SHORT ANSWER Type Questions

9 If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, then find the value of x. **[All India 2014]**

10 If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x. **[Delhi 2013]**

11 If $x \in N$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x. **[Delhi 2016C]**

12 If A is a 3×3 invertiable matrix, then what will be the value of k, if $\det(A^{-1}) = (\det A)^k$? **[Delhi 2017]**

13 If A is an invertible matrix of order 2 and $\det(A) = 4$, then write the value of $\det(A^{-1})$. **[All India 2017C]**

14 Find the area of triangle, whose vertices are (2, 7), (1, 1) and (10, 8).

15 If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} \cdot A_{32}$. **[All India 2013]**

16 For what value of x, matrix $\begin{bmatrix} 6-x & 4 \\ 3-x & 1 \end{bmatrix}$ is a singular matrix? **[Delhi 2011C]**

SHORT ANSWER Type I Questions

17 If $f(x)=\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then show that $f(0)=0$. **[NCERT Exemplar]**

18 If $f(x)=\begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$ $= A+Bx+Cx^2+\ldots$, then find the value of A. **[NCERT Exemplar]**

19 Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a. **[NCERT Exemplar]**

[**Hint** Show that the given points are not collinear for any value of a]

20 If the value of a third order determinant is 12, then find the value of the determinant formed by replacing each element by its cofactor. **[NCERT Exemplar]**

21 If for the non-singular matrix A, $A^2=I$, then find A^{-1}.

22 If A is a non-singular symmetric matrix, then write whether A^{-1} is symmetric or skew-symmetric.

SHORT ANSWER Type II Questions

23 Find the values of k, if the area of triangle is 4 sq units and vertices are $(-2, 0)$, $(0, 4)$, $(0, k)$. **[NCERT]**

24 Show that the points $A(a, b+c)$, $B(b, c+a)$ and $C(c, a+b)$ are collinear. **[NCERT]**

25 If $A=\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, then find the value of λ so that $A^2=\lambda A-2I$. Hence, find A^{-1}.

26 For the matrix $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, show that $A^3-6A^2+5A+11I=O$. Then, find A^{-1}. **[NCERT]**

27 Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0\le\theta\le 2\pi$, then prove that $\det(A)\in[2, 4]$. **[NCERT Exemplar]**

28 If $f(t)=\begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then find $\lim_{t\to 0}\dfrac{f(t)}{t^2}$. **[NCERT Exemplar]**

29 Find the value of θ satisfying $\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix}=0$. **[NCERT Exemplar]**

30 If A, B and C are angles of a triangle, then find the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$. **[NCERT Exemplar]**

LONG ANSWER Type Questions

31 If $A=\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that $A\cdot\text{adj}(A)=|A|I$. Also, find A^{-1}. **[NCERT]**

32 Find the inverse of the matrix $A=\begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ and show that $aA^{-1}=(a^2+bc+1)I-aA$.

33 Suppose $A=\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$. **[NCERT]**

34 Let $A=\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, verify that $[\text{adj}\, A]^{-1}=\text{adj}(A^{-1})$. **[NCERT]**

35 Using matrices, solve the system of equations $4x+3y+2z=60$, $x+2y+3z=45$ and $6x+2y+3z=70$. **[All India 2011]**

36 If $A=\begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix}$, then find A^{-1} and hence solve the following system of equations $3x-4y+2z=-1$, $2x+3y+5z=7$ and $x+z=2$ **[Delhi 2011C]**

37 If $A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 2\\ -3 & 1 & -1\end{bmatrix}$, find A^{-1} and hence solve the system of equations $2x+y-3z=13$, $3x+2y+z=4$ and $x+2y-z=8$. **[Delhi 2017]**

38 If $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x+y+z=6$, $x+2z=7$ and $3x+y+z=12$. **[Delhi 2019]**

39 Use product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ to solve the system of equations $x+3z=9$, $-x+2y-2z=4$ and $2x-3y+4z=3$. **[Delhi 2017; Foreign 2011]**

40 The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹ 70. Find the cost of each item per kg by matrix method. **[NCERT]**

CASE BASED Questions

41. On her birthday, Seema decided to donate some money to children of an orphanage home.

If there were 8 children less, everyone would have got ₹10 more. However, if there were 16 children more, everyone would have got ₹ 10 less. Let the number of children be x and the amount distributed by Seema for one child be y (in ₹). **[CBSE Question Bank]**

Based on the information given above, answer the following questions.

(i) The equations in terms x and y are

(a) $5x-4y=40, 5x-8y=-80$

(b) $5x-4y=40, 5x+8y=80$

(c) $5x-4y=40, 5x+8y=-80$

(d) $5x+4y=40, 5x-8y=-80$

(ii) Which of the following matrix equations represent the information given above?

(a) $\begin{bmatrix}5 & 4\\ 5 & 8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}40\\ -80\end{bmatrix}$

(b) $\begin{bmatrix}5 & -4\\ 5 & -8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}40\\ 80\end{bmatrix}$

(c) $\begin{bmatrix}5 & -4\\ 5 & -8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}40\\ -80\end{bmatrix}$

(d) $\begin{bmatrix}5 & 4\\ 5 & -8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}40\\ -80\end{bmatrix}$

(iii) The number of children who were given some money by Seema, is

(a) 30 (b) 40

(c) 23 (d) 32

(iv) How much amount is given to each child by Seema?

(a) ₹ 32 (b) ₹ 30

(c) ₹ 62 (d) ₹26

(v) How much amount Seema spends in distributing the money to all the students of the Orphanage?

(a) ₹ 609 (b) ₹ 960

(c) ₹ 906 (d) ₹ 690

42. Manjit wants to donate a rectangular plot of land for a school in his village. When he was asked to give dimensions of the plot, he told that if its length is decreased by 50 m and breadth is increased by 50m, then its area will remain same, but if length is decreased by 10m and breadth is decreased by 20m, then its area will decrease by 5300 m^2. **[CBSE Question Bank]**

Based on the information given above, answer the following questions.

(i) The equations in terms of x and y are

(a) $x-y=50, 2x-y=550$

(b) $x-y=50, 2x+y=550$

(c) $x+y=50, 2x+y=550$

(d) $x+y=50, 2x-y=550$

(ii) Which of the following matrix equation represent the information given above?

(a) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 50 \\ 550 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 50 \\ 550 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 50 \\ 550 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -50 \\ -550 \end{bmatrix}$

(iii) The value of x (length of rectangular field) is

(a) 150 m (b) 400 m

(c) 200 m (d) 320 m

(iv) The value of y (breadth of rectangular field) is

(a) 150 m (b) 200 m

(c) 430 m (d) 350 m

(v) How much is the area of rectangular field?

(a) 60000 m^2 (b) 30000 m^2

(c) 30000 m (d) 3000 m

ANSWERS

1. (c) **2.** (c) **3.** (c) **4.** (d)

5. (a) **6.** (b) **7.** (a) **8.** (c)

9. $x = -2$ **10.** $x = 2$ **11.** $x = 2$ **12.** $k = -1$

13. $\frac{1}{4}$ **14.** 23.5 sq units **15.** 110 **16.** $x = 2$

18. $A = 0$ **20.** 144 **21.** A **22.** Symmetric

23. $k = 0, 8$ **25.** $\lambda = 1$; $A^{-1} = \begin{bmatrix} -1 & 1 \\ -2 & 3/2 \end{bmatrix}$ **26.** $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$ **28.** 0

29. $n\pi$ or $n\pi + (-1)^n \frac{\pi}{6}$ **30.** 0 **31.** $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **32.** $\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

35. $x = 5, y = 8$ and $z = 8$ **36.** $A^{-1} = \frac{-1}{9}\begin{bmatrix} 3 & 4 & -26 \\ 3 & 1 & -11 \\ -3 & -4 & 17 \end{bmatrix}$; $x = 3, y = 2$ and $z = -1$

37. $A^{-1} = -\frac{1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$; $x = 1, y = 2$ and $z = -3$ **38.** $A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$; $x = 3, y = 1$ and $z = 2$

39. $x = 36, y = 11$ and $z = -9$ **40.** Price of onion per kg is ₹ 5, wheat per kg is ₹ 8 and rice per kg is ₹ 8.

41. (i) → (a), (ii) → (c), (iii) → (d), (iv) → (b), (v) → (b) **42.** (i) → (b), (ii) → (a), (iii) → (c), (iv) → (a), (v) → (b)