# 06

In this chapter, we will study the use of derivatives
(i) to determine rate of change of quantities.
(ii) to find interval on which a given function is increasing or decreasing.
(iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value of a function occurs.
(iv) to find maximum and minimum values of a function.

# APPLICATION OF DERIVATIVES

## |TOPIC 1|

## Rate of Change of Quantities

**CHAPTER CHECKLIST**

- Rate of Change of Quantities
- Increasing and Decreasing Functions
- Maxima and Minima



By the derivative $\frac{ds}{dt}$, we mean the rate of change of distance $s$ with respect to time $t$. Here, $s$ is dependent quantity and $t$ is independent quantity. So, we can say that the rate of change of a quantity is the derivative of this dependent quantity with respect to that quantity at which it depends.

Suppose one quantity $y$ (dependent quantity) varies with another quantity, say $x$ (independent) satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ or $f'(x)$ represents the rate of change of $y$ with respect to $x$.

### RATE OF CHANGE OF A QUANTITY AT A POINT

Suppose quantity $y$ varies with another quantity $x$, then rate of change of $y$ with respect to $x$ at point $x = x_0$ is given by $\left(\frac{dy}{dx}\right)_{x=x_0}$ or $f'(x_0)$, i.e. firstly, find the derivative of $y$ with respect to $x$ and then put $x_0$ in place of $x$ in $\frac{dy}{dx}$.

### Rate of Change of Two Variables

Suppose two variables are varying with respect to another variable $t$, i.e. $y = f(t)$ and $x = g(t)$. Then, rate of change of $y$ with respect to $x$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ provided that } \frac{dx}{dt} \neq 0 \text{ or } \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}.$$

[by chain rule of derivative]

On differentiating both sides w.r.t. $r$, we get

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2 \quad \text{...(i)}$$

Now, we have to find rate of change of volume, when radius is 7 cm. So, putting $r = 7$ in Eq. (i), we get

$$\left(\frac{dV}{dr}\right)_{r=7} = 4\pi(7)^2 = 4\pi(49) = 196\pi \text{ cm}^3/\text{cm}$$

Hence, the volume is increasing with respect to its radius

Thus, the rate of change of $y$ with respect to $x$ can be calculated using the rate of change of $y$ and that of $x$ both with respect to $t$.

**Note**

(i) The term 'rate of change' means the instantaneous rate of change.

(ii) $\frac{dy}{dx}$ is positive, if $y$ increases as $x$ increases and is negative, if $y$ decreases as $x$ increases.

(iii) For rate of change of shadow, use the concept of similar triangles to establish the relationship between the variables.

## METHOD OF FINDING THE RATE OF CHANGE

To find the rate of change, we generally use the following steps

I. First, identify the quantity for which we have to find rate of change, i.e. dependent quantity, generally this quantity is given as area or volume of geometrical figure, etc.

II. Identify that quantity at which quantity obtained in step I is depend, i.e. independent quantity as radius, side, etc.

III. Differentiate the quantity obtained in step I with respect to quantity obtained in step II and find required rate of change.

IV. If we have to calculate rate of change at a particular value of quantity obtained in step II, then put that particular value in rate of change obtained in step III and get required answer.

**Note** Sometimes, rate of change of independent quantity is given, then we find the rate of change of dependent quantity with respect to that quantity at which independent quantity depend.

**EXAMPLE |1|** A balloon which always remains spherical has a variable radius, find the rate at which its volume is increasing with respect to its radius, when the radius is 7 cm.

*Sol.* Here, we have to find rate of change of volume with respect to radius, so volume is dependent quantity and radius is independent quantity.

Let $r$ be the radius and $V$ be the volume of spherical balloon, then $V = \frac{4}{3}\pi r^3$.

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r\left(\frac{3}{4\pi r^2}\right) \quad \text{[using Eq. (iii)]}$$

$$\Rightarrow \quad \left(\frac{dS}{dt}\right) = \frac{6}{r}$$

When $r = 2$, then

$$\frac{dS}{dt} = \frac{6}{2} = 3 \text{ cm}^2/\text{s}$$

**EXAMPLE |4|** The radius $r$ of a right circular cone is decreasing at the rate of 3 cm/min and the height $h$ is increasing at the rate of 2 cm/min. When $r = 9$ cm and $h = 6$ cm, find the rate of change of its volume.

at the rate of $196\,\pi$ cm$^3$/cm, when the radius is 7 cm.

**EXAMPLE |2|** The radius of a circle increases at a rate of 0.2 cm/s. Calculate the rate of the increase of the area, when the radius is 5 cm.

*Sol.* Here, we have to calculate rate of change of area with respect to radius.

So, let $x$ be the radius and $y$ be the area of circle. Then,

$$y = \pi x^2 \quad \text{...(i)}$$

Also, rate of change of radius with respect to time is 0.2 cm/s.

$$\therefore \quad \frac{dx}{dt} = 0.2 \quad \text{...(ii)}$$

[$\because$ radius changing with time, so area depends on time]

Now, differentiating both sides of Eq. (i) w.r.t. $t$, we get

$$\frac{dy}{dt} = \pi \cdot 2x \cdot \frac{dx}{dt} \quad \text{[by chain rule]}$$

$$= 2\pi x(0.2) = 0.4\pi x \quad \text{[using Eq. (ii)]}$$

When radius $x = 5$ cm, then

$$\left(\frac{dy}{dt}\right) = 0.4\,\pi \times 5 = 2\pi \text{ cm}^2/\text{s}$$

Hence, the required rate of increase of the area is $2\pi$ cm$^2$/s.

**EXAMPLE |3|** The volume of a sphere is increasing at the rate of 3 cm$^3$/s. Find the rate of increase of its surface area, when the radius is 2 cm. **[Delhi 2017]**

*Sol.* Let $r$ be the radius of sphere and $V$ be its volume.

Then, $$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} \quad \text{...(i)}$$

Given, $$\frac{dV}{dt} = 3 \text{ cm}^3/\text{s} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$3 = \frac{4}{3}(3\pi r^2)\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2} \quad \text{... (iii)}$$

Now, let $S$ be the surface area of sphere, then

$$S = 4\pi r^2$$

$$\Rightarrow \quad \frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt}$$

On putting $y = 4$ in Eq. (ii), we get

$$8x = 16 \Rightarrow x = 2$$

Hence, the required point is (2, 4).

**EXAMPLE |6|** A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 cm/s. How fast is its height on the wall decreasing, when the foot of the ladder is 4 m away from the wall? **[NCERT; All India 2012]**

*Sol.* Let $AC = 5$m be the length of ladder and $AB$ be the distance of the ladder from the wall at any time $t$, such that $AB = x$ and $BC = y$ ($BC$ be the wall)

[Delhi 2017C]

**Sol.** We have, $\frac{dr}{dt} = -3$ cm/min ...(i)

and $\frac{dh}{dt} = 2$ cm/min ...(ii)

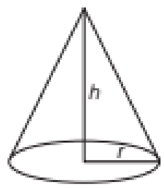
Let $V$ be the volume of a cone. Then,

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{1}{3}\pi\left(r^2 \cdot \frac{dh}{dt} + h \cdot 2r\frac{dr}{dt}\right)$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{1}{3}\pi\,[2r^2 + 2rh(-3)]$$

[using Eqs. (i) and (ii)]




Now, $\frac{dV}{dt}$ when $r = 9$ and $h = 6$, is given by

$$\left(\frac{dV}{dt}\right)_{\text{at } r=9,\, h=6} = \frac{1}{3}\pi[2(9)^2 - 6(9)(6)] = \frac{1}{3}\pi(162 - 324)$$

$$= \frac{1}{3}\pi(-162) = -54\pi \text{ cm}^3\text{/min}$$

Thus, the volume is decreasing at the rate $54\pi$ cm$^3$/min.

**EXAMPLE |5|** Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.

**Sol.** Let the required point be $(x, y)$.

Given, $\frac{dy}{dt} = \frac{dx}{dt}$ ...(i)

and $y^2 = 8x$ ...(ii)

On differentiating both sides of Eq. (ii) w.r.t. $t$, we get

$$2y\frac{dy}{dt} = 8 \cdot \frac{dx}{dt} \Rightarrow 2y = 8 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad y = 4$$

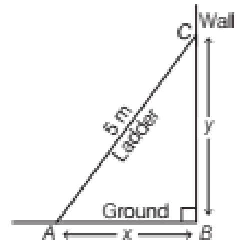
(iii) Area of a trapezium $= \frac{1}{2} \times$ Sum of parallel sides $\times$ Perpendicular distance between them

(iv) Area of a circle $= \pi r^2$ and circumference of a circle $= 2\pi r$ where, $r$ is the radius of a circle.

(v) Volume of a sphere $= \frac{4}{3}\pi r^3$ and surface area $= 4\pi r^2$ where, $r$ is the radius of sphere.

(vi) Total surface area of a right circular cylinder $= 2\pi rh + 2\pi r^2$. Curved surface area of a right circular cylinder $= 2\pi rh$ and volume $= \pi r^2 h$, where $r$ is the radius and $h$ is the height of the cylinder.

(vii) Volume of a right circular cone $= \frac{1}{3}\pi r^2 h$




Given, $x = 4$ m, $\frac{dx}{dt} = 2$ cm/s $= 0.02$ m/s

In right angled $\Delta ABC$, we get

$AB^2 + BC^2 = AC^2$ [by Pythagoras theorem]

$$\therefore \quad x^2 + y^2 = (5)^2$$

$$\Rightarrow \quad x^2 + y^2 = 25 \quad \text{...(i)}$$

When $x = 4$, then $16 + y^2 = 25$

$$\Rightarrow \quad y = \sqrt{25 - 16} = \sqrt{9} = 3$$

On differentiating both sides of Eq. (i) w.r.t. $t$, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow 4 \times 0.02 + 3 \times \frac{dy}{dt} = 0 \Rightarrow 3\frac{dy}{dt} = -0.08$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-0.08}{3} \text{ m/s}$$

$$= \frac{-0.08 \times 100}{3} \text{ cm/s} \quad [\because 1 \text{ m} = 100 \text{ cm}]$$

$$= \frac{-8}{3} \text{ cm/s}$$

Hence, the height of the wall decreasing at the rate of $\frac{8}{3}$ cm/s, when the ladder is 4 m away from the wall.

## Some Useful Results

(i) Area of a square $= x^2$ and perimeter $= 4x$ where, $x$ is the side of the square.

(ii) Area of a rectangle $= xy$ and perimeter $= 2(x + y)$ where, $x$ and $y$ are length and breadth of rectangle.

## Marginal Revenue

Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant.

If $R(x)$ represents the revenue function for $x$ units sold, then marginal revenue denoted by MR, is given by

$$\text{MR} = \frac{d}{dx}\{R(x)\}.$$

**Note** Total cost = Fixed cost + Variable cost
i.e. $C(x) = f(c) + v(x)$.

**EXAMPLE |8|** The total revenue (in ₹) received from the sale of $x$ units of a product is given by $R(x) = 3x^2 + 6x + 5$. Find the marginal revenue, when $x = 5$. [NCERT]

Curved surface area = $\pi rl$

and total surface area = $\pi r^2 + \pi rl$

where, $r$ is the radius, $h$ is the height and $l$ is the slant height of the cone.

(viii) Volume of a parallelopiped (cuboid) = $xyz$

and surface area = $2(xy + yz + zx)$, where $x$, $y$ and $z$ are the dimensions of a parallelopiped.

(ix) Volume of a cube = $x^3$ and surface area = $6x^2$

where, $x$ is the side of the cube.

(x) Area of an equilateral triangle = $\frac{\sqrt{3}}{4}(\text{Side})^2$

## Marginal Cost

Marginal cost represents the instantaneous rate of change of the total cost with respect to the number of items produced at an instant.

If $C(x)$ represents the cost function for $x$ units produced, then marginal cost denoted by MC, is given by

$$MC = \frac{d}{dx}\{C(x)\}.$$

**EXAMPLE |7|** The total cost $C(x)$ of producing $x$ items in a firm is given by $C(x) = 0.005x^3 - 0.02\,x^2 + 30x + 6000$. Find the marginal cost when 4 units are produced.

**Sol.** We have, $C(x) = 0.005\,x^3 - 0.02x^2 + 30x + 6000$

$\therefore$ Marginal cost,

$$MC = \frac{dC}{dx} = 0.005 \times 3x^2 - 0.02 \times 2x + 30$$

$$\Rightarrow \quad [MC]_{x=4} = (0.005 \times 3 \times 16) - (0.02 \times 2 \times 4) + 30$$
$$= 0.24 - 0.16 + 30 = 30.08$$

Hence, the required marginal cost is ₹ 30.08.

**Sol.** Given, $R(x) = 3x^2 + 6x + 5$.

We know that marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal Revenue,

$$MR = \frac{dR}{dx} = 6x + 6$$

When $x = 5$, then

$$MR = 6(5) + 6 = 36.$$

Hence, the required marginal revenue is ₹ 36.

# TOPIC PRACTICE 1

## OBJECTIVE TYPE QUESTIONS

**1** If the sides of an equilateral triangle are increasing at the rate of 4 cm/s, then the rate at which the area increases, when side is 5 cm, is

(a) $10\ \text{cm}^2/\text{s}$ (b) $\sqrt{3}\ \text{cm}^2/\text{s}$

(c) $10\sqrt{3}\ \text{cm}^2/\text{s}$ (d) $\frac{10}{3}\ \text{cm}^2/\text{s}$

**2** A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/s, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 m from the wall is **[NCERT Exemplar]**

(a) $\frac{1}{10}$ rad/s (b) $\frac{1}{20}$ rad/s

(c) 20 rad/s (d) 10 rad/s

**3** The radius of the base of a cone is increasing at the rate of 3 cm/min and the altitude is decreasing at the rate of 4 cm/min. The rate of change of lateral surface when the radius = 7 cm and altitude 24 cm, is

(a) $54\pi\ \text{cm}^2/\text{min}$

(b) $7\pi\ \text{cm}^2/\text{min}$

(c) $27\ \text{cm}^2/\text{min}$

(d) None of the above

**4** A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, then the rate at which the string is being let out when the kite is 250 m away from the boy who is flying the kite and the height of the boy 1.5 m is

(a) 4 m/s (b) 6 m/s

**11** The total cost $C(x)$ in rupees, associated with the production of $x$ units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, whereby marginal cost, we mean the instantaneous rate of change of total cost at any level of output. **[2018; NCERT]**

**12** The total revenue in rupees received from the sale of $x$ units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue, when $x = 7$. **[NCERT]**

## SHORT ANSWER Type I Questions

**13** If the area of a circle increase at a uniform rate, then prove that perimeter varies inversely as the radius. **[NCERT Exemplar]**

(c) 7 m/s (d) 8 m/s

**5** The total cost $C(x)$ (in ₹) associated with the production of $x$ units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$.
The marginal cost when 17 units are produced, is

(a) ₹ 20.967 (b) ₹ 21.967
(c) ₹ 81.968 (d) ₹ 11.967

## VERY SHORT ANSWER Type Questions

**6** The rate of change of the area of a circle with respect to its radius $r$, when $r = 3$ cm, is ...... **[Delhi 2020]**

**7** The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of bubble increasing, when the radius is 1 cm? **[NCERT]**

**8** Find an angle $\theta$, where $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

**9** The volume of a cube is increasing at the rate of $8\ cm^3/s$. How fast is the surface area increasing when the length of its edge is 12 cm? **[All India 2019]**

**10** A balloon which always remains spherical on inflation is being inflated by pumping in 900 cu cm of gas per second. Find the rate at which the radius of the balloon increases, when the radius is 15 cm. **[NCERT]**

**14** The volume of a cube is increasing at a rate of $9\ cm^3/s$. How fast is the surface area increasing when the length of an edge is 10 cm? **[All India 2017]**

**15** The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side. **[NCERT Exemplar]**

**16** For the curve $y = 5x - 2x^3$, if $x$ increase at the rate of 2 units/s, then find the rate of change of the slope of curve changing when $x = 3$. **[Delhi 2017; NCERT Exemplar]**

**17** The radius $r$ of a right circular cylinder is increasing at the rate of 5 cm/min and its height $h$, is decreasing at the rate of 4 cm/min. Find the rate of change of volume when its $r = 8$ cm and $h = 6$ cm. **[All India 2017C]**

**18** A balloon which always remains spherical has a variable diameter $\frac{3}{2}(2x+1)$. Then, find the rate of change of its volume with respect to $x$. **[All India 2017C]**

**19** A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the $y$-coordinate is changing 2 times as fast as the $x$-coordinate. **[All India 2017C]**

**20** A spherical ball of salt is dissolving in water in such a manner that the rate of decreasing of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate. **[NCERT Exemplar]**

## SHORT ANSWER Type II Questions

**21** The length $x$ of a rectangle is decreasing at the rate of 5 cm/min and the width $y$ is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rate of change of

(i) the perimeter.
(ii) the area of the rectangle.

**[NCERT; All India 2017]**

**22** A man 2 m tall, walks at a uniform speed of 6 km/h away from a lamp post 6 m high.
Find the rate at which the length of his shadow increases.

**23** A man is moving away from a tower 41.6 m high at the rate of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 m from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.

**24** A swimming pool is to be drained for cleaning. If $L$ represents the number of litres of water in the pool $t$ seconds after the pool has been

Given that the visible area $A$ at height $h$ is given by $A = 2\pi r^2 \frac{h}{r+h}$.

**28** Water is leaking from a conical funnel at the rate of $5\ cm^3/s$. If the radius of the base of funnel is 5 cm and height is 10 cm, then find the rate at which the water level is dropping when it is 2.5 cm from the top.

**29** Water is running into a conical vessel, 15 cm deep and 5 cm in radius at the rate of $0.1\ cm^3/s$, when the water is 6 cm deep. Find at what rate is

(i) the water level rising?
(ii) the water surface area increasing?
(iii) the wetted surface of the vessel increasing?

**30** Water is dripping out at a steady rate of $1\ cm^3/s$ through a tiny hole at the vertex of the conical vessel, whose axis is vertical. When the slant height of water in the vessel is 4 cm, find the rate of decrease of slant height, where the semi-vertical angle of the conical vessel is $\frac{\pi}{6}$.

plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 s and what is the average rate at which the water flows out during the first 5 s? [NCERT Exemplar]

**25** A car starts from a point $P$ at time $t = 0$ s and stops at $Q$. The distance $x$ (in metres), covered by it in $t$ seconds is given by $x = t^2\left(2 - \frac{t}{3}\right)$. Find the time taken by it to reach $Q$ and also the distance $P$ and $Q$.

## LONG ANSWER Type Questions

**26** Sand is pouring from a pipe at the rate of $12\ \text{cm}^3/\text{s}$. The falling sand form a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

**27** An Airforce plane is ascending vertically at the rate of 100 km/h. If the radius of the Earth is $r$ km, then how fast is the area of the Earth visible from the plane increasing at 3 min after it started ascending?

**31** A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of $5\ \text{m}^3/\text{h}$. Find the rate at which the level of the water is rising at the instant, when the depth of water in the tank is 4 m.

**32** A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, then how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite, if the height of boy is 1.5 m? [NCERT Exemplar]

**33** Two men $A$ and $B$ start with velocities $v$ at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, then find the rate at which they are being separated. [NCERT Exemplar]

# HINTS & SOLUTIONS



**1.** (c) Let the side of an equilateral triangle be $x$ cm.

$\therefore$ Area of equilateral triangle,

$$A = \frac{\sqrt{3}}{4}x^2 \quad \ldots\text{(i)}$$

Also, $\frac{dx}{dt} = 4$ cm/s

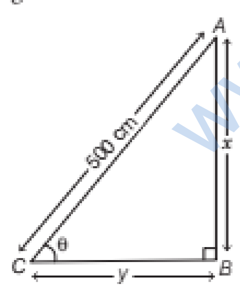
On differentiating Eq. (i) w.r.t. $t$, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\cdot 2x \cdot \frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4}\cdot 2\cdot 5\cdot 4 \qquad \left[\because x = 5 \text{ and } \frac{dx}{dt} = 4\right]$$

$$= 10\sqrt{3}\ \text{cm}^2/\text{s}$$

**2.** (b) Let the angle between floor and the ladder be $\theta$.




Let $AB = x$ cm and $BC = y$ cm

$\therefore \quad \sin\theta = \frac{x}{500}$ and $\cos\theta = \frac{y}{500}$

$\Rightarrow \quad x = 500\sin\theta$ and $y = 500\cos\theta$

Also, $\frac{dx}{dt} = 10$ cm/s

$\Rightarrow \quad 500\cdot\cos\theta\cdot\frac{d\theta}{dt} = 10$ cm/s

$\Rightarrow \quad \frac{d\theta}{dt} = \frac{10}{500\cos\theta} = \frac{1}{50\cos\theta}$

Let $S$ denote the lateral surface area, then

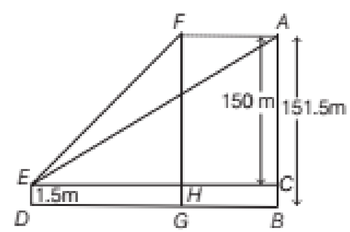
$$S = \pi r l$$

$$\Rightarrow \quad \frac{dS}{dt} = \pi\left(\frac{dr}{dt}l + r\frac{dl}{dt}\right)$$

$$= \pi(3\times 25 + 7\times(-3))$$

$$= 54\pi\ \text{cm}^2/\text{min}$$

**4.** (d) Let $AB$ be the height of kite and $DE$ be the height of the boy.




Let $DB = x = EC$

$\therefore \quad \frac{dx}{dt} = 10$ m/s

Let $AE = y$

$\because \quad AB = 151.5$ m

$\therefore \quad AC = AB - BC$

$= 151.5 - 1.5$ m $= 150$ m

$AC^2 + EC^2 = AE^2$ [by Pythagoras theorem]

$\Rightarrow \quad 150^2 + x^2 = y^2$

On differentiating both sides w.r.t. $x$, we have

$$0 + 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

For $y = 2\text{ m} = 200\text{ cm}$,

$$\frac{d\theta}{dt} = \frac{1}{50\cdot\frac{y}{500}} = \frac{10}{y}$$

$$= \frac{10}{200} = \frac{1}{20}\text{ rad/s}$$

**3.** (a) Let $r$, $l$ and $h$ denote respectively the radius, slant height and height of the cone at any time $t$. Then,

$$l^2 = r^2 + h^2$$

$$\Rightarrow \quad 2l\frac{dl}{dt} = 2r\frac{dr}{dt} + 2h\frac{dh}{dt}$$

$$\Rightarrow \quad l\frac{dl}{dt} = r\frac{dr}{dt} + h\frac{dh}{dt}$$

$$\Rightarrow \quad l\frac{dl}{dt} = 7\times 3 + 24\times(-4) \left[\because \frac{dh}{dt} = -4 \text{ and } \frac{dr}{dt} = 3\right]$$

$$\Rightarrow \quad l\frac{dl}{dt} = -75$$

When $r = 7$ and $h = 24$, then we have

$$l^2 = 7^2 + 24^2$$

$$\Rightarrow \quad l = 25$$

$$\therefore \quad l\frac{dl}{dt} = -75 \Rightarrow \frac{dl}{dt} = -3$$

**7.** Let $r$ be the radius and $V$ be the volume of the air bubble.

Given, $r = 1$ cm, $\frac{dr}{dt} = \frac{1}{2}$ cm/s ...(i)

Volume of air bubble, $V = \frac{4}{3}\pi r^3$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 4\pi(1^2)\frac{1}{2}$$

$$\Rightarrow \quad \frac{dV}{dt} = 2\pi\text{ cm}^3\text{/s} \quad \text{[from Eq. (i)]}$$

Hence, the volume of bubble is increasing at the rate of $2\pi\text{ cm}^3\text{/s}$, when the radius is 1cm.

**8.** Suppose $\theta$ increases twice as fast as its sine.

$$\therefore \quad \theta = 2\sin\theta$$

Now, differentiating both sides w.r.t. $t$, we get

$$\frac{d\theta}{dt} = 2\cdot\cos\theta\cdot\frac{d\theta}{dt}$$

$$\Rightarrow \quad 1 = 2\cos\theta \Rightarrow \frac{1}{2} = \cos\theta$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$$

So, the required angle is $\frac{\pi}{3}$.

**9.** Let $x$ be the length of an edge of the cube, $V$ be the volume and $S$ be the surface area at any time $t$. Then, $V = x^3$ and $S = 6x^2$. It is given that

$$\frac{dV}{dt} = 8\text{ cm}^3\text{/s} \Rightarrow \frac{d}{dt}(x^3) = 8$$

$$\Rightarrow \quad \frac{x\,dx}{dt} = y\frac{dy}{dt}$$

Now,

$$x = \sqrt{y^2 - (150)^2} \quad \text{[when } y = 250\text{]}$$

$$= \sqrt{62500 - 22500}$$

$$= \sqrt{40000} = 200\text{ m}$$

$$\therefore \quad 200\times 10 = 250\times\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{2000}{250} = 8\text{ m/s}$$

**5.** (a) Similar as Example 7.

**6.** Area of circle $= \pi r^2$

i.e. $\quad A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$\left(\frac{dA}{dr}\right)_{r=3} = 2\pi(3)$$

$$= 6\pi\text{ cm}^2\text{ / cm}$$

$$\Rightarrow \quad \frac{d}{dt}(\pi r^2) = k$$

$$\Rightarrow \quad 2\pi r\frac{dr}{dt} = k$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{k}{2\pi r} \quad \text{...(i)}$$

Now, $\quad \frac{dp}{dt} = \frac{d}{dt}(2\pi r) = 2\pi\frac{dr}{dt}$

$$= 2\pi\times\frac{k}{2\pi r} = \frac{k}{r} \quad \text{[from Eq. (i)]}$$

Thus, $\frac{dp}{dt} \propto \frac{1}{r}$

**Hence proved.**

**14.** Let the volume of a cube be $V\text{ cm}^3$.

Then, we have

$$\frac{dV}{dt} = 9\text{ cm}^3\text{/s}$$

and edge of cube, $x = 10$ cm

$$\because \quad V = x^3$$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \quad 9 = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{3}{x^2}$$

Also, surface area of a cube, $S = 6x^2$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dS}{dt} = 12x\frac{dx}{dt} = 12\times 10\times\frac{3}{x^2}$$

$$= 12\times 10\times\frac{3}{10\times 10} = 3.6\text{ cm}^2\text{/s}$$

$$\Rightarrow \quad 3x^2 \frac{dx}{dt} = 8 \Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$$

Now, $S = 6x^2$

$$\Rightarrow \frac{dS}{dt} = 12x \frac{dx}{dt} \Rightarrow \frac{dS}{dt} = 12x \times \frac{8}{3x^2}$$

$$\Rightarrow \frac{dS}{dt} = \frac{32}{x} \Rightarrow \left(\frac{dS}{dt}\right)_{x=12} = \frac{32}{12} \text{ cm}^2/\text{s} = \frac{8}{3} \text{ cm}^2/\text{s}$$

**10.** **Hint** $\frac{dV}{dt} = 900 \text{ cm}^3/\text{s}$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 900$$

$$\Rightarrow \quad 4\pi r^2 \frac{dr}{dt} = 900$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{4\pi r^2} \left[\textbf{Ans. } \frac{1}{\pi} \text{cm/s}\right]$$

**11.** Similar as Example 7. [**Ans.** ₹ 30.015]

**12.** Similar as Example 8. [**Ans.** ₹ 208]

**13.** Let $r$ be the radius, $A$ be the area and $P$ be the perimeter.

Then, we have $\frac{dA}{dt} = \text{constant} = k$ (say)

**17.** Similar as Example 4. [**Ans.** $224\,\pi \text{ cm}^3/\text{s}$ ]

**18.** Given, diameter of the balloon $= \frac{3}{2}(2x+1)$

$\therefore$ Radius of the balloon $= \frac{\text{Diameter}}{2}$

$$= \frac{1}{2}\left[\frac{3}{2}(2x+1)\right] = \frac{3}{4}(2x+1)$$

For the volume $V$, the balloon is given by

$$V = \frac{4}{3}\pi(\text{radius})^3 = \frac{4}{3}\pi\left[\frac{3}{4}(2x+1)\right]^3 = \frac{9\pi}{16}(2x+1)^3$$

For the rate of change of volume, differentiate w.r.t. $x$, we get

$$\frac{dV}{dx} = \frac{9\pi}{16} \times 3(2x+1)^2 \times 2 = \frac{27\pi}{8}(2x+1)^2$$

Thus, the rate of change of volume is $\frac{27\pi}{8}(2x+1)^2$.

**19.** Similar as Example 5. $\left[\textbf{Ans. } (-2, -1) \text{ and } \left(2, \frac{5}{3}\right)\right]$

**20.** Let the radius of spherical ball of the salt be $r$.

$\therefore$ Volume of the ball, $V = \frac{4}{3}\pi r^3$

and surface area, $S = 4\pi r^2$

$$\therefore \quad \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -\frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = -4\pi r^2 \frac{dr}{dt}$$

[here, we take negative sign because salt is dissolving]

According to the given condition,

$\frac{dV}{dt} \propto S \Rightarrow \frac{dV}{dt} = kS$, where $k$ is proportionality constant.

$$\Rightarrow \quad -4\pi r^2 \frac{dr}{dt} = k \cdot 4\pi r^2 \Rightarrow \frac{dr}{dt} = -k$$

Hence, the radius of ball is decreasing at constant rate.

**21.** Let $P$ be the perimeter and $A$ be the area of the rectangle of length $x$ and width $y$.

Given, $\frac{dx}{dt} = -5 \text{ cm/min}$,

[negative $(-)$ sign for decreasing rate]

$$x = 8 \text{ cm}, \frac{dy}{dt} = 4 \text{ cm/min and } y = 6 \text{ cm}$$

**15.** **Hint** $\frac{dV}{dt} = k$ (constant)

$$\Rightarrow \quad 3a^2 \frac{da}{dt} = k \Rightarrow \frac{da}{dt} = \frac{k}{3a^2}$$

Now, $\frac{dS}{dt} = \frac{d}{dt}(6a^2) = 12a\frac{da}{dt}$

$$= 12a \cdot \frac{k}{3a^2} = 4\frac{k}{a} \Rightarrow \frac{dS}{dt} \propto \frac{1}{a}$$

**16.** Given curve is $y = 5x - 2x^3$ and $\frac{dx}{dt} = 2 \text{ units/s}$ ...(i)

Now, slope of the curve $\frac{dy}{dx} = 5 - 6x^2 = M$ (say)

Rate of change of the slope,

$$\frac{dM}{dt} = -6\frac{d}{dt}(x^2)$$

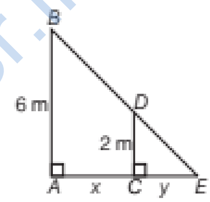
$$\Rightarrow \quad \frac{dM}{dt} = -12x\frac{dx}{dt}$$

When $x = 3$, then

$$\frac{dM}{dt} = -12 \times 3 \times 2 = -72 \text{ unit/s}$$

Thus, the slope of decreasing at a rate of 72 units/s.

**22.** Let $AB$ be the lamp post and a man $CD$ be at a distance $x$ from the lamp post and let $CE = y$ be his shadow.




Given that, $\frac{dx}{dt} = 6 \text{ km/h}$, $AB = 6 \text{ m} = \frac{6}{1000}$ km and

$$CD = 2 \text{ m} = \frac{2}{1000} \text{ km} \quad [\because 1 \text{ km} = 1000 \text{ m}]$$

Here, $\Delta BAE \sim \Delta DCE$

$$\therefore \quad \frac{AB}{CD} = \frac{AE}{CE} \quad \text{[by property of similar triangles]}$$

$$\Rightarrow \quad \frac{\frac{6}{1000}}{\frac{2}{1000}} = \frac{x+y}{y} \Rightarrow 3 = \frac{x+y}{y}$$

$$\Rightarrow \quad 3y = x + y \Rightarrow 3y - y = x \Rightarrow 2y = x$$

On differentiating both sides w.r.t. $t$, we get

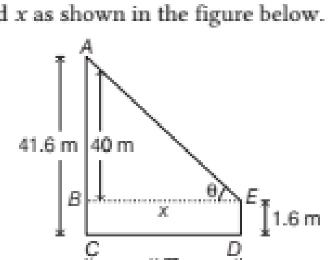
$$2\frac{dy}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \quad 2\frac{dy}{dt} = 6 \quad \left[\because \frac{dx}{dt} = 6\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{6}{2} = 3 \text{ km/h}$$

Hence, the shadow increases at the rate of 3 km/h.

**23.** Let $\theta$ and $x$ as shown in the figure below.



(i) Perimeter of the rectangle, $P = 2(x + y)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-5 + 4) = -2 \text{ cm/min}$$

So, perimeter decreases at the rate of 2 cm/min.

(ii) Area of the rectangle, $A = xy$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$= 8 \times 4 + 6 \times (-5) = 32 - 30 = 2 \text{ cm}^2\text{/min}$$

Hence, area increases at the rate of 2 cm$^2$/min.

Now, when $x = 30$ m, then $\tan\theta = \frac{4}{3}$ [using Eq. (i)]

$$\Rightarrow \quad \cot\theta = \frac{3}{4}$$

$$\Rightarrow \quad \operatorname{cosec}\theta = \sqrt{1 + \cot^2\theta} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

$\therefore$ From Eq. (ii), we get $\frac{d\theta}{dt} = \frac{-1}{20 \cdot \frac{25}{16}} = \frac{-4}{125}$ rad/s

**24.** Given that $L$ represents the number of litres of water in the pool, $t$ seconds after the pool has been plugged off to drain, then

$$L = 200(10 - t)^2$$

$\therefore$ Rate at which the water is running out,

$$\frac{dL}{dt} = -200 \cdot 2(10 - t) \cdot (-1) = 400(10 - t)$$

[here, we take negative sign, because the water is decreasing]

Rate at which the water is running out at the end of 5 s

$$= 400(10 - 5) = 2000 \text{ L/s} = \text{Final rate}$$

$$\text{Initial rate (at } t = 0) = \left(\frac{dL}{dt}\right)_{t=0} = 4000 \text{ L/s}$$

$$\therefore \text{Average rate during 5 s} = \frac{\text{Initial rate} + \text{Final rate}}{2}$$

$$= \frac{4000 + 2000}{2} = 3000 \text{ L/s}$$

Hence, the water flows out during the first 5 s with average rate of 3000 L/s.

**25.** We have, $x = t^2\left(2 - \frac{t}{3}\right) = 2t^2 - \frac{t^3}{3}$

Let $v$ be the velocity of car.

Then, $\quad v = \frac{dx}{dt} = 4t - t^2$

Now, $\quad v = 0 \Rightarrow \frac{dx}{dt} = 0$

$$\Rightarrow \quad 4t - t^2 = 0 \Rightarrow t(4 - t) = 0$$

$$\Rightarrow \quad t = 0 \quad \text{or} \quad t = 4$$

Thus, the car takes 4 s to reach at $Q$.

Then, we have

$$\frac{dx}{dt} = 2 \text{ m/s}$$

$$\therefore \text{ In } \Delta ABE, \ \tan\theta = \frac{40}{x} \quad \text{...(i)}$$

$$\Rightarrow \quad x = 40\cot\theta$$

$$\Rightarrow \quad \frac{dx}{dt} = 40(-\operatorname{cosec}^2\theta)\frac{d\theta}{dt}$$

$$\Rightarrow \quad 2 = -40\operatorname{cosec}^2\theta\frac{d\theta}{dt}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{-1}{20\operatorname{cosec}^2\theta} \quad \text{...(ii)}$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(6h)^2 h \quad [\because r = 6h]$$

$$\Rightarrow \quad V = \frac{1}{3}\pi \times 36h^2 \times h = 12\pi h^3$$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dV}{dt} = 12\pi \times 3h^2\frac{dh}{dt} = 36\pi h^2\frac{dh}{dt}$$

$$\Rightarrow \quad 12 = 36\pi(4)^2\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi \times 16} = \frac{1}{48\pi} \text{ cm/s}$$

Hence, the height of the sand cone is increasing at the rate of $\frac{1}{48\pi}$ cm/s, when the height is 4 cm.

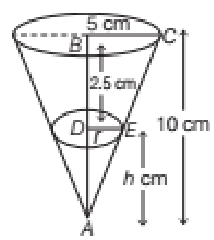
**27. Hint** Given, $\frac{dh}{dt} = 100$ km/h, then find $\frac{dA}{dt}$.

Height of the plane after 3 min $= 100 \times \frac{3}{60} = 5$ km.

$$\left[\textbf{Ans. } \frac{200\pi r^3}{(r+5)^2}\right]$$

**28.** Let $r$ be the radius of base, $h$ be the height at any stage and $V$ be the volume of water in conical funnel.

Here, the radius of base of the conical funnel is 5 cm and height of the conical funnel is 10 cm.




$$\therefore \quad \frac{dV}{dt} = -5 \text{ cm}^3\text{/s}$$

[here, negative $(-)$ sign taken for leaking of water]

and $\quad h = 10 - 2.5 = 7.5$ cm

Since, $\Delta ABC \sim \Delta ADE$

$$\therefore \quad \frac{AB}{AD} = \frac{BC}{DE} \quad \text{[by property of similar triangles]}$$

Hence, the distance between $P$ and $Q$ is the value of $x$ at $t = 4$, which is $\frac{32}{3}$ m.

$$\left[\because x = 2(4)^2 - \frac{(4)^3}{3} = 32 - \frac{64}{3} = \frac{32}{3}\right]$$

**26.** Let $r$ be the radius, $h$ be the height and $V$ be the volume of the sand cone.

Also given that, $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$, $h = \frac{1}{6}r$

$\Rightarrow \quad r = 6h$ and $h = 4$ cm

Volume of sand cone,

$$V = \frac{1}{3}\pi r^2 h$$

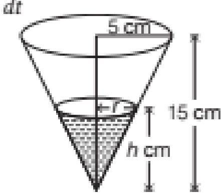
$$\Rightarrow \quad \frac{dh}{dt} = -5 \times \frac{4}{\pi} \times \frac{1}{56.25}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{-20 \times 100}{\pi \times 5625} = \frac{-20 \times 4}{\pi \times 225} = \frac{-16}{45\pi} \text{ cm/s}$$

Hence, the rate at which the water level dropping is $\frac{16}{45\pi}$ cm/s.

**29.** **Hint** $\because \frac{dV}{dt} = 0.1 \text{ cm}^3/\text{s}$

(i) To find $\frac{dh}{dt}$, where $h = 6$ cm




$\left[\textbf{Ans. } \frac{1}{40\pi} \text{cm/s}\right]$

(ii) Water surface area $(A)$ at any time $t = \pi r^2$

$$= \pi\frac{h^2}{9}$$

$$\left[\text{from part (i), } \frac{5}{r} = \frac{15}{h} \Rightarrow r = \frac{h}{3}\right]$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2\pi}{9} h \frac{dh}{dt}$$

Now, $\left(\frac{dA}{dt}\right)_{h=6} = \frac{2\pi}{9} \times 6 \times \frac{1}{40\pi}$

$$\left[\because \frac{dh}{dt} = \frac{1}{40\pi}, \text{ from part (i)}\right]$$

$\left[\textbf{Ans. } \frac{1+\sqrt{10}}{30} \text{ cm}^2/\text{s}\right]$

(iii) Wetted surface area $(S)$ of the vessel at any time

$$\Rightarrow \quad \frac{10}{h} = \frac{5}{r} \Rightarrow 10r = 5h \Rightarrow r = h/2$$

Now, we know that

Volume of cone $= \frac{1}{3}\pi r^2 h$

$\therefore$ Volume of water in conical funnel

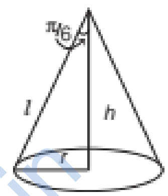
$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12}h^3$$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dV}{dt} = \frac{\pi}{12} \times 3h^2 \times \frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\Rightarrow \quad -5 = \frac{\pi}{4} \times (7.5)^2 \frac{dh}{dt}$$

**30.** **Hint** $r = l\sin\frac{\pi}{6} = \frac{l}{2}$ and $h = l\cos\frac{\pi}{6} = \frac{l\sqrt{3}}{2}$.

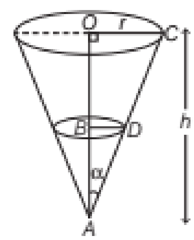



$$V = \frac{\sqrt{3}}{24}\pi l^3$$

$$\Rightarrow \quad \frac{dV}{dt} = -1 = \frac{\sqrt{3}}{8}\pi l^2 \frac{dl}{dt} \qquad \left[\textbf{Ans. } \frac{1}{2\sqrt{3\pi}} \text{ cm/s}\right]$$

**31.** **Hint** Let $r$ be the radius, $h$ be the height, $\alpha$ be semi-vertical angle of the cone and $V$ be the volume of the cone.

Given, $\frac{dV}{dt} = 5 \text{ m}^3/\text{h}$ and $h = 4$ m




In $\Delta OAC$, $\tan\alpha = \frac{r}{h} \Rightarrow \alpha = \tan^{-1}\left(\frac{r}{h}\right)$

But $\alpha = \tan^{-1}(0.5)$

$\therefore \quad \tan^{-1}\left(\frac{r}{h}\right) = \tan^{-1}(0.5) \Rightarrow r = \frac{h}{2}$

$\therefore$ Volume of the tank is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \qquad \left[\textbf{Ans. } \frac{35}{88} \text{ m/h}\right]$$

**32.** Let $CD$ be the height of kite and $AB$ be the height of boy, then height of kite $CD = h = 151.5$ m.

$$t = \pi r l = \pi \frac{h}{3}\sqrt{r^2 + h^2}$$

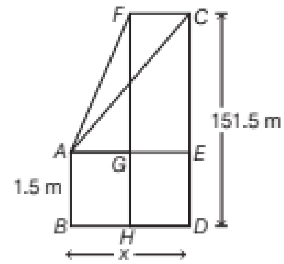
$$= \pi \frac{h}{3}\sqrt{\frac{h^2}{9} + h^2}$$

$$= \pi \cdot \frac{h}{3} \cdot \frac{\sqrt{10}h}{3} = \frac{\pi}{9}\sqrt{10}h^2$$

Now, $$\frac{dS}{dt} = \frac{2\pi\sqrt{10}}{9} h \frac{dh}{dt}$$

$$\Rightarrow \left(\frac{dS}{dt}\right)_{at\ h=6} = \frac{2\pi\sqrt{10}}{9} \times 6 \times \frac{1}{40\pi}$$

$$\left[\textbf{Ans. } \frac{\sqrt{10}}{30}\ \text{cm}^2/\text{s}\right]$$




Let $DB = x$ m $= EA$

Since, the kite is 250 m away from the boy. So,

$$AC = y = 250 \text{ m}$$

Speed of the kite, $v = \frac{dx}{dt} = 10$ m/s

From the figure, we get

$$EC = 151.5 - 1.5 = 150 \text{ m}$$

In right angled $\Delta CEA$, we get

$$AE^2 + EC^2 = AC^2 \quad \text{[by Pythagoras theorem]}$$

$$\Rightarrow x^2 + (150)^2 = y^2 \quad \text{...(i)}$$

$$\Rightarrow x^2 + (150)^2 = (250)^2 \Rightarrow x^2 = (250)^2 - (150)^2$$

$$\Rightarrow x^2 = (250 + 150)(250 - 150) = 400 \times 100$$

$$\Rightarrow x = 20 \times 10 = 200$$

On differentiating both sides of Eq. (i) w.r.t. $t$, we get

$$2x \cdot \frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$$

$$\Rightarrow 2y \cdot \frac{dy}{dt} = 2x\frac{dx}{dt}$$

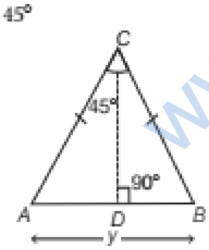
$$\Rightarrow \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$$

$$= \frac{200}{250} \cdot 10 = 8 \text{ m/s} \quad \left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

Hence, the required rate at which the string is being let out is 8 m/s.

**33.** Let two men start from the point $C$ with velocity $v$ each at the same time.

Also, $\angle BCA = 45^\circ$




Since, $A$ and $B$ are moving with same velocity $v$, so they will cover same distance in same time.

Therefore, $\Delta ABC$ is an isosceles triangle with $AC = BC$.

Now, draw $CD \perp AB$.

Let at any instant $t$, the distance between them is $AB$.

Let $AC = BC = x$ and $AB = y$

In $\Delta ACD$ and $\Delta DCB$, we get

$$\angle CAD = \angle CBD \quad [\because BC = AC \Rightarrow \angle A = \angle B]$$

$$\angle CDA = \angle CDB = 90^\circ$$

$$\therefore \angle ACD = \angle DCB \text{ or } \angle ACD = \frac{1}{2} \times \angle ACB$$

$$\Rightarrow \angle ACD = \frac{1}{2} \times 45^\circ \Rightarrow \angle ACD = 22.5^\circ$$

$$\therefore \sin 22.5^\circ = \frac{AD}{AC}$$

$$\Rightarrow \sin 22.5^\circ = \frac{y/2}{x}$$

$$\Rightarrow \frac{y}{2} = x \sin 22.5^\circ$$

$$\Rightarrow y = 2x \cdot \sin 22.5^\circ$$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = 2 \cdot \sin 22.5^\circ \frac{dx}{dt} = 2 \cdot \sin 22.5^\circ\, v \quad \left[\because v = \frac{dx}{dt}\right]$$

$$= 2v \cdot \frac{\sqrt{2 - \sqrt{2}}}{2} = v\sqrt{2 - \sqrt{2}} \quad \left[\because \sin 22.5^\circ = \frac{\sqrt{2 - \sqrt{2}}}{2}\right]$$

which is the required rate at which $A$ and $B$ are being separated.

# |TOPIC 2|

# Increasing and Decreasing Functions

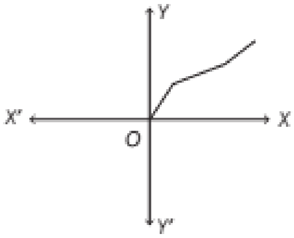
Let $I$ be an open interval contained in the domain of a real valued function $f$.

## INCREASING FUNCTION

### Strictly Increasing Function

A function $f$ is called a strictly increasing function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$. The graphical representation of such function is given below

A function $f$ is called a increasing function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in I$. The graphical representation of such function is given below




**Sol.** We have, $f(x) = 3x + 5$

Let $x_1, x_2 \in R$, such that $x_1 < x_2$.

Then, $x_1 < x_2$

$\Rightarrow \quad 3x_1 < 3x_2$ [multiplying both sides by 3]

$\Rightarrow \quad 3x_1 + 5 < 3x_2 + 5$ [adding 5 on both sides]

$\Rightarrow \quad f(x_1) < f(x_2)$

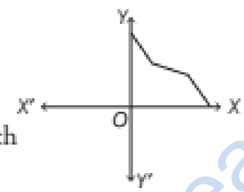
Thus, $x_1 < x_2$

$\Rightarrow \quad f(x_1) < f(x_2), \forall x_1, x_2 \in R$

Hence, $f(x)$ is strictly increasing on $R$.

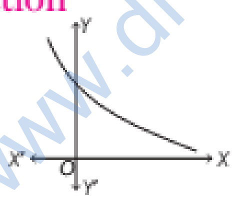
## DECREASING FUNCTION

A function $f$ is called a decreasing function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in I$. The graphical representation of such function is given below




### Strictly Decreasing Function

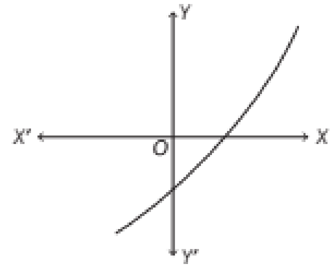
A function $f$ is called a strictly decreasing function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$. The graphical representation of such function is given below




## INCREASING AND DECREASING FUNCTIONS AT A POINT

Let $x_0$ be a point in the domain of definition of a real valued function $f$. Then, $f$ is said to be increasing or strictly increasing, decreasing or strictly decreasing at $x_0$, if there exists an open interval $I$ containing $x_0$, such that $f$ is increasing or strictly increasing, decreasing or strictly decreasing respectively, in $I$.

In other words,




**EXAMPLE |1|** Show that $f(x) = 3x + 5$ is a strictly increasing function on $R$.

**EXAMPLE |2|** If $a$ is a real number such that $0 < a < 1$. Show that the function $f(x) = a^x$ is strictly decreasing on $R$.

**Sol.** We have, $f(x) = a^x,\ 0 < a < 1$

Let $x_1, x_2 \in R$ such that $x_1 < x_2$.

Then, $\quad x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}$

$[\because 0 < a < 1$, when $x_1 < x_2$, then $a^{x_1} > a^{x_2}]$

$\Rightarrow \quad f(x_1) > f(x_2), \ \forall x_1, x_2 \in R$

Hence, $f(x)$ is strictly decreasing on $R$.

## FIRST DERIVATIVE TEST FOR INCREASING AND DECREASING FUNCTIONS

To check or find that given function is increasing or decreasing, we use first derivative test as a theorem which is given below

**Theorem** Let $f$ be continuous on $[a, b]$ and differentiable on the open interval $(a, b)$. Then,

(i) $f$ is increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$.

(ii) $f$ is decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$.

(iii) $f$ is a constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.

**Proof** (i) We can prove this test by using Mean Value theorem.

Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$. Then,

$f$ is continuous and differentiable and $f'(x) > 0$, so by Mean Value theorem, there exists a point $c$ between $x_1$ and $x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$\Rightarrow \quad f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$

$\Rightarrow \quad f(x_2) - f(x_1) > 0 \qquad$ [as $f'(c) > 0$]

(i) $f$ is said to be increasing at $x = x_0$, if there exists an interval $I = (x_0 - h, x_0 + h)$, $h > 0$ such that for all $x_1, x_2 \in I$; $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$.

(ii) $f$ is said to be strictly increasing at $x = x_0$, if there exists an interval $I = (x_0 - h, x_0 + h)$, $h > 0$ such that for all $x_1, x_2 \in I$; $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

(iii) $f$ is said to be decreasing at $x = x_0$, if there exists an interval $I = (x_0 - h, x_0 + h)$, $h > 0$ such that for all $x_1, x_2 \in I$; $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$.

(iv) $f$ is said to be strictly decreasing at $x = x_0$, if there exists an interval $I = (x_0 - h, x_0 + h)$, $h > 0$ such that for all $x_1, x_2 \in I$; $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

**EXAMPLE |3|** Show that the function $f(x) = x^3 - 6x^2 + 12x - 18$ is an increasing function on $R$.

**Sol.** We have, $f(x) = x^3 - 6x^2 + 12x - 18$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$$
$$= 3(x-2)^2 \ge 0, \forall x \in R$$

Thus, $f'(x) \ge 0, \forall x \in R$

Hence, $f(x)$ is an increasing function on $R$.

**EXAMPLE |4|** Show that the function $f(x) = \cos^2 x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

**Sol.** We have, $f(x) = \cos^2 x$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = -2\cos x \sin x = -\sin 2x < 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$
$$\left[\because \sin 2x > 0 \text{ in } \left(0, \frac{\pi}{2}\right)\right]$$

$\Rightarrow f'(x) < 0 \, \forall x \in \left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos^2 x$ is strictly decreasing function on $\left(0, \frac{\pi}{2}\right)$

**EXAMPLE |5|** Show that $f(x) = \log \sin x$ is

(i) strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

(ii) strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$. **[NCERT]**

**Sol.** We have, $f(x) = \log \sin x$

On differentiating both sides w.r.t. $x$, we get

$\Rightarrow \qquad f(x_2) > f(x_1)$

Thus, $x_1 < x_2$

$\Rightarrow f(x_1) < f(x_2), \forall x_1, x_2 \in [a, b]$

Hence, $f$ is an increasing function in $[a, b]$.

Similarly, we can prove other parts.

**Note**

(i) $f$ is strictly increasing in $(a, b)$, if $f'(x) > 0$ for each $x \in (a, b)$.

(ii) $f$ is strictly decreasing in $(a, b)$, if $f'(x) < 0$ for each $x \in (a, b)$.

(iii) A function will be increasing (decreasing) in $R$, if it is so as every interval of $R$.

(iv) If for a given interval $I \subseteq R$, function $f$ increases for some values in $I$ and decreases for the other values in $I$, then we say function is neither increasing nor decreasing.

## WORKING RULE TO FIND THE INTERVAL IN WHICH FUNCTION IS INCREASING OR DECREASING

Suppose a function $f(x)$, where $x \in R$ or ($x \in$ an interval) is given to us and we have to find the interval in which it is increasing or decreasing. Then, we use the following steps

I. First, write the given function $f(x)$, where $x \in R$ or an interval and then find $f'(x)$.

II. Put $f'(x) = 0$ and find the value (or values) of $x$ in $R$ or given interval.

III. Divide the number line ($R$) or given interval in which $x$ lies into disjoint sub intervals with the help of values of $x$ obtained in step II.

IV. In each sub interval, check that $f'(x) > 0$ or $f'(x) < 0$.

(i) If $f'(x) > 0$, then $f(x)$ is strictly increasing in that interval.

(ii) If $f'(x) < 0$, then $f(x)$ is strictly decreasing in that interval.

V. From step IV, we get those intervals in which function $f(x)$ is increasing or decreasing. If $f(x)$ is increasing in some sub interval and decreasing in some other sub interval, then in whole interval or number line, function $f(x)$ is neither increasing nor decreasing.

**Note** A strictly increasing function is an increasing function but an increasing function may or may not be strictly increasing (same is true for decreasing function).

**EXAMPLE |6|** Find the intervals in which the function $f$ given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is strictly increasing and strictly decreasing. Also, check on whole real line. **[NCERT]**

**Sol.** Given, $f(x) = 4x^3 - 6x^2 - 72x + 30, \forall x \in R$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = \frac{1}{\sin x} \cdot \cos x$$

$\Rightarrow \quad f'(x) = \cot x$

(i) We know that, for each $x \in \left(0, \frac{\pi}{2}\right)$, $\cot x > 0$

$\Rightarrow \quad f'(x) > 0$

So, $f(x)$ is strictly increasing in $(0, \pi/2)$.

(ii) We know that, for each $x \in \left(\frac{\pi}{2}, \pi\right)$,

$\cot x < 0 \Rightarrow f'(x) < 0$

So, $f(x)$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

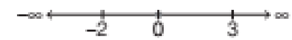
$$f'(x) = 12x^2 - 12x - 72$$

On putting $f'(x) = 0$, we get

$$12x^2 - 12x - 72 = 0 \Rightarrow 12(x^2 - x - 6) = 0$$

$\Rightarrow \quad 12(x-3)(x+2) = 0 \Rightarrow x = 3, -2$

We have, $x = -2$ and $x = 3$, which divides the number line into three disjoint sub intervals.



Thus, we have sub intervals $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Now, check that $f'(x) > 0$ or $f'(x) < 0$.

In the interval $(-\infty, -2)$, $f'(x) > 0$

$\left[\text{as at } x = -4 \in (-\infty, -2), (x-3) < 0, (x+2) < 0, \text{ so } (x-3)(x+2) > 0\right]$

$\Rightarrow \quad f(x)$ is strictly increasing in $(-\infty, -2)$.

In the interval $(-2, 3)$, $f'(x) < 0$

$[\text{as at } x = 1 \in (-2, 3), (x-3) < 0, (x+2) > 0 \text{ so, } (x-3)(x+2) < 0]$

$\Rightarrow \quad f(x)$ is strictly decreasing in $(-2, 3)$.

In the interval $(3, \infty)$, $f'(x) > 0$

$[\text{as at } x = 4 \in (3, \infty), (x-3) > 0, (x+2) > 0 \text{ so, } (x-3)(x+2) > 0]$

$\Rightarrow \quad f(x)$ is strictly increasing in $(3, \infty)$.

Since, the function $f(x)$ is strictly increasing in $(-\infty, -2)$ and $(3, \infty)$ and strictly decreasing in $(-2, 3)$, so $f$ is neither increasing nor decreasing in $R$.

$$f'(x) = 36 + 6x - 6x^2$$
$$= -6(x^2 - x - 6)$$
$$= -6(x+2)(x-3)$$

(i) For $f(x)$ to be increasing, $f'(x) \ge 0$

$\Rightarrow \quad -6(x+2)(x-3) \ge 0$

$(x+2)(x-3) \le 0$

$\Rightarrow \quad -2 \le x \le 3$

Hence, $f(x)$ is increasing in $[2, 3]$.

(ii) For $f(x)$ to be decreasing, $f'(x) \le 0$

$\therefore \quad -6(x+2)(x-3) \le 0$

$\therefore \quad (x+2)(x-3) \ge 0$

$\Rightarrow \quad x \le -2$ or $x \ge 3$

Hence, $f(x)$ is decreasing on $(-\infty, -2] \cup [3, \infty)$.

**EXAMPLE |7|** Find interval(s) in which the function $f(x) = \sin x + \cos x$, $x \in \left(0, \frac{\pi}{2}\right)$ is strictly increasing or decreasing.

**Sol.** Given function is $f(x) = \sin x + \cos x$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = \cos x - \sin x$$

Put $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin x = 0$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Now, $\frac{\pi}{4}$ divides the interval $\left(0, \frac{\pi}{2}\right)$ into two sub intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

In interval $\left(0, \frac{\pi}{4}\right)$, $\cos x > \sin x$

$\therefore \quad \cos x - \sin x > 0$

$\Rightarrow \quad f'(x) > 0$

So, $f(x)$ is strictly increasing function in $\left(0, \frac{\pi}{4}\right)$

In interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x < \sin x$

# TOPIC PRACTICE 2

## OBJECTIVE TYPE QUESTIONS

**1** The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is **[NCERT Exemplar]**

(a) $[-1, \infty)$ (b) $[-2, -1]$ (c) $(-\infty, -2]$ (d) $[-1, 1]$

**2** If $y = x(x-3)^2$ decreases for the values of $x$ given by **[NCERT Exemplar]**

(a) $1 < x < 3$ (b) $x < 0$

(c) $x > 0$ (d) $0 < x < \frac{3}{2}$

**3** The function $f(x) = \tan x - x$ **[NCERT Exemplar]**

(a) always increases

(b) always decreases

(c) never increases

(d) sometimes increases and sometimes decreases

**4** The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

$\therefore \quad \cos x - \sin x < 0$

$\Rightarrow \quad f'(x) < 0$

So, $f(x)$ is strictly decreasing function in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

**EXAMPLE |8|** Find the intervals on which the function $f(x) = 5 + 36x + 3x^2 - 2x^3$ is

(i) increasing. (ii) decreasing.

**Sol.** We have, $f(x) = 5 + 36x + 3x^2 - 2x^3$

On differentiating both sides w.r.t. $x$, we get

## VERY SHORT ANSWER Type Questions

**6** Show that the function given by $f(x) = 7x - 3$ is strictly increasing on $R$. **[NCERT]**

**7** Prove that $f(x) = ax + b$, where $a$ and $b$ are constants and $a > 0$ is strictly increasing function for all real values of $x$ without using the derivative.

**8** Show that $f(x) = x^2$ is strictly decreasing in $(-\infty, 0)$.

**9** Prove that $f(x) = \frac{3}{x} + 7$ is strictly decreasing for $x \in R, (x \neq 0)$.

## SHORT ANSWER Type I Questions

**10** Show that the function $f$ defined by $f(x) = (x-1)e^x + 1$ is an increasing function for all $x > 0$. **[Delhi 2020]**

**11** Show that the function $f$ given by $f(x) = x^3 - 3x^2 + 4x, x \in R$ is strictly increasing on $R$. **[NCERT]**

**12** Let $I$ be an interval disjoint from $(-1, 1)$. Prove that function $f(x) = \left(x + \frac{1}{x}\right)$ is strictly increasing on $I$.

**13** Show that the function $f(x) = \frac{x}{3} + \frac{3}{x}$ decreases in the intervals $(-3, 0) \cup (0, 3)$. **[Delhi 2020]**

**14** Show that the function $f$ given by $f(x) = \tan^{-1}(\sin x + \cos x), x > 0$ is always an strictly increasing function in $\left(0, \frac{\pi}{4}\right)$. **[NCERT]**

**15** Show that $f(x) = \frac{x}{\sin x}$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

(a) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) decreasing in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (d) decreasing in $\left[0, \frac{\pi}{2}\right]$

**5** Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$? **[NCERT Exemplar]**

(a) $\sin 2x$ (b) $\tan x$

(c) $\cos x$ (d) $\cos 3x$

**18** Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in $R$. **[NCERT Exemplar]**

**19** Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$ is increasing in $R$. **[NCERT Exemplar]**

**20** Find the intervals on which the function $f(x) = (x-1)^3 (x-2)^2$ is strictly increasing and strictly decreasing. **[Delhi 2020]**

**21** Find the value(s) of $x$ for which $y = [x(x-2)]^2$ is an increasing function. **[NCERT; All India 2014]**

**22** Find the interval in which $y = x^2 e^{-x}$ is increasing. **[NCERT]**

**23** Find the intervals in which the following functions is strictly increasing or strictly decreasing. (Each part carries 4 Marks)

(i) $f(x) = -2x^3 - 9x^2 - 12x + 1$ **[NCERT; Delhi 2011]**

(ii) $f(x) = 2x^3 - 3x^2 - 36x + 7$ **[All India 2017C]**

(iii) $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36x}{5} + 11$ **[All India 2014C]**

(iv) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ **[Delhi 2014]**

(v) $f(x) = \sin^4 x + \cos^4 x, 0 \leq x \leq \frac{\pi}{2}$

**24** Find the intervals in which the following functions is increasing or decreasing. (Each part carries 4 Marks)

(i) $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ **[NCERT]**

(ii) $f(x) = x^4 - \frac{4x^3}{3}$

(iii) $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$

## SHORT ANSWER Type II Questions

**16** Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$. (Each part carries 4 Marks) **[NCERT]**

(i) $\cos x$ (ii) $\cos 2x$
(iii) $\cos 3x$ (iv) $\tan x$

**17** Prove that the function $f$ given by $f(x) = \log(\cos x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$. **[NCERT]**

**25** Prove that the function $f$ defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence find the interval in which $f(x)$, is

(i) strictly increasing. (ii) strictly decreasing.

**26** Prove that $\frac{x}{1+x} < \log(1+x) < x$ for $x > 0$. **[Delhi 2014]**

## LONG ANSWER Type Questions

**27** Find the intervals in which the function $f$ given by $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing. **[NCERT; Delhi 2017C, 2010]**

**28** Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. **[Delhi 2016]**

**29** Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of $\theta$ on $\left[0, \frac{\pi}{2}\right]$. **[All India 2016]**

# HINTS & SOLUTIONS

**1.** (b) We have, $f(x) = 2x^3 + 9x^2 + 12x - 1$

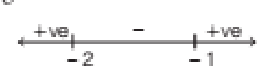
$\therefore \quad f'(x) = 6x^2 + 18x + 12$

$= 6(x^2 + 3x + 2)$

$= 6(x+2)(x+1)$

So, $f'(x) \le 0$, for decreasing.

On drawing number lines as below




We see that $f'(x)$ is decreasing in $[-2, -1]$.

(iv) $f(x) = x^x, x > 0$

(v) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ **[All India 2012]**

(vi) $f(x) = (x-1)^3(x-2)^2$ **[All India 2011]**

(vii) $f(x) = 2x^3 + 9x^2 + 12x + 20$ **[Delhi 2011]**

(viii) $f(x) = 2x^3 - 9x^2 + 12x - 15$ **[Delhi 2011]**

(ix) $f(x) = x^4 - 2x^2$

(x) $f(x) = 2\log(x-2) - x^2 + 4x + 1$

(xi) $f(x) = \frac{x}{\log x}$

$\therefore \quad f'(x) = \sec^2 x - 1 \Rightarrow f'(x) \ge 0, \forall\ x \in R$

So, $f(x)$ always increases.

**4.** (b) We have,

$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$

$\therefore f'(x) = 12\sin^2 x \cdot \cos x - 12\sin x \cdot \cos x + 12\cos x$

$= 12[\sin^2 x \cdot \cos x - \sin x \cdot \cos x + \cos x]$

$= 12\cos x[\sin^2 x - \sin x + 1]$

$\Rightarrow f'(x) = 12\cos x[\sin^2 x + (1 - \sin x)]$ ...(i)

$\because \quad 1 - \sin x \ge 0$ and $\sin^2 x \ge 0$

$\therefore \sin^2 x + 1 - \sin x \ge 0$

Hence, $f'(x) > 0$, when $\cos x > 0$ *i.e.*, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So, $f(x)$ is increasing when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $f'(x) < 0$, when $\cos x < 0$ *i.e.*, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Since, $\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

**5.** (c) In the interval $\left(0, \frac{\pi}{2}\right)$, $f(x) = \cos x$

$\Rightarrow \quad f'(x) = -\sin x$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

**6.** Here, $f'(x) = 7 > 0, \forall x \in R$

$\therefore f$ is strictly increasing on $R$.

**2.** (a) We have, $y = x(x-3)^2$

$$\therefore \quad \frac{dy}{dx} = x \cdot 2(x-3) \cdot 1 + (x-3)^2 \cdot 1$$
$$= 2x^2 - 6x + x^2 + 9 - 6x$$
$$= 3x^2 - 12x + 9$$
$$= 3(x^2 - 3x - x + 3)$$
$$= 3(x-3)(x-1)$$


So, $y = x(x-3)^2$ decreases for (1, 3).

[since, $y' < 0$ for all $x \in (1, 3)$, hence $y$ is decreasing on (1, 3)]

**3.** (a) We have, $f(x) = \tan x - x$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = 3(-1)x^{-2} + 0 = \frac{-3}{x^2} < 0, \forall x \in R - \{0\}$$

$\therefore f(x)$ is strictly decreasing for $x \in R, (x \neq 0)$.

**Hence proved.**

**10.** We have, $f(x) = (x-1)e^x + 1$

On differentiating w.r.t. $x$, we get

$$f'(x) = (x-1)e^x + e^x$$
$$f'(x) = xe^x$$

For all $\quad x > 0 \Rightarrow f'(x) > 0$

$\therefore f(x)$ is an increasing for all $x > 0$.

**11. Hint** $f'(x) = 3x^2 - 6x + 4 = 3(x^2 - 2x + 1) + 1$

$$= 3(x-1)^2 + 1 > 0, \forall x \in R$$

**12.** We have, $f(x) = x + \dfrac{1}{x}$ and $I = R - (-1, 1)$

$$= (-\infty, -1] \cup [1, \infty).$$

Clearly, $\quad f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2}$

$> 0, \ \forall \ x \in I$

$[\because x^2 > 1, \ \forall \ x \in I$ and $x^2 - 1 > 0, \ \forall \ x \in I]$

Hence, $f(x)$ is strictly increasing on $I$.

**13.** Given, $f(x) = \dfrac{x}{3} + \dfrac{3}{x}$

$$\Rightarrow \quad f'(x) = \frac{1}{3} - \frac{3}{x^2} \Rightarrow f'(x) = \frac{x^2 - 9}{3x^2}$$

When $\quad x \in (-3, 0) \cup (0, 3)$

$$f'(x) < 0$$

$\therefore f(x)$ is decreasing function in $(-3, 0) \cup (0, 3)$.

**14.** Here, $f'(x) = \dfrac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$

$$= \frac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2}$$

**7.** Let $x_1, x_2 \in R$.

Now, consider $x_1 < x_2 \quad \Rightarrow \ ax_1 < ax_2$

[multiplying both sides by $a$ as $a > 0$, given]

$\Rightarrow \quad ax_1 + b < ax_2 + b \qquad$ [adding $b$ on both sides]

$\Rightarrow \quad f(x_1) < f(x_2) \qquad [\because f(x) = ax + b]$

Thus, $\quad x_1 < x_2$

$\Rightarrow \quad f(x_1) < f(x_2)$

Hence, $f$ is strictly increasing on $R$.

**8.** Here, $f'(x) = 2x < 0, \forall x \in (-\infty, 0)$

$\therefore f$ is strictly decreasing function in $(-\infty, 0)$.

**9.** Given function is $f(x) = \dfrac{3}{x} + 7, \ x \neq 0$

$$\Rightarrow \quad f(x) = 3x^{-1} + 7$$

Clearly, $\sin^2 x > 0, \ \forall x \in \left(0, \dfrac{\pi}{2}\right)$

Now, $\quad f'(x) \geq 0$

$\Rightarrow \quad \sin x - x\cos x \geq 0$

$\Rightarrow \quad \sin x > x \cos x$

$\Rightarrow \quad \tan x > x,$

which is true, $\forall x \in \left(0, \dfrac{\pi}{2}\right)$.

Thus, $f'(x) \geq 0, \ \forall \ x \in \left(0, \dfrac{\pi}{2}\right)$.

So, $f(x)$ is increasing on $\left(0, \dfrac{\pi}{2}\right)$.

**16. Hint** Find $\dfrac{dy}{dx}$ and see which is smaller than 0 in $\left(0, \dfrac{\pi}{2}\right)$.

**[Ans.** (i) and (ii) are strictly decreasing]

**17.** Similar as Example 5.

**18.** Given, for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = \sqrt{3}\cos x - (-\sin x) - 2a$$
$$= \sqrt{3}\cos x + \sin x - 2a$$
$$= 2\left[\frac{\sqrt{3}}{2} \cdot \cos x + \frac{1}{2} \cdot \sin x\right] - 2a$$

[multiply and divide by 2]

$$= 2\left[\cos\frac{\pi}{6} \cdot \cos x + \sin\frac{\pi}{6} \cdot \sin x\right] - 2a$$
$$= 2\cos\left(\frac{\pi}{6} - x\right) - 2a$$

$[\because \cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B]$

$$= 2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right]$$

We know that, $\cos x \in [-1, 1]$ and $a \geq 1$.

$\therefore \qquad \cos x - \sin x > 0$

or $\qquad \dfrac{\cos x - \sin x}{1+(\sin x + \cos x)^2} > 0$

$\Rightarrow \qquad f'(x) > 0, \forall\, x \in \left(0, \dfrac{\pi}{4}\right)$

Hence, $f(x)$ is strictly increasing in $\left(0, \dfrac{\pi}{4}\right)$.

**15.** We have, $f(x) = \dfrac{x}{\sin x}$

$\Rightarrow \qquad f'(x) = \dfrac{\sin x \cdot 1 - x \cdot \cos x}{\sin^2 x}$

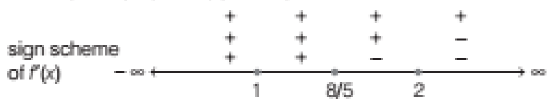
$= \dfrac{\sin x - x\cos x}{\sin^2 x}$

**20.** Given function is $f(x) = (x-1)^3(x-2)^2$.

On differentiating w.r.t. $x$, we get

$f'(x) = 3(x-1)^2(x-2)^2 + 2(x-2)(x-1)^3$

$= (x-1)^2(2-x)(8-5x)$




For strictly increasing $f'(x) > 0$, we get

positive $f'(x)$ in the interval $\left(-\infty, \dfrac{8}{5}\right) \cup (2, \infty)$.

and for strictly decreasing $f'(x) < 0$, we get

negative $f'(x)$ in the interval $\left(\dfrac{8}{5}, 2\right)$.

**21.** Given, $y = [x(x-2)]^2 = [x^2 - 2x]^2$

On differentiating both sides w.r.t. $x$, we get

$\dfrac{dy}{dx} = 2(x^2 - 2x)\cdot\dfrac{d}{dx}(x^2 - 2x)$

$= 2(x^2 - 2x)(2x - 2) = 2[2x^3 - 6x^2 + 4x]$

$= 4x[x^2 - 3x + 2] = 4x(x-2)(x-1)$

On putting $\dfrac{dy}{dx} = 0$, we get

$4x(x-2)(x-1) = 0 \Rightarrow x = 0, 1$ and $2$


These values divide the real line into four disjoint intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$.
In each interval, nature of $y(x)$ is given below

| Interval | Sign of $f(x)$ | Nature of $y(x)$ |
|---|---|---|
| $(-\infty, 0)$ | $(-)(-)(-) = -$ve | Strictly decreasing |
| $(0, 1)$ | $(+)(-)(-) = +$ ve | Strictly increasing |
| $(1, 2)$ | $(+)(-)(+) = -$ ve | Strictly decreasing |
| $(2, \infty]$ | $(+)(+)(+) = +$ ve | Strictly increasing |

Therefore, $y(x)$ is increasing in $(0, 1)$ and $(2, \infty)$.

Hence, $f(x)$ is a decreasing function in $R$.

**19. Hint** $f'(x) = \dfrac{1+2x^2}{1+x^2} - \dfrac{1}{\sqrt{1+x^2}} = \dfrac{1+2x^2-\sqrt{1+x^2}}{1+x^2}$

Now, $f'(x) \geq 0$

$\Rightarrow \qquad \dfrac{1+2x^2-\sqrt{1+x^2}}{1+x^2} \geq 0$

$\Rightarrow \qquad 1+2x^2 \geq \sqrt{1+x^2}$

$\Rightarrow \qquad 1+4x^4+4x^2 \geq 1+x^2$

$\Rightarrow \qquad 4x^4+3x^2 \geq 0,$

which is true for all $x \in R$.

So, $f'(x) \geq 0, \forall\, x \in R$.

The points, $x = -2$ and $x = -1$ divide the real line into their disjoint intervals $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.


The nature of function in these intervals are given below

| Interval | Sign of $f'(x)$ $f'(x) = -6(x+2)(x+1)$ | Nature of function |
|---|---|---|
| $(-\infty, -2)$ | $(-)(-)(-) = (-) < 0$ | Strictly decreasing |
| $(-2, -1)$ | $(-)(+)(-) = (+) > 0$ | Strictly increasing |
| $(-1, \infty)$ | $(-)(+)(+) = (-) < 0$ | Strictly decreasing |

Hence, $f(x)$ is strictly increasing in the interval $(-2, -1)$ and $f(x)$ is strictly decreasing in the interval $(-\infty, -2) \cup (-1, \infty)$.

(ii) Similar as Example 6.

[**Ans.** Strictly increasing in $(-\infty, -2) \cup (3, \infty)$ and strictly decreasing in $(-2, 3)$]

(iii) Given, $f(x) = \dfrac{3}{10}x^4 - \dfrac{4}{5}x^3 - 3x^2 + \dfrac{36x}{5} + 11$

On differentiating both sides w.r.t. $x$, we get

$f'(x) = \dfrac{12x^3}{10} - \dfrac{12x^2}{5} - 6x + \dfrac{36}{5} + 0$

Put $\quad f'(x) = 0$

$\Rightarrow \qquad \dfrac{12x^3}{10} - \dfrac{12x^2}{5} - 6x + \dfrac{36}{5} = 0$

$\Rightarrow \qquad \dfrac{6x^3 - 12x^2 - 30x + 36}{5} = 0$

$\Rightarrow \qquad x^3 - 2x^2 - 5x + 6 = 0 \quad \left[\text{divide by } \dfrac{6}{5}\right]$

$\Rightarrow \qquad (x-1)(x^2 - x - 6) = 0$

$\Rightarrow \qquad (x-1)(x+2)(x-3) = 0$

$\Rightarrow \qquad x - 1 = 0$

or $\qquad x + 2 = 0$ or $x - 3 = 0$

$\therefore \qquad x = -2, 1, 3$

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

$\Rightarrow \ 0 < x < 1$ and $x > 2$.

**22.** **Hint** $\frac{dy}{dx} = -e^{-x}\ x(x-2)$. Solve $\frac{dy}{dx} \ge 0$. [**Ans.** [0, 2]]

**23.** (i) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = -6x^2 - 18x - 12$$

$$\Rightarrow \quad f'(x) = -6(x^2 + 3x + 2)$$

$$\Rightarrow \quad f'(x) = -6(x^2 + 2x + x + 2)$$

$$= -6[x(x+2) + 1(x+2)]$$

$$= -6(x+2)(x+1)$$

Now, put $f'(x) = 0$

$\Rightarrow -6(x+2)(x+1) = 0$

$\Rightarrow \quad x = -2, -1$

| Interval | Sign of $f'(x)$ <br> $f'(x) = (x-1)(x+2)(x-3)$ | Nature of function |
|---|---|---|
| $(-\infty, -2]$ | $(-)(-)(-) = (-) < 0$ | Strictly decreasing |
| $[-2, 1]$ | $(-)(+)(-) = (+) > 0$ | Strictly increasing |
| $[1, 3]$ | $(+)(+)(-) = (-) < 0$ | Strictly decreasing |
| $[3, \infty)$ | $(+)(+)(+) = (+) > 0$ | Strictly increasing |

(a) $f(x)$ is strictly increasing in the interval $(-2, 1) \cup (3, \infty)$.

(b) $f(x)$ is strictly decreasing in the interval $(-\infty, -2) \cup (1, 3)$.

(iv) Similar as part (iii).

[**Ans.** Strictly increasing $(-1, 0) \cup (2, \infty)$ and strictly decreasing $(-\infty, -1) \cup (0, 2)$]

(v) Given, $f(x) = \sin^4 x + \cos^4 x,\ x \in \left[0, \frac{\pi}{2}\right]$.

On differentiating both sides w.r.t. $x$, we get

$f'(x) = 4\sin^3 x \cos x + 4\cos^3 x(-\sin x)$

[by chain rule of derivative]

$= 4\sin^3 x \cos x - 4\cos^3 x \sin x$

$= -4\sin x \cos x(\cos^2 x - \sin^2 x)$

$= -2\sin 2x \cos 2x$

[$\because \cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2\sin x \cos x$]

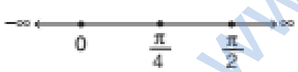
$= -\sin 4x$

Now, put $f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow \sin 4x = 0$

$\Rightarrow \quad 4x = 0, \pi, 2\pi, \ldots \qquad \left[\because x \in \left[0, \frac{\pi}{2}\right] \Rightarrow 4x \in [0, 2\pi]\right]$

$\Rightarrow \quad x = \frac{\pi}{4}$ as $\frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$

The point $x = \pi/4$ divides the interval $\left[0, \frac{\pi}{2}\right]$ into two disjoint intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$




The nature of function in these intervals are given below

| Interval | Test value | Sign of $f'(x)$, $f'(x) = -\sin 4x$ | Nature of function |
|---|---|---|---|
| $\left[0, \frac{\pi}{4}\right)$ | $x = \frac{\pi}{6}$ | $f'\left(\frac{\pi}{6}\right) = -\sin\frac{4\pi}{6}$ $= -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2} < 0$ | Strictly decreasing |
| $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ | $x = \frac{\pi}{3}$ | $f'\left(\frac{\pi}{3}\right) = -\sin\frac{4\pi}{3}$ $= -\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} > 0$ | Strictly increasing |

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = 4\left\{\frac{(2+\cos x)\cos x - \sin x(0 - \sin x)}{(2+\cos x)^2}\right\} - 1$$

$$= 4\left\{\frac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2}\right\} - 1$$

[$\because \cos^2 x + \sin^2 x = 1$]

$$= \frac{8\cos x + 4}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4 - (2+\cos x)^2}{(2+\cos x)^2}$$

$$= \frac{8\cos x + 4 - 4 - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$= \frac{\cos x(4 - \cos x)}{(2+\cos x)^2}$$

We know that, $-1 \le \cos x \le 1$

$\Rightarrow \ 4 - \cos x > 0$ and $(2 + \cos x)^2 > 0$

(a) For $f(x)$ to be increasing,

$f'(x) \ge 0$, when $\cos x \ge 0$

[$\because \cos x$ is positive in I and IV quadrants]

Thus, $f(x)$ is increasing in the interval

$\left(0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right)$.

(b) For $f(x)$ to be decreasing,

$f'(x) \le 0$, when $\cos x \le 0$

[$\because \cos x$ is negative in II and III quadrants]

Hence, $f(x)$ is decreasing in the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

(ii) Similar as Example 8.

[**Ans.** Increasing in $[1, \infty)$, decreasing in $(-\infty, 1]$]

(iii) Similar as Example 8.

So, $f(x)$ is strictly increasing in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right]$ and

$f(x)$ is strictly decreasing in the interval $\left[0, \frac{\pi}{4}\right)$.

**24.** (i) Given, $f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$

$= \frac{4\sin x - x(2+\cos x)}{2+\cos x}$

$= \frac{4\sin x}{2+\cos x} - \frac{x(2+\cos x)}{2+\cos x}$

$= \frac{4\sin x}{2+\cos x} - x$

(vi) Similar as Example 8.

$\left[\text{Ans. Increasing in } (-\infty, 1] \cup \left[1, \frac{8}{5}\right] \cup [2, \infty), \text{ decreasing in } \left[\frac{8}{5}, 2\right]\right]$

(vii) Similar as Example 8.

[**Ans.** Increasing in $(-\infty, 2] \cup [-1, \infty)$, decreasing in $[-2, -1]$]

(viii) Similar as Example 8.

[**Ans.** Increasing in $(-\infty, 1] \cup [2, \infty)$, decreasing in $[1, 2]$]

(ix) Similar as Example 8.

[**Ans.** $f(x)$ is increasing in $[-1, 0] \cup [1, \infty]$, $f(x)$ is decreasing in $[-\infty, -1] \cup [0, 1]$]

(x) Similar as Example 8.

[**Ans.** $f(x)$ is increasing in $(2, 3]$, $f(x)$ is decreasing in $[3, \infty)$]

(xi) **Hint** $f(x) = \frac{x}{\log x}$

$\therefore \quad f'(x) = \frac{\log x - 1}{(\log x)^2}, \; x \neq 1$

Now, $f'(x) \geq 0, \; x \geq e$ and $f'(x) \leq 0, \; x \in (0, e] - \{1\}$

[**Ans.** $f(x)$ is increasing in $[e, \infty)$, $f(x)$ is decreasing in $(0, e] - \{1\}$]

**25.** **Hint** $f'(x) = 2x - 1$, which is both positive and negative in $(-1, 1)$.

Hence, $f(x)$ is neither increasing nor decreasing, further proceed as Example 6.

$\left[\text{Ans. (i) } \left(\frac{1}{2}, 1\right) \text{ (ii) } \left(-1, \frac{1}{2}\right)\right]$

**26.** Let $f(x) = \log(1+x) - \frac{x}{1+x}$.

Then, $f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0$ for $x > 0$

[**Ans.** Increasing in $x \leq -1$ or $x \geq 1$, decreasing in $-1 \leq x \leq 1$]

(iv) **Hint** Let $y = x^x$, $\log y = x \log x$

$\Rightarrow \quad \frac{dy}{dx} = x^x(1 + \log x) = 0 \Rightarrow x = \frac{1}{e}$

$\left[\text{Ans. Increasing in } \left[\frac{1}{e}, \infty\right), \text{ decreasing in } \left(0, \frac{1}{e}\right]\right]$

(v) Similar as Example 8.

[**Ans.** Increasing in $[1, 2] \cup [3, \infty)$ and decreasing in $(-\infty, 1] \cup [2, 3]$]

Also, $g(0) = 0$

Now, $\quad x > 0 \Rightarrow g(x) > g(0)$

$\Rightarrow \quad g(x) > 0 \quad [\because g(0) = 0]$

$\Rightarrow \quad [x - \log(1+x)] > 0$

$x > \log(1+x) \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$\frac{x}{1+x} < \log(1+x) < x$ for $x > 0$

**27.** Similar as Example 7.

$\left[\text{Ans. } f \text{ is strictly increasing in the intervals } \left[0, \frac{\pi}{4}\right) \text{ and } \left(\frac{5\pi}{4}, 2\pi\right], \text{ and } f \text{ is strictly decreasing in the interval } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)\right]$

**28.** Given, $f(x) = \sin 3x - \cos 3x, 0 < x < \pi$

On differentiating both sides w.r.t. $x$, we get

$f'(x) = 3\cos 3x + 3\sin 3x$

On putting $f'(x) = 0$

$\Rightarrow \quad \sin 3x = -\cos 3x \Rightarrow \tan 3x = -1$

$\Rightarrow \quad 3x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

[$\because \tan\theta$ is negative in II and IV quadrants]

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$

Now, we find intervals and check in which interval $f(x)$ is strictly increasing or strictly decreasing.

| Interval | Test value | $f'(x) = 3(\cos 3x + \sin 3x)$ | Sign of $f'(x)$ |
|---|---|---|---|
| $0 < x < \frac{\pi}{4}$ | At $x = \frac{\pi}{6}$ | $3\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 3(0+1) = 3$ | + ve |

Since, $f'(x) > 0,\ \forall x > 0$ and $f'(0) = 0$, it follows that $f(x)$ is increasing on $(0, \infty)$.

Now, $x > 0 \Rightarrow f(x) > f(0) \Rightarrow f(x) > 0 \quad [\because f(0) = 0]$

$\Rightarrow \left[\log(1 + x) - \dfrac{x}{1 + x}\right] > 0$

$\Rightarrow \quad \log(1 + x) > \dfrac{x}{1 + x}$ ...(i)

Again, let $g(x) = x - \log(1 + x)$,

then $g'(x) = \left[1 - \dfrac{1}{1 + x}\right] = \dfrac{x}{1 + x} > 0$ for $x > 0$

Now, $g'(x) > 0,\ \forall x > 0$ and $g'(0) = 0$

$\therefore$ $g(x)$ is strictly increasing on $(0, \infty)$.

| | | | |
|---|---|---|---|
| $\frac{\pi}{4} < x < \frac{7\pi}{12}$ | At $x = \frac{\pi}{3}$ | $3(\cos \pi + \sin \pi)$ $= 3(-1 + 0) = -3$ | – ve |
| $\frac{7\pi}{12} < x < \frac{11\pi}{12}$ | At $x = \frac{3\pi}{4}$ | $3\left(\cos \frac{9\pi}{4} + \sin \frac{9\pi}{4}\right)$ $= 3\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 3\sqrt{2}$ | + ve |
| $\frac{11\pi}{12} < x < \pi$ | At $x = \frac{23\pi}{24}$ | $3\left(\cos \frac{23\pi}{8} + \sin \frac{23}{8}\pi\right)$ $< 0$ | – ve |

Here, we see that $f'(x) > 0$, for $0 < x < \frac{\pi}{4}$ and $\frac{7\pi}{12} < x < \frac{11\pi}{12}$, so $f(x)$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$.

While $f'(x) < 0$ in $\frac{\pi}{4} < x < \frac{7\pi}{12}$ and $\frac{11\pi}{12} < x < \pi$.

So, $f(x)$ is strictly decreasing in the intervals $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$ and $\left(\frac{11\pi}{12}, \pi\right)$.

**29.** Let $y = f(\theta) = \dfrac{4 \sin \theta}{2 + \cos \theta} - \theta$

On differentiating both sides w.r.t. $\theta$, we get

$$f'(\theta) = \frac{(2 + \cos \theta)(4 \cos \theta) - 4 \sin \theta\,(0 - \sin \theta)}{(2 + \cos \theta)^2} - 1$$

[by quotient rule of derivative]

$$= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} - 1 \qquad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$= \frac{8 \cos \theta + 4 - 4 - \cos^2 \theta - 4 \cos \theta}{(2 + \cos \theta)^2}$$

$$= \frac{4 \cos \theta - \cos^2 \theta}{(2 + \cos \theta)^2}$$

$$= \frac{\cos \theta\,(4 - \cos \theta)}{(2 + \cos \theta)^2}$$

For all $x \in \left(0, \frac{\pi}{2}\right)$, $\dfrac{\cos \theta\,(4 - \cos \theta)}{(2 + \cos \theta)^2} > 0$

$\Rightarrow \quad f'(\theta) > 0,\ \forall\, \theta \in \left(0, \frac{\pi}{2}\right)$

Therefore, $f(\theta)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$. Also, the given function is continuous at $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Hence, $f(\theta)$ is increasing in $\left[0, \frac{\pi}{2}\right]$.

**Hence proved.**

# |TOPIC 3|
# Maxima and Minima



Here, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. We will find the turning points of the graph of a function and then find points at which the graph reaches its highest (or lowest) locally. Also, find the absolute maximum and absolute minimum of a function.

## Maximum and Minimum Values in an Interval

Let $f$ be a function defined on an interval $I$ (i.e. $I$ is the domain of $f$).

(i) **Maximum Value of a Function** A function $f(x)$ is

or minimum value in the interval. Then, $f(c)$ is called the extreme value of $f$ in that interval and $c$ is the extreme point.

**Note** Graphs of certain particular functions help us to find maximum value and minimum value at a point. Also, from graph, we can find maximum/minimum value of a function at a point at which it is not even differentiable.

**EXAMPLE |1|** Find the maximum and minimum values of $f$, if any of the function given by $f(x) = |x|,\ x \in R$. Also, find the maximum value in $[-2, 1]$.

*Sol.* We have, $f(x) = |x|$

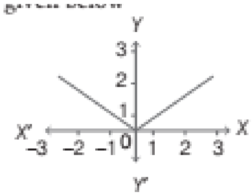
Its graph is given below

said to have a maximum value in $I$, if there exists a point $c$ in interval $I$, such that $f(c) > f(x)$ for all $x \in I$. Here, the number $f(c)$ is called the maximum value of $f$ in $I$ and point $c$ is called a point of maximum value of $f$ in $I$.

(ii) **Minimum Value of a Function** A function $f(x)$ is said to have a minimum value in $I$, if there exists a point $c$ in interval $I$, such that $f(c) < f(x)$ for all values of $x$ in $I$. Here, the number $f(c)$ is called the minimum value of $f$ in $I$ and the point $c$ is called a point of minimum value of $f$ in $I$.

(iii) **Extreme Value of a Function** A function $f(x)$ is said to have an extreme value in $I$, if there exists a point $c$ in interval $I$, such that $f(c)$ is either a maximum value

its graph is given below




Here, $f(x) \geq 0$, $\forall x \in R$ and $f(x) = 0$, if $x = 0$.

$\therefore$ Minimum value of $f(x) = 0$ at point $x = 0$.

Also, $f$ has no maximum value in $R$, so no point of maximum value in $R$. Now, if domain of $f = [-2, 1]$, then maximum value of $f$ is $|-2| = 2$ and minimum value is 0.

**Note** Here, function $f(x)$ is not differentiable at $x = 0$, but minimum value exist.

## IMPORTANT RESULTS

(i) Every monotonic function, i.e. either increasing or decreasing function assumes its maximum/minimum value at the end points of the domain of the function.

(ii) Every continuous function on a closed interval has a maximum and a minimum value.

**e.g.** Let $f(x) = x$, $x \in (0, 1)$. In $(0, 1)$, $f(x)$ has neither the maximum value nor the minimum value. If we include the points 0 and 1 in the domain, i.e. if domain is $[0,1]$, then $f(x) = x$ has minimum value 0 at $x = 0$ and maximum value 1 at $x = 1$.

**Note** If a function $f$ defined on an interval $I$, then following possibilities are arises

(i) It may attain the maximum value at a point in $I$ but not the minimum value at any point in $I$.

(ii) It may attain the minimum value at a point in $I$ but not the maximum value at any point in $I$.

(iii) It may attain both the maximum and minimum values at some point in $I$.

(iv) It may not attain both the maximum and minimum values at any point in $I$.

# LOCAL MAXIMA AND LOCAL MINIMA

Consider points $P, Q, R$ and $S$ on the graph, at which the function changes its nature from decreasing to increasing or *vice-versa*.

These points may be called **turning points** of the given function. At turning points, the graph has either a little hill or a little valley. Here, the points $P$ and $R$ may be regarded as points of local minimum value or relative minimum value and points $Q$ and $S$ may be regarded as points of local maximum value or relative maximum value for the function.

value $f(c)$ is called the local maximum value (or relative maximum value) of $f$.

(ii) $c$ is called a point of local minima, if there is an $h > 0$ such that $f(c) < f(x)$ for all $x$ in $(c - h, c + h)$. Here, value $f(c)$ is called the local minimum value (or relative minimum value) of $f$.

## GEOMETRICAL INTERPRETATION

(i) Suppose $x = c$ is a point of local maxima of $f$, then the graph of $f$ around $c$ will be as shown in Fig. (i).

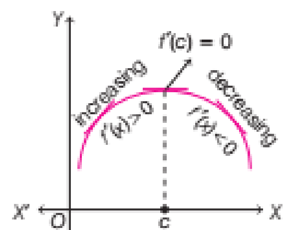



Fig. (i)

(ii) Similarly, if $c$ is a point of local minima of $f$, then the graph of $f$ around $c$ will be as shown in the Fig. (ii).

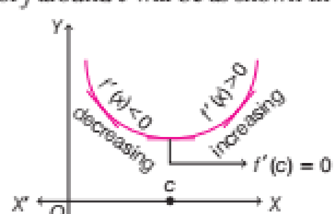
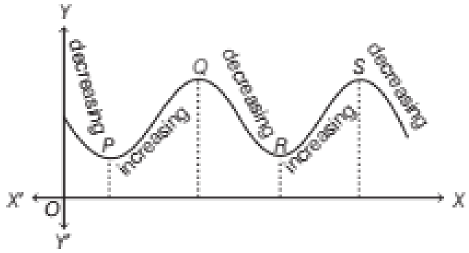



Fig. (ii)

Thus, the nature of $f$ in intervals is given below

| Interval | $f$ in Fig. (i) | $f$ in Fig. (ii) |
|---|---|---|
| $(c - h, c)$ | increasing [i.e. $f'(x) > 0$] | decreasing [i.e. $f'(x) < 0$] |
| $(c, c + h)$ | decreasing [i.e. $f'(x) < 0$] | increasing [i.e. $f'(x) > 0$] |




The local maximum value and local minimum value of the function are referred to as local maxima and local minima, respectively of the function.

**Definition** Let $f$ be a real valued function and $c$ be an interior point in the domain of $f$, then

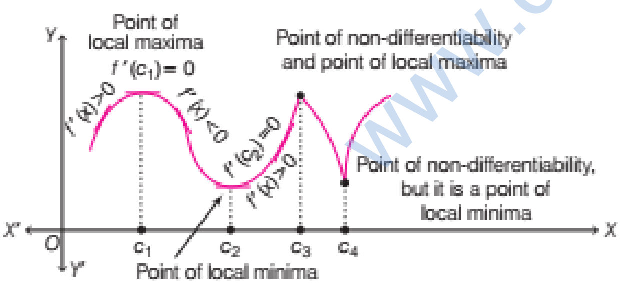
(i) $c$ is called a point of local maxima, if there is an $h > 0$ such that $f(c) > f(x)$ for all $x$ in $(c - h, c + h)$. Here,

**Theorem 2 (First Derivative Test)** Let $f$ be a function defined on an open interval $I$ and $f$ be continuous at a critical point $c$ in $I$.

(i) If $f'(x)$ changes sign from positive to negative as $x$ increases through point $c$, i.e. if $f'(x) > 0$ at every point sufficiently close to and to the left of $c$ and $f'(x) < 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of local maxima.

(ii) If $f'(x)$ changes sign from negative to positive as $x$ increases through point $c$, i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of $c$ and $f'(x) > 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of local minima.

(iii) If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Infact, such a point is called **point of inflection**.




**Note**

(i) If c is a point of local maxima of $f$, then $f(c)$ is a local maximum value of $f$. Similarly, if c is a point of local minima of $f$, then $f(c)$ is a local minimum value of $f$.

(ii) Condition for a point to be a point of inflection of a function: A function $f(x)$ has one of its points $x = c$ as an inflection point, if

(a) $f'(c)$ exists

(b) $f''(x) < 0$, if $x > c$ and $f''(x) > 0$, if $x < c$
or $f''(x) > 0$, if $x > c$ and $f''(x) < 0$, if $x < c$

## WORKING RULE TO FIND LOCAL MAXIMA/MINIMA BY FIRST DERIVATIVE TEST

Thus, this suggest that $f'(c)$ must be zero.

## Some Important Theorems

**Theorem 1** Let $f$ be a function defined on an open interval $I$. Suppose $c \in I$ be any point. If $f$ has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or $f$ is not differentiable at $c$.

**Note** The converse of above theorem need not be true that is a point at which the derivative vanishes need not be a point of local maxima or local minima. e.g. If $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f'(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima.

**Critical Point** A point $c$ in the domain of a function $f$ at which either $f'(c) = 0$ or $f$ is not differentiable, is called a critical point of $f$. If $f$ is continuous at point $c$ and $f'(c) = 0$, then there exists an $h > 0$ such that $f$ is differentiable in the interval $(c - h, c + h)$.

| | Values of $x$ | Sign of $f'(x)$ | Nature of point |
|---|---|---|---|
| Close to $a$ | to the right of $a$<br>and to the left of $a$ | $> 0$<br>$< 0$ | Local minima |
| Close to $b$ | to the right of $b$<br>and to the left of $b$ | $< 0$<br>$> 0$ | Local maxima |

IV. Now, find the local maximum or minimum value of function by putting corresponding point in given function in place of $x$.

**Note** If we get two or more than two critical points, then for making above Step III easier, we can divide the real number line into intervals and then check the nature of function in each interval.

**EXAMPLE [2]** Find the points of local maxima and local minima of the function $f(x) = (x-1)^3 (x+1)^2$.

***Sol.*** We have, $f(x) = (x-1)^3 (x+1)^2$

$$\therefore\ f'(x) = 3(x-1)^2 (x+1)^2 + (x-1)^3\, 2(x+1)$$
$$= 3(x-1)^2 (x+1)[3(x+1) + 2(x-1)]$$
$$= (x-1)^2 (x+1)(5x+1)$$

For local maxima or local minima, put $f'(x) = 0$,

$$\Rightarrow (x-1)^2 (x+1)(5x+1) = 0 \Rightarrow x = -\frac{1}{5}, -1, 1$$

For $x = -1$, $f'(x) > 0$ to the left of $x = -1$ and $f'(x) < 0$ to the right of $x = -1$.

$\therefore\ x = -1$ is a point of local maxima.

For $x = -\frac{1}{5}$, $f'(x) < 0$ to the left of $x = -\frac{1}{5}$ and $f'(x) > 0$ to the right of $x = -\frac{1}{5}$.

$\therefore\ x = -\frac{1}{5}$ is a point of local minima.

For $x = 1$, $f'(x) > 0$ to the left of $x = 1$ and $f'(x) > 0$ to the right of $x = 1$.


$\therefore\ x = 1$ is neither a point of local maxima nor local minima. Thus, we have the following table

Suppose $f(x)$ be the given function, then to find local maxima and local minima of $f(x)$ by first derivative test, we use the following steps

I. Firstly, write the given function $f(x)$ and differentiate it w.r.t. $x$, i.e. find $f'(x)$.

II. Put $f'(x) = 0$ and solve it to find critical points (say $a$ and $b$) which could possibly be the points of local maxima or local minima of $f$.

III. Take these points one-by-one and check that it is a point of local maxima/minima, i.e. check the following.

At $x = \frac{-1}{5}$,

$$f\left(-\frac{1}{5}\right) = \left(-\frac{1}{5}-1\right)^3\left(-\frac{1}{5}+1\right)^2 = \left(\frac{-6}{5}\right)^3\left(\frac{4}{5}\right)^2 = -\frac{3456}{3125}$$

At $x = -1$, $f(-1) = (-1-1)^3(-1+1)^2 = 0$

Hence, the point of local maxima is $-1$ and the local maximum value $= 0$ and the point of local minima is $-\frac{1}{5}$ and the local minimum value $= \frac{-3456}{3125}$. Also, point $x = 1$ is the point of inflection.

**EXAMPLE |3|** Find all the points of local maxima and minima of the function $f(x) = x^3 - 6x^2 + 9x - 8$.

**Sol.** We have, $f(x) = x^3 - 6x^2 + 9x - 8$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = 3x^2 - 12x + 9$$

Put $f'(x) = 0$

$\Rightarrow \quad 3x^2 - 12x + 9 = 0$

$\Rightarrow \quad 3(x^2 - 4x + 3) = 0$

$\Rightarrow \quad 3(x-1)(x-3) = 0 \quad$ ...(i)

$\Rightarrow \quad x = 1, 3$

These are the critical points.

Now, for values close to 1 and to the right of 1, $f'(x) < 0$ and for values close to 1 and to the left of 1, $f'(x) > 0$.

$\therefore$ $x = 1$ is a point of local maxima.

For values close to 3 and to the right of 3, $f'(x) > 0$ and for values close to 3 and to the left of 3, $f'(x) < 0$.

$\therefore \quad x = 3$ is a point of local minima.

Hence, $x = 1$ is a point of local maxima and $x = 3$ is a point of local minima for given function.

**EXAMPLE |4|** Find the local maxima or local minima, if any of the function $f(x) = \frac{1}{x^2 + 2}$.

| | Value of $x$ | Sign of $f'(x)$ | Nature of point |
|---|---|---|---|
| Close to $-1$ | to the right of $-1$ | $< 0$ | Local maxima |
| | and to the left of $-1$ | $> 0$ | |
| Close to $\frac{-1}{5}$ | to the right of $\frac{-1}{5}$ | $> 0$ | Local minima |
| | and to the left of $\frac{-1}{5}$ | $< 0$ | |
| Close to 1 | to the right of 1 | $> 0$ | Neither local maxima nor local minima |
| | and to the left of 1 | $> 0$ | |

Thus, $f'(x)$ changes sign from positive to negative.

So, $x = 0$ is a point of local maxima and local maximum value is $f(0) = \frac{1}{0+2} = \frac{1}{2}$.

**Theorem 3** **(Second Derivative Test)** Let $f$ be a function defined on an interval $I$ and $c \in I$ and $f$ be twice differentiable at $c$. Then,

(i) $x = c$ is a point of local maxima, if

$$f'(c) = 0 \quad \text{and} \quad f''(c) < 0$$

The value $f(c)$ is local maximum value of $f$.

(ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.

The value $f(c)$ is local minimum value of $f$.

(iii) The test fails, if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether $c$ is a point of local maxima, local minima or a point of inflection.

**Note** As $f$ is twice differentiable at $c$, we mean second order derivative of $f$ exists at $c$.

## WORKING RULE TO FIND LOCAL MAXIMA/ MINIMA BY SECOND DERIVATIVE TEST

Suppose $f(x)$ be the given function. Then, to find local maxima and local minima by second derivative test, we use the following steps

I. First, write the given function and differentiate it w.r.t. $x$, i.e. find first derivative.

II. Put first derivative, i.e. $f'(x) = 0$ and find the critical points which could be possibly the points of local maxima or local minima.

III. Find second derivative of $f(x)$, i.e. again differentiate $f'(x)$ w.r.t. $x$ to get $f''(x)$.

IV. Check the sign of $f''(x)$ at each critical point obtained in step II to get local maxima/minima, i.e. Suppose $x = a$ is a critical point, then if at $x = a$,

(i) $f''(a) < 0$, then $a$ is point of local maxima.

(ii) $f''(a) > 0$, then $a$ is point of local minima.

**Sol.** We have, $f(x) = \dfrac{1}{x^2+2} \Rightarrow f'(x) = \dfrac{-2x}{(x^2+2)^2}$

For a local maxima or local minima, put $f'(x) = 0$

$$\Rightarrow -\frac{2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$$

Now, when $x$ is slightly less than 0, i.e. when $x$ is negative, then $f'(x) = -\dfrac{2x}{(x^2+2)^2}$ is positive.

When $x$ is slightly greater than 0, then

$f'(x) = -\dfrac{2x}{(x^2+2)^2}$ is negative.

**Sol.** (i) Given, $f(x) = x\sqrt{1-x}$

On differentiating twice w.r.t. $x$, we get

$$f'(x) = \frac{-x}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{(2-3x)}{2\sqrt{1-x}}$$

and $$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}\,(0-3)-(2-3x)\times\dfrac{-1}{2\sqrt{1-x}}}{(\sqrt{1-x})^2}\right]$$

$$= \frac{(3x-4)}{4(1-x)^{3/2}}$$

For local maxima and local minima, put $f'(x) = 0$

$$\Rightarrow (2-3x) = 0 \Rightarrow x = \frac{2}{3}$$

and $$f''\left(\frac{2}{3}\right) = \frac{\left(3\times\dfrac{2}{3}-4\right)}{4\left(1-\dfrac{2}{3}\right)^{3/2}} = \frac{-3^{3/2}}{2} < 0$$

$\therefore$ $x = \dfrac{2}{3}$ is a point of local maxima.

Hence, local maximum value $= f\left(\dfrac{2}{3}\right) = \dfrac{2}{3\sqrt{3}}$.

(ii) Given, $f(x) = \dfrac{x}{(x-1)(x-4)} = \dfrac{x}{(x^2-5x+4)}$

On differentiating twice w.r.t. $x$, we get

$$f'(x) = \frac{(x^2-5x+4)1 - x(2x-5)}{(x^2-5x+4)^2} = \frac{4-x^2}{(x^2-5x+4)^2}$$

and $$f''(x) = \frac{\left[\begin{array}{r}(x^2-5x+4)^2(-2x)-(4-x^2)2\\ (x^2-5x+4)(2x-5)\end{array}\right]}{(x^2-5x+4)^4}$$

$$\Rightarrow f''(x) = \frac{2[(x^2-5x+4)(-x)-(4-x^2)(2x-5)]}{(x^2-5x+4)^3}$$

and $$f''(x) = \frac{\left[\begin{array}{r}(x^2-5x+4)^2(-2x)-(4-x^2)2\\ (x^2-5x+4)(2x-5)\end{array}\right]}{(x^2-5x+4)^4}$$

$$\Rightarrow f''(x) = \frac{2[(x^2-5x+4)(-x)-(4-x^2)(2x-5)]}{(x^2-5x+4)^3}$$

$$= \frac{2[-x^3+5x^2-4x-(8x-20-2x^3+5x^2)]}{(x^2-5x+4)^3}$$

(iii) $f''(a) = 0$, then test fails.

V. Now, find the local maximum or minimum value of function by putting corresponding point in the given function in place of $x$.

**EXAMPLE |5|** Find the local maxima and local minima and the corresponding local maximum and local minimum values of the following functions.

(i) $f(x) = x\sqrt{1-x}$, where $x \le 1$ **[NCERT]**

(ii) $f(x) = \dfrac{x}{(x-1)(x-4)}$, where $1 < x < 4$.

**EXAMPLE |6|** Find all the points of local maxima and local minima of $f(x) = -x + 2\sin x$ on $[0, 2\pi]$. Also, find local maximum and minimum values.

**Sol.** We have, $f(x) = -x + 2\sin x$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = -1 + 2\cos x \quad \text{...(i)}$$

For local maxima and local minima, put $f'(x) = 0$

$$\Rightarrow -1 + 2\cos x = 0 \Rightarrow \cos x = \frac{1}{2}$$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

On differentiating both sides of Eq. (i) w.r.t. $x$, we get

$$f''(x) = -2\sin x$$

At $x = \pi/3$, $f''\left(\dfrac{\pi}{3}\right) = -2\sin\dfrac{\pi}{3} = -\sqrt{3} < 0$

$\therefore$ $x = \dfrac{\pi}{3}$ is a point of local maxima.

At $x = \dfrac{5\pi}{3}$, $f''\left(\dfrac{5\pi}{3}\right) = -2\sin\dfrac{5\pi}{3} = \sqrt{3} > 0$

$\therefore$ $x = \dfrac{5\pi}{3}$ is a point of local minima.

Hence, the points of local maxima is $\dfrac{\pi}{3}$ and local minima is $\dfrac{5\pi}{3}$.

On putting $x = \dfrac{\pi}{3}$ in $f(x)$, we get

$$f\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} + 2\sin\frac{\pi}{3} = -\frac{\pi}{3} + 2\times\frac{\sqrt{3}}{2} = \frac{-\pi}{3} + \sqrt{3}$$

which is the required local maximum value.

On putting $x = \dfrac{5\pi}{3}$ in $f(x)$, we get

$$f\left(\frac{5\pi}{3}\right) = -\frac{5\pi}{3} + 2\sin\frac{5\pi}{3}$$

which is the required local maximum value.

On putting $x = \dfrac{5\pi}{3}$ in $f(x)$, we get

$$f\left(\frac{5\pi}{3}\right) = -\frac{5\pi}{3} + 2\sin\frac{5\pi}{3}$$

$$= \frac{-5\pi}{3} + 2\times\left(\frac{-\sqrt{3}}{2}\right) = \frac{-5\pi}{3} - \sqrt{3}$$

$$= \frac{2[x^3 - 12x + 20]}{(x^2 - 5x + 4)^3}$$

For local maxima and local minima, put

$$f'(x) = 0$$

$\Rightarrow \quad (4 - x^2) = 0 \Rightarrow x = \pm 2$

But $1 < x < 4 \Rightarrow x = 2$ $\quad [\because x = -2 \notin (1, 4)]$

Now, $f''(2) = \frac{2[2^3 - (12 \times 2) + 20]}{(2^2 - (5 \times 2) + 4)^3} = \frac{2(4)}{(-2)^3} = -1 < 0.$

$\therefore$ $x = 2$ is a point of local maxima.

Thus, local maximum value $= f(2)$

$$= \frac{2}{(2-1)(2-4)} = \frac{2}{-2} = -1$$

which is the required local minimum value.

**EXAMPLE |7|** Find two positive numbers $x$ and $y$ such that $x + y = 60$ and $xy^3$ is maximum.

**Sol.** Let $x + y = 60$ and $P = xy^3$.

Then, $P = (60 - y)y^3$ $\quad [\because x = 60 - y]$

On differentiating twice w.r.t. $x$, we get

$$\frac{dP}{dy} = 3y^2(60 - y) + y^3(-1)$$

$$= (180y^2 - 4y^3) = 4y^2(45 - y)$$

and $\frac{d^2P}{dy^2} = (360y - 12y^2) = 12y(30 - y)$

For maximum, put

$$\frac{dP}{dy} = 0 \Rightarrow 4y^2(45 - y) = 0$$

$\Rightarrow \quad y = 0$ or $y = 45$

Neglecting $y = 0$, we are left with $y = 45$

Now, $\left[\frac{d^2P}{dy^2}\right]_{y=45} = (12 \times 45)(30 - 45) = -8100 < 0$

So, $y = 45$ is a point of maximum and another positive maximum number is $x = 60 - 45 = 15$. $\quad [\because x + y = 60]$

Hence, the numbers are 45 and 15.

**EXAMPLE |8|** Find the minimum value of $(ax + by)$, where $xy = c^2$. **[All India 2020, Foreign 2015]**

**Sol.** Let $f(x) = ax + by$, whose minimum value is required.

Then, $f(x) = ax + \frac{bc^2}{x}$ $\quad \left[\because xy = c^2 \Rightarrow y = \frac{c^2}{x}\right]$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = a - \frac{bc^2}{x^2}$$

For maximum or minimum value of $f(x)$, put $f'(x) = 0$

$\Rightarrow \quad a - \frac{bc^2}{x^2} = 0 \Rightarrow a = \frac{bc^2}{x^2} \Rightarrow x^2 = \frac{bc^2}{a}$

$\Rightarrow \quad x = \pm\sqrt{\frac{b}{a}}\,c$

Now, $f''(x) = 0 + \frac{2bc^2}{x^3}$

At $x = +\sqrt{\frac{b}{a}}\,c$, $f''(x) = \frac{2bc^2}{\left(\sqrt{\frac{b}{a}}\,c\right)^3} = +\text{ve}$

Hence, $f(x)$ has minimum value at $x = \sqrt{\frac{b}{a}}\,c$.

At $x = -\sqrt{\frac{b}{a}}\,c$, $f''(x) = \frac{2bc^2}{\left(-\sqrt{\frac{b}{a}}\,c\right)^3} = -\text{ve}$

**Sol.** Let $V$ be the fixed volume of a closed cuboid with length $x$, breadth $x$ and height $y$.

Let $S$ be its surface area, then

$$V = x \times x \times y$$

$\Rightarrow \quad y = \frac{V}{x^2}$ $\quad$ ...(i)

Now, $\quad S = 2(x^2 + xy + xy)$

$\Rightarrow \quad S = 2(x^2 + 2xy) = 2\left(x^2 + \frac{2V}{x}\right)$ [using Eq. (i)]

$\Rightarrow \quad S = 2\left(x^2 + \frac{2V}{x}\right)$

On differentiating w.r.t. $x$, we get

$$\frac{dS}{dx} = 2\left(2x - \frac{2V}{x^2}\right)$$

and $\quad \frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right)$

Now, put $\frac{dS}{dx} = 0 \Rightarrow 2\left(2x - \frac{2V}{x^2}\right) = 0$

$\Rightarrow \quad 2x = \frac{2V}{x^2} \Rightarrow x^3 = V$

$\Rightarrow \quad V = x^3$

$\Rightarrow \quad x \times x \times y = x^3 \Rightarrow y = x$

Now, when $y = x$, then we have $V = x^3$

$\therefore \quad \left(\frac{d^2S}{dx^2}\right)_{y=x} = 4 + \frac{8x^3}{x^3} = 12 > 0$

So, $S$ is minimum when length $= x$, breadth $= x$ and height $= x$, i.e. when it is cube.

**EXAMPLE |10|** All the closed right circular cylindrical cans of volume $128\pi\ \text{cm}^3$, find the dimensions of the can which has minimum surface area. **[Delhi 2014]**

**Sol.** Let $r$ cm be the radius of the base and $h$ cm be the height of the cylindrical can. Let its volume be $V$ and $S$ be its total surface area.

At $x = -\sqrt{\frac{b}{a}}\,c$, $f''(x) = \dfrac{2bc^2}{\left(-\sqrt{\frac{b}{a}}\,c\right)^3} = -\text{ve}$

Hence, $f(x)$ has maximum value at $x = -\sqrt{\frac{b}{a}}\,c$.

When $x = \sqrt{\frac{b}{a}}\,c$, then $y = \dfrac{c^2}{x} = \dfrac{c^2}{\left(\sqrt{\frac{b}{a}}\,c\right)} = \sqrt{\frac{a}{b}}\,c$

$\therefore$ Minimum value of $f(x) = a\sqrt{\frac{b}{a}}\cdot c + b\sqrt{\frac{a}{b}}\cdot c$

$$= \sqrt{ab}\cdot c + \sqrt{ab}\cdot c = 2\sqrt{ab}\cdot c$$

**EXAMPLE |9|** Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. **[All India 2017]**

which has minimum surface area. **[Delhi 2014]**

**Sol.** Let $r$ cm be the radius of the base and $h$ cm be the height of the cylindrical can. Let its volume be $V$ and $S$ be its total surface area.

Then, $\quad V = 128\,\pi\text{ cm}^3 \quad$ [given]

$\Rightarrow \quad \pi r^2 h = 128\pi$

$\Rightarrow \quad h = \dfrac{128}{r^2} \quad$ ...(i)

Also, $\quad S = 2\pi r^2 + 2\pi r h \quad$ ...(ii)

$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\dfrac{128}{r^2}\right) \quad$ [using Eq.(i)]

$\Rightarrow \quad S = 2\pi r^2 + \dfrac{256\pi}{r} \quad$ ...(iii)

On differentiating both sides of Eq. (iii) w.r.t. $r$, we get

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} \quad \text{...(iv)}$$

For maxima or minima, put $\dfrac{dS}{dr} = 0$

$\Rightarrow \quad 4\pi r = \dfrac{256\pi}{r^2} \Rightarrow r^3 = 64$

Taking cube root on both sides, we get

$$r = (64)^{1/3} \Rightarrow r = 4$$

Again, differentiating Eq. (iv) w.r.t. $r$, we get

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

At $r = 4$, $\dfrac{d^2S}{dr^2} = \dfrac{512\pi}{64} + 4\pi = 8\pi + 4\pi = 12\pi > 0$

By second derivative test, the surface area is minimum, when the radius of cylinder is 4 cm.

On putting the value of $r$ in Eq. (i), we get

$$h = \frac{128}{(4)^2} = \frac{128}{16} = 8\text{ cm}$$

Hence, for the minimum surface area of can, the dimensions are $r = 4$ cm and $h = 8$ cm.

**EXAMPLE |11|** The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}$, $x$ and $2x$ is constant. Show that the sum of their volumes is minimum, if $x$ is equal to three times the radius of the shpere. **[All India 2015C]**

**Sol.** Let $r$ be the radius of the sphere and dimensions of cuboid are $\frac{x}{3}$, $x$, $2x$, respectively.

$\Rightarrow \quad 4\pi r^2 + 2\left[\dfrac{x}{3}\times x + x\times 2x + 2x\times\dfrac{x}{3}\right] = k$ (constant)

For maximum or minimum value, put $\dfrac{dV}{dx} = 0$

$\Rightarrow \quad (-6x)\sqrt{\dfrac{k-6x^2}{4\pi}} + 2x^2 = 0$

$\Rightarrow \quad 2x^2 = 6x\sqrt{\dfrac{k-6x^2}{4\pi}} \Rightarrow x = 3\sqrt{\dfrac{k-6x^2}{4\pi}}$

$\Rightarrow \quad x = 3r \quad$ [using Eq. (i)]

Again, differentiating $\dfrac{dV}{dx}$ w.r.t. $x$, we get

$$\frac{d^2V}{dx^2} = \frac{-6}{2\sqrt{\pi}}\frac{d}{dx}\left(x\sqrt{k-6x^2}\right) + 4x$$

$$= \frac{-3}{\sqrt{\pi}}\left(\sqrt{k-6x^2} + x\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{k-6x^2}}(-12x)\right) + 4x$$

$$= \frac{-3}{\sqrt{\pi}}\left(\frac{k-6x^2-6x^2}{\sqrt{k-6x^2}}\right) + 4x$$

$$= \frac{3(12x^2-k)}{\sqrt{\pi}\sqrt{k-6x^2}} + 4x$$

$$\left[\frac{d^2V}{dx^2}\right]_{x=\frac{3}{2\sqrt{\pi}}\sqrt{k-6x^2}} = \frac{3\left(12x^2 - \frac{4\pi x^2}{9} - 6x^2\right)}{\sqrt{\pi}\sqrt{\frac{4\pi x^2}{9} + 6x^2 - 6x^2}} + 4x$$

$$= \frac{3\left(6x^2 - \frac{4\pi x^2}{9}\right)}{\sqrt{\pi}\left(\frac{2\sqrt{\pi}\,x}{3}\right)} + 4x = \frac{3\left(\frac{54x^2 - 4\pi x^2}{9}\right)}{\frac{2\pi x}{3}} + 4x$$

[given]

$\Rightarrow \quad 4\pi r^2 + 6x^2 = k \Rightarrow r^2 = \dfrac{k - 6x^2}{4\pi}$

$$\Rightarrow \quad r = \sqrt{\frac{k - 6x^2}{4\pi}} \quad \text{...(i)}$$

Sum of the volumes, $V = \dfrac{4}{3}\pi r^3 + \dfrac{x}{3} \times x \times 2x$

$$\Rightarrow \quad V = \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{3/2} + \frac{2}{3}x^3$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dV}{dx} = \frac{4}{3}\pi \times \frac{3}{2}\left(\frac{k - 6x^2}{4\pi}\right)^{1/2}\left(\frac{-12x}{4\pi}\right) + \frac{2}{3} \times 3x^2$$

$$= 2\pi\sqrt{\frac{k - 6x^2}{4\pi}}\left(\frac{-3x}{\pi}\right) + 2x^2$$

$$= (-6x)\sqrt{\frac{k - 6x^2}{4\pi}} + 2x^2$$

Then, $4a = x$ and $2\pi r = 25 - x \Rightarrow a = \dfrac{x}{4}$ and $r = \dfrac{25 - x}{2\pi}$

$\therefore$ Area of the square $= a^2 = \left(\dfrac{x^2}{16}\right)$ sq m

and area of the circle $= \pi r^2 = \pi\left(\dfrac{25 - x}{2\pi}\right)^2 = \dfrac{(25 - x)^2}{4\pi}$

Now, combined area, $A = \dfrac{x^2}{16} + \dfrac{(25 - x)^2}{4\pi}$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dA}{dx} = \frac{x}{8} - \frac{(25 - x)}{2\pi} = \frac{(\pi + 4)x - 100}{8\pi} \text{ and } \frac{d^2A}{dx^2} = \frac{(\pi + 4)}{8\pi}$$

For maxima or minima, put $\dfrac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{(\pi + 4)x - 100}{8\pi} = 0 \Rightarrow x = \frac{100}{(\pi + 4)}$$

and $\dfrac{d^2A}{dx^2} = \dfrac{(\pi + 4)}{8\pi} > 0$ for all values of $x$.

$\therefore \quad x = \dfrac{100}{(\pi + 4)}$ is a point of local minima.

Hence, the lengths of pieces are $\left(\dfrac{100}{\pi + 4}\right)$ m and $\left(\dfrac{25\pi}{\pi + 4}\right)$ m, respectively.

**EXAMPLE |13|** Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius $R$ is $\dfrac{2R}{\sqrt{3}}$. Find the volume of the largest cylinder inscribed in a

$$= \frac{54x^2 - 4\pi x^2}{2\pi x} + 4x = \frac{2x^2(27 - 2\pi)}{2\pi x} + 4x$$

$$= \frac{x(27 - 2\pi)}{\pi} + 4x > 0$$

$\therefore V$ is minimum, when $x = 3\sqrt{\dfrac{k - 6x^2}{4\pi}}$.

But $r = \sqrt{\dfrac{k - 6x^2}{4\pi}} \Rightarrow x = 3r$

Hence, $V$ is minimum when $x$ is equal to three times the radius of the sphere.

**EXAMPLE |12|** A wire of length 25 m is to be cut into two pieces. One of the wires is to be made into a square and the other into a circle. What should be the lengths of the two pieces, so that the combined area of the square and the circle is minimum?

**Sol.** Let $x$ m and $(25 - x)$ m be the required lengths.

Again, let $a$ be the side of the square formed and $r$ be the radius of the circle formed.

Volume of cylinder, $V = \pi r^2 h$

$$\Rightarrow \quad V = \pi\left[\frac{1}{4}(4R^2 - h^2)\right]h \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{\pi}{4}(4R^2h - h^3) \quad \text{...(ii)}$$

On differentiating $V$ w.r.t. $h$, we get

$$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2) \quad \text{...(iii)}$$

For maxima or minima, put $\dfrac{dV}{dh} = 0$

$$\Rightarrow \frac{\pi}{4}(4R^2 - 3h^2) = 0$$

$$\Rightarrow \quad 4R^2 = 3h^2$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}} \quad [\because \text{height cannot be negative}]$$

On differentiating both sides of Eq. (iii) w.r.t. $h$, we get

$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$$

At $h = \dfrac{2R}{\sqrt{3}}$, $\dfrac{d^2V}{dh^2} = -\dfrac{3}{2}\pi \times \dfrac{2R}{\sqrt{3}} = -\sqrt{3}\,\pi R < 0 \quad [\because R > 0]$

$\therefore$ Volume of cylinder is maximum at $h = \dfrac{2R}{\sqrt{3}}$.

From Eq. (i), we get $r^2 = \dfrac{1}{4}\left(4R^2 - \dfrac{4R^2}{3}\right) = \dfrac{2R^2}{3}$

$\therefore$ Volume of largest cylinder inscribed in a sphere,

$$V = \pi\left(\frac{2R^2}{3}\right) \times \frac{2R}{\sqrt{3}}$$

sphere of radius $R$. **[NCERT, All India 2019]**

Firstly, find the radius of cylinder by using Pythagoras theorem, then eliminate $r$ from volume of cylinder. After this, apply second derivative test to find maximum height and maximum volume.

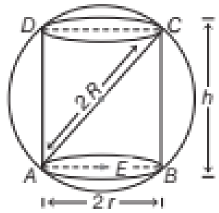
***Sol.*** Let we have a sphere of radius $R$ and a cylinder is inscribed in it. $EB = r$ be the radius of the cylinder and $BC = h$ be the height of cylinder.

In right angled $\Delta ABC$, we have

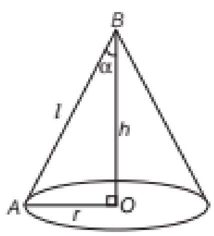
$$(AC)^2 = (AB)^2 + (BC)^2 \quad \text{[using Pythagoras theorem]}$$

$$\Rightarrow \quad (2R)^2 = (2r)^2 + h^2 \Rightarrow 4R^2 = 4r^2 + h^2$$

$$\Rightarrow \quad r^2 = \frac{1}{4}(4R^2 - h^2) \quad \text{...(i)}$$




***Sol.*** Let radius of cone $= r$, Height of cone $= h$,
Slant height of cone $= l$, Semi-vertical angle $= \alpha$,
Surface area $= S$ and volume of cone,




$$V = \frac{1}{3}\pi r^2 h \quad \text{...(i)}$$

On squaring both sides, we get

$$V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \quad \text{...(ii)}$$

$$[\because \text{ in } \Delta AOB, \text{ using Pythagoras theorem, } l^2 = r^2 + h^2]$$

Given, surface area of cone, $S = \pi rl + \pi r^2$

$$\Rightarrow \quad \pi rl = S - \pi r^2$$

$$\Rightarrow \quad l = \frac{S - \pi r^2}{\pi r} \quad \text{...(iii)}$$

On putting the value of $l$ in Eq. (ii), we get

$$V^2 = \frac{1}{9}\pi^2 r^4 \left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 \left[\frac{S^2 - 2\pi S r^2 + \pi^2 r^4}{\pi^2 r^2} - r^2\right]$$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 \left[\frac{S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$\Rightarrow \quad V^2 = \frac{1}{9} r^2 [S^2 - 2\pi S r^2]$$

$$= \frac{1}{\sqrt{3}} \times \left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{1}{\sqrt{3}} \times \text{Volume of sphere}$$

Hence, height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$ and volume of largest cylinder inscribed in sphere is $\frac{1}{\sqrt{3}}$ times volume of sphere.

**Note** For maximum volume, the diagonal of cylinder passes through the centre of the sphere i.e. diagonal is a diameter of sphere.

**EXAMPLE [14]** Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$. **[NCERT]**

First, establish a relation between volume and surface area of cone, then differentiate it and find the critical points and apply the second derivative test to find the radius. Further, find a semi-vertical angle.

$$\therefore \quad S = 4\pi r^2 \Rightarrow r^2 = \frac{S}{4\pi} \Rightarrow r = \sqrt{\frac{S}{4\pi}}$$

[$\because$ radius cannot be negative]

Again, differentiating both sides of Eq. (v) w.r.t. $r$, we get

$$f''(r) = \frac{1}{9}(2S^2 - 24\pi S r^2)$$

At $r = \sqrt{\frac{S}{4\pi}}$, $[f''(r)]_{r=\sqrt{\frac{S}{4\pi}}} = \frac{1}{9}\left[2S^2 - 24\pi S \cdot \frac{S}{4\pi}\right]$

$$= \frac{1}{9}[2S^2 - 6S^2] = \frac{-4}{9}S^2 < 0$$

$\therefore$ Volume of cone is maximum, when $r = \sqrt{\frac{S}{4\pi}}$

$$\because \quad S = 4\pi r^2$$

On putting the value of $S$ in Eq. (iii), we get

$$l = \frac{4\pi r^2 - \pi r^2}{\pi r} = \frac{3\pi r^2}{\pi r} = 3r$$

In $\Delta OBA$, $\sin\alpha = \frac{r}{l}$

$$\Rightarrow \quad \sin\alpha = \frac{r}{3r} = \frac{1}{3} \Rightarrow \alpha = \sin^{-1}\left(\frac{1}{3}\right)$$

Hence, the semi-vertical angle of cone is $\sin^{-1}\left(\frac{1}{3}\right)$.

## Maximum and Minimum Values of a Function in a Closed Interval

A function may have a number of local maxima or local minima in a given interval. A local maximum value may not be the greatest and a local minimum value may not be the least value of the function in any given interval.

$$\Rightarrow \quad V^2 = \frac{1}{9}[S^2r^2 - 2\pi Sr^4] \quad \text{...(iv)}$$

Let $V^2 = f(r)$, then $f(r)$ is maximum or minimum, accordingly as volume $(V)$ is maximum or minimum.

Now, $$f(r) = \frac{1}{9}(S^2r^2 - 2\pi Sr^4)$$

On differentiating both sides w.r.t. $r$, we get

$$f'(r) = \frac{1}{9}(2S^2r - 8\pi Sr^3) \quad \text{...(v)}$$

For maximum or minimum value put $f'(r) = 0$

$$\Rightarrow \quad \frac{1}{9}(2S^2r - 8\pi Sr^3) = 0$$

$$\Rightarrow \quad 2Sr(S - 4\pi r^2) = 0$$

$$\Rightarrow \quad 2Sr = 0 \text{ and } S - 4\pi r^2 = 0 \Rightarrow S \neq 0$$

$$\therefore \quad r = 0 \text{ and } S - 4\pi r^2 = 0$$

$$\Rightarrow \quad r = 0 \text{ and } S = 4\pi r^2$$

Since, $r = 0$ is not possible.

## IMPORTANT THEOREMS

**Theorem 1** Let $f$ be a continuous function on an interval $I = [a, b]$. Then, $f$ has the absolute maximum value and $f$ attains it atleast once in $I$. Also, $f$ has the absolute minimum value and attains it atleast once in $I$.

**Theorem 2** Let $f$ be a differentiable function on a closed interval $I$ and $c$ be any interior point of $I$. Then,

(i) $f'(c) = 0$, if $f$ attains its absolute maximum value at $c$.

(ii) $f'(c) = 0$, if $f$ attains its absolute minimum value at $c$.

### Method to Find Absolute Maximum or Absolute Minimum Values in an Interval $[a, b]$

Suppose $f(x)$ be the given function. Then, to find absolute maximum and absolute minimum values in the given interval, we use the following steps

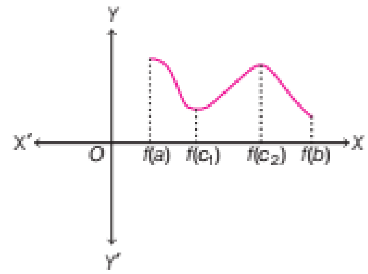
I. Find all critical points of $f$ in the interval, i.e. find points at which either $f'(x) = 0$ or $f$ is not differentiable.

II. Take the end points of the interval.

III. At all these points listed in steps I and II, calculate the values of $f$.

IV. Identify the maximum and minimum values of $f$ out of the values calculated in step III. This maximum value will be the absolute maximum (greatest) value of $f$ and the minimum value will be the absolute minimum (least) value of $f$.

**EXAMPLE |15|** Find absolute maximum and minimum




If a function $f(x)$ is continuous on a closed interval, say $[a, b]$, then it attains the absolute maximum value or global maximum (absolute minimum value or global minimum) at critical points or at the end points of the interval $[a, b]$. Thus, to find the absolute maximum (absolute minimum) value of the function, we choose the largest and smallest amongst the numbers $f(a), f(c_1), f(c_2), f(b)$, where $c_1$ and $c_2$ are the critical points.

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3}$$

$$= 12\left(\frac{1}{16}\right) - 6\left(\frac{1}{2}\right) = \frac{-9}{4}$$

and $$f(1) = 12(1)^{4/3} - 6(1)^{1/3} = 6$$

Hence, absolute maximum value of function $f$ is 18 at $x = -1$ and absolute minimum value of function $f$ is $\frac{-9}{4}$ at $x = \frac{1}{8}$.

**EXAMPLE |16|** Find the maximum and minimum values of $(x + \sin 2x)$ on $[0, \pi]$.

***Sol.*** Let $f(x) = x + \sin 2x$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = (1 + 2\cos 2x)$$

For critical points, put $f'(x) = 0$

$$\Rightarrow \quad \cos 2x = \frac{-1}{2}, \text{ where } x \in [0, \pi]$$

$$\Rightarrow \quad 2x = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Now, $$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \frac{2\pi + 3\sqrt{3}}{6},$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \sin\frac{\pi}{3} = \frac{4\pi - 3\sqrt{3}}{6},$$

$$f(0) = 0 \text{ and } f(\pi) = \pi + \sin 2\pi = \pi$$

Hence, the minimum value of $f(x)$ is 0 at $x = 0$ and the maximum value of $f(x)$ is $\pi$ at $x = \pi$.

# TOPIC PRACTICE 3

values of a function $f$ given by

$$f(x) = 12x^{4/3} - 6x^{1/3},\ x \in [-1, 1].$$ [NCERT]

**Sol.** We have, $f(x) = 12x^{4/3} - 6x^{1/3}$

On differentiating both sides w.r.t. $x$, we get

$$f'(x) = 12 \times \frac{4}{3}x^{1/3} - 6 \times \frac{1}{3}x^{-2/3}$$

$$= 16x^{1/3} - \frac{2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}}$$

For critical points, put $f'(x) = 0$

$$\Rightarrow \quad \frac{2(8x-1)}{x^{2/3}} = 0 \Rightarrow 8x - 1 = 0 \Rightarrow x = \frac{1}{8}$$

Also, $f'(x)$ is not defined at $x = 0$, so critical points are $x = 0$ and $x = \frac{1}{8}$ and end points of interval are $-1$ and 1.

Now, $f(-1) = 12(-1)^{4/3} - 6(-1)^{1/3} = 12 + 6 = 18$

$f(0) = 12(0) - 6(0) = 0$

**4** The functin $f(x) = x^x$ has a stationary point at

(a) $x = e$ (b) $x = \frac{1}{e}$

(c) $x = 1$ (d) $x = \sqrt{e}$

**5** A right circular cylinder which is open at the top and has a given surface area, will have the greatest volume, if its height $h$ and radius $r$ are related by

(a) $2h = r$ (b) $h = 4r$

(c) $h = 2r$ (d) $h = r$

## SHORT ANSWER Type II Questions

**6** Find all the points of local maxima and local minima of the function

$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$ [NCERT Exemplar]

**7** Find the points at which the function $f$ given by $f(x) = (x-4)^4 (x+1)^3$ has

(i) local maxima. (ii) local minima.

(iii) point of inflection. [NCERT]

**8** It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains maximum value on the interval $[0, 2]$. Find the value of $a$. [NCERT]

**9** Find the least value of $f(x) = e^x + e^{-x}$.

**10** Find the maximum and minimum values, if any of the following functions. (Each part carries 4 Marks)

(i) $f(x) = |x+2| - 1$

(ii) $g(x) = -|x+1| + 3$

(iii) $h(x) = \sin(2x) + 5$

(iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1,\ x \in (-1, 1)$ [NCERT]

## OBJECTIVE TYPE QUESTIONS

**1** If $x$ is real, then the minimum value of $x^2 - 8x + 17$ is

(a) $-1$ (b) 0

(c) 1 (d) 2

**2** The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(a) two points of local maximum

(b) two points of local minimum

(c) one maxima and one minima

(d) no maxima or minima

**3** The maximum slope of curve $y = -x^3 + 3x^2 + 9x - 27$ is

(a) 0 (b) 12

(c) 16 (d) 32

**13** Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume. [Delhi 2020]

**14** Show that $\sin^p\theta \cos^q\theta$ attains a maximum value, when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

**15** Prove that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

**16** Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.

**17** Prove that the largest rectangle with a given perimeter is a square.

**18** Let $AP$ and $BQ$ be two vertical poles at points $A$ and $B$, respectively. If $AP = 16$ m, $BQ = 22$ m and $AB = 20$ m, then find the distance of a point $R$ on $AB$ from the point $A$ such that $RP^2 + RQ^2$ is minimum.

**19** Manufacturer can sell $x$ items at a price of ₹$\left(5 - \frac{x}{100}\right)$ each. The cost price is ₹$\left(\frac{x}{5} + 500\right)$. Then, find the number of items he should sell to earn maximum profit.

**20** Find the points of absolute maximum and minimum of $f(x) = (x-1)^{1/3}(x-2);\ 1 \le x \le 9$.

**21** Find the absolute maximum and minimum values of $f(x) = 2x^3 - 24x + 57$ in the interval $[1, 3]$.

**22** Find the absolute maximum and absolute minimum values of the function $f$ given by $f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$.

11 Find the local maximum and local minimum values of $\frac{(x-1)(x-6)}{x-10}, x \neq 10$.

12 Find the local maximum and local minimum value of the following functions.

(i) $f(x) = (\sin x - \cos x)$, where $0 < x < \frac{\pi}{2}$ **[NCERT]**

(ii) $f(x) = (2\cos x + x)$, where $0 < x < 2\pi$

25 Show that the right circular cone of least curved surface area and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **[NCERT; All India 2017C; Delhi 2011]**

26 Show that the height of a closed right circular cylinder of given surface and maximum volume is equal to diameter of base. **[Delhi 2012]**

27 Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. **[All India 2016; Delhi 2014]**

28 Find the point $P$ on the curve $y^2 = 4ax$, which is nearest to the point $(11a, 0)$. **[All India 2014C]**

29 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier placed at $(3, 7)$ wants to shoot down the helicopter, when it is nearest to him. Find the nearest distance. **[NCERT Exemplar]**

30 If the length of three sides of a trapezium other than the base are each equal to 10 cm, then find the area of the trapezium, when it is maximum. **[All India 2014, 2010; Delhi 2013]**

31 If an open box with square base is to be made of a given quantity of card board of area $c^2$, then show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cu units. **[All India 2012]**

32 A square piece of tin of side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to cut off, so that the volume of the box is the maximum possible? Find maximum volume. **[NCERT]**

33 A metal box with a square base and vertical sides is to contain 1024 $cm^3$. If the material for the top and bottom costs ₹ 5 per $cm^2$ and the material for the sides costs ₹ 2.50 per $cm^2$. Then, find the least cost of the box. **[Delhi 2017, 2016C; NCERT Exemplar]**

34 The sum of the perimeter of a circle and square is $k$, where $k$ is some constant. Prove that the sum of their areas is least, when the side of

23 Find the point on the curve $y^2 = 4x$, which is nearest to the point $(2, -8)$. **[All India 2019]**

## LONG ANSWER Type Questions

24 Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also, find the corresponding local maximum and local minimum values. **[All India 2016C]**

square and the other in the form of an equilateral triangle. Find the length of each pieces, so that the sum of the areas of the two be minimum.

36 Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius $r$ is $\frac{4r}{3}$. Also, show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere. **[Delhi 2016C; All India 2014]**

*Or*

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius $r$ is $\frac{4r}{3}$. Also, find the maximum volume in terms of volume of the sphere. **[Delhi 2019, 16]**

37 Prove that radius of right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. **[All India 2012]**

38 Show that the height of the cylinder of greatest volume which can be inscribed in a circular cone of height $h$ and having semi-vertical angle $\alpha$ is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$. **[Delhi 2017C; NCERT]**

39 A window is in the form of a rectangle surmounted by a semi-circle opening. The perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. **[NCERT; All India 2017; Foreign 2014]**

40 A figure consists of a semi-circle with a rectangle on its diameter. Given the perimeter of the figure, find its dimensions in order that the area may be maximum. **[All India 2016C]**

41 $AB$ is a diameter of a circle and $C$ is any point on the circle. Show that the area of $\Delta ABC$ is maximum, when it is isosceles. **[All India 2017, 2014C]**

square is double the radius of the circle.
**[NCERT; Foreign 2014; Delhi 2014C]**

**35** A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a

**43** Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
**[All India 2013]**

**44** Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

**45** An isosceles triangle of vertical angle $2\theta$ is inscribed in a circle of radius $a$. Show that the area of triangle is maximum when $\theta=\frac{\pi}{6}$.

**46** If the sum of the lengths of the hypotenuse and a side of a right triangle is given, then show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$.
**[NCERT Exemplar; Delhi 2017; All India 2014]**

**47** A point on the hypotenuse of a triangle is at distances $a$ and $b$ from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3}+b^{2/3})^{3/2}$. **[Delhi 2015C]**

**48** A tank with rectangular base and rectangular sides open at the top is to be constructed, so that its depth is 3 m and volume is 75 $m^3$.
If building of tank costs ₹ 100 per square metre for the base and ₹ 50 per square metre for the sides, then find the cost of least expensive tank.
**[Delhi 2015C]**

**42** Prove that the area of a right angled triangle of given hypotenuse is maximum, when the triangle is isosceles.

# HINTS & SOLUTIONS

**1.** (c) Let $f(x)=x^2-8x+17$

$\therefore$ $f'(x)=2x-8$

So, $f'(x)=0$, gives $x=4$

Now, $f''(x)=2>0, \forall\ x$

So, $x=4$ is the point of local minima.

$\therefore$ Minimum value of $f(x)$ at $x=4$,

$f(4)=4\times4-8\times4+17=1$

**2.** (c) We have, $f(x)=2x^3-3x^2-12x+4$

$\therefore$ $f'(x)=6x^2-6x-12$

Now, $f'(x)=0 \Rightarrow 6(x^2-x-2)=0$

$\Rightarrow$ $6(x+1)(x-2)=0$

$\Rightarrow$ $x=-1$ and $x=+2$

On number line for $f'(x)$, we get


Hence $x=-1$ is point of local maxima and $x=2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

**3.** (b) We have, $y=-x^3+3x^2+9x-27$

$\therefore$ $\frac{dy}{dx}=-3x^2+6x+9=$ Slope of the curve

and $\frac{d^2y}{dx^2}=-6x+6=-6(x-1)$

$\therefore$ $\frac{d^2y}{dx^2}=0$

$\Rightarrow$ $-6(x-1)=0 \Rightarrow x=1>0$

Now, $\frac{d^3y}{dx^3}=-6<0$

So, the maximum slope of given curve is at $x=1$.

$\therefore$ $\left(\frac{dy}{dx}\right)_{(x=1)}=-3\cdot1^2+6\cdot1+9=12$

**4.** (b) We have, $f(x)=x^x$

Let $y=x^x$

and $\log y=x\log x$

$\therefore$ $\frac{1}{y}\cdot\frac{dy}{dx}=x\cdot\frac{1}{x}+\log x\cdot1$

$\Rightarrow$ $\frac{dy}{dx}=(1+\log x)\cdot x^x$

$\therefore$ $\frac{dy}{dx}=0$

$\Rightarrow$ $(1+\log x)\cdot x^x=0 \Rightarrow \log x=-1$

$\Rightarrow$ $\log x=\log e^{-1} \Rightarrow x=e^{-1}$

$\Rightarrow$ $x=\frac{1}{e}$

Hence, $f(x)$ has a stationary point at $x=\frac{1}{e}$.

**5.** (d) Surface area, $S=2\pi rh+\pi r^2$ ...(i)

and $V=\pi r^2h$ ...(ii)

From Eq. (i), $h=\frac{S-\pi r^2}{2\pi r}$

From Eq. (ii), $V=\frac{r}{2}(S-\pi r^2)$

$\Rightarrow$ $\frac{dV}{dr}=\frac{1}{2}(S-3\pi r^2)=0$

$\Rightarrow$ $S-3\pi r^2=0$

$\Rightarrow$ $S=3\pi r^2$

On putting the value of $S$ in Eq. (i), we get

$3\pi r^2=2\pi rh+\pi r^2$

$\Rightarrow$ $r=h$

**6.** Similar as Example 3.

[**Ans.** Local maxima $=0,-5$; Local minima $=-3$]

**7.** Similar as Example 2.

[**Ans.** (i) Local maxima at $x = 8/7$.

(ii) Local minima at $x = 4$.

(iii) Point of inflection at $x = -1$.]

**8.** **Hint** $f'(x) = 4x^3 - 124x + a$, put $f'(1) = 0$

[**Ans.** $a = 120$]

**9.** **Hint** $\frac{dy}{dx} = e^x - e^{-x} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$ [**Ans.** 2]

**10.** Similar as Example 5.

[**Ans.** (i) Minimum value = $-1$, no maximum value

(ii) Maximum value = 3, no minimum value

(iii) Minimum value = 4, maximum value = 6

(iv) Minimum value = 2, maximum value = 4

(v) Neither minimum nor maximum value.]

**11.** Let $y = \frac{(x-1)(x-6)}{x-10} = \frac{x^2 - 7x + 6}{x-10}$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{(x-10)(2x-7)-(x^2-7x+6)\cdot 1}{(x-10)^2}$$

[using quotient rule of derivative]

$$= \frac{2x^2 - 7x - 20x + 70 - x^2 + 7x - 6}{(x-10)^2}$$

$$= \frac{x^2 - 20x + 64}{(x-10)^2} = \frac{(x-4)(x-16)}{(x-10)^2}$$

For local maxima and local minima, put $dy/dx = 0$

$$\Rightarrow \frac{(x-4)(x-16)}{(x-10)^2} = 0 \Rightarrow (x-4)(x-16) = 0$$

$\therefore$ $x = 4, 16$

For $x = 4$,

When $x$ is slightly $< 4$, then $\frac{dy}{dx} = \frac{(-)(-)}{(+)} =$ Positive

When $x$ is slightly $> 4$, then $\frac{dy}{dx} = \frac{(+)(-)}{(+)} =$ Negative

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $y$ is local maxima at $x = 4$.

$\therefore$Local maximum value at $x = 4$ is

$$y = \frac{(4-1)(4-6)}{4-10} = \frac{3 \times -2}{-6} = 1$$

For $x = 16$,

When $x$ is slightly $< 16$, then $\frac{dy}{dx} = \frac{(+)(-)}{(+)} =$ Negative

When $x$ is slightly $> 16$, then $\frac{dy}{dx} = \frac{(+)(+)}{(+)} =$ Positive

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $y$ is local minima at $x = 16$.

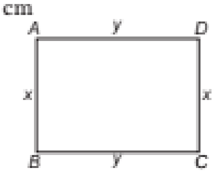
$\therefore$ Local minimum value at $x = 16$ is

$$y = \frac{(16-1)(16-6)}{16-10} = \frac{15 \times 10}{6} = 25$$

**12.** Similar as Example 6.

[**Ans.** (i) Local maximum value = $\sqrt{2}$ and no local minimum value exist (ii) Local maximum value $= \left(\sqrt{3} + \frac{\pi}{6}\right)$ and local minimum value $= \left(\frac{5\pi}{6} - \sqrt{3}\right)$]

**13.** Here $ABCD$ is a rectangle with length $AD = y$ cm and breadth $= x$ cm




The rectangle is rotated about $AD$. Let $V$ be the volume of the cylinder so formed

$\therefore$ $V = \pi x^2 y$ ...(i)

Perimeter of rectangle $= 2(x + y)$

$\Rightarrow$ $36 = 2(x + y)$

$\Rightarrow$ $y = 18 - x$

Now, $V = \pi x^2(18 - x)$

$V = \pi(18x^2 - x^3)$

$\Rightarrow$ $\frac{dV}{dx} = \pi(36x - 3x^2)$

For maxima or minima, put $\frac{dV}{dx} = 0$

$$\pi(36x - 3x^2) = 0 \Rightarrow x = 12, x \neq 0$$

Now, $\frac{d^2V}{dx^2} = \pi(36 - 6x)$

$$\left(\frac{d^2V}{dx^2}\right)_{x=12} = \pi(36 - 72) = -36\pi < 0$$

$\therefore$ Volume is maximum when $x = 12$ cm

$\therefore$ $y = 18 - x = 18 - 12 = 6$ cm

Hence, the dimension of rectangle, which have maximum volume, when revolved about of its side is $12 \times 6$.

Putting the value of $x$ and $y$ in Eq. (i), we get

$$V = 864\pi \text{ cm}^3$$

**14.** **Hint** $\frac{dy}{d\theta} = \sin^p\theta \cdot q\cos^{q-1}\theta \times (-\sin\theta)$

$+ \cos^q\theta \cdot p\sin^{p-1}\theta \times (\cos\theta) = 0$

$\Rightarrow$ $\sin^{p-1}\theta\cos^{q-1}\theta(-q\sin^2\theta + p\cos^2\theta) = 0$

$\Rightarrow$ $\tan\theta = \pm\sqrt{p/q}$

**15.** **Hint** Let $y = x^{-x} \Rightarrow \frac{dy}{dx} = -x^{-x}(1 + \log x)$

$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{e}$

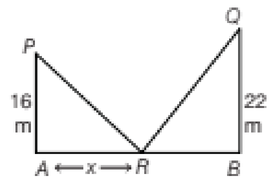
**16.** Similar as Example 7. [**Ans.** 8, 8]

**17.** **Hint** Let $x$ be length and $y$ be breadth of rectangle.

Given, $2(x+y)=p$, then $A = xy = x\left(\frac{p}{2}-x\right)$

$\because \frac{dA}{dx}=0$, we get $x=y$.

**18.** **Hint** Let $AR = x$, then $BR = 20 - x$




$\because \; RP^2 + RQ^2 = x^2 + (20-x)^2 + 740$ [**Ans.** 10 m]

**19.** **Hint** Profit, $P = SP - CP = \frac{24x}{5} - \frac{x^2}{100} - 500$

Put $\frac{dP}{dx} = 0$ [**Ans.** 240]

**20.** We have, $f(x) = (x-1)^{1/3}(x-2)$

On differentiating both sides w.r.t. $x$, we get

$f'(x) = (x-1)^{1/3}\cdot 1 + (x-2)\cdot\frac{1}{3}(x-1)^{-2/3}$

[by product rule of derivative]

$= (x-1)^{-2/3}\left[x-1+\frac{1}{3}(x-2)\right]$

$= (x-1)^{-2/3}\left[\frac{3x-3+x-2}{3}\right] = (x-1)^{-2/3}\left(\frac{4x-5}{3}\right)$

Put $f'(x)=0$

$\Rightarrow \frac{1}{3}(x-1)^{-2/3}(4x-5)=0 \Rightarrow x = \frac{5}{4}$

Now, $f(1) = (1-1)^{1/3}(1-2) = 0$

$f\left(\frac{5}{4}\right) = \left(\frac{5}{4}-1\right)^{1/3}\left(\frac{5}{4}-2\right) = \left(\frac{1}{4}\right)^{1/3}\left(\frac{-3}{4}\right) = \frac{-3}{4\cdot\sqrt[3]{4}}$

and $f(9) = (9-1)^{1/3}(9-2) = (8)^{1/3}\cdot 7 = 2\times 7 = 14$

Hence, $f(x)$ is absolute maximum at $x = 9$ and absolute minimum at $x = \frac{5}{4}$.

**21.** Similar as Example 15. [**Ans.** Maximum value is 39 at $x = 3$, minimum value is 25 at $x = 2$]

**22.** We have, $f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$

On differentiating both sides w.r.t. $x$, we get

$f'(x) = -2\cos x \cdot \sin x + \cos x$

For critical points, put $f'(x) = 0$

$\Rightarrow -2\cos x \sin x + \cos x = 0$

$\Rightarrow \cos x(1 - 2\sin x) = 0$

$\Rightarrow \cos x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$ or $\sin x = \sin\frac{\pi}{6}$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$ or $x = n\pi + (-1)^n\cdot\frac{\pi}{6}$,

where $n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Then, we evaluate the value of $f$ at critical points $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ and at the end points of the interval $[0, \pi]$.

At $x = 0$, $f(0) = \cos^2 0 + \sin 0 = 1$

At $x = \frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right) = \cos^2\frac{\pi}{6} + \sin\frac{\pi}{6}$

$= \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

At $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$

At $x = \frac{5\pi}{6}$, $f\left(\frac{5\pi}{6}\right) = \cos^2\frac{5\pi}{6} + \sin\frac{5\pi}{6}$

$= \cos^2\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$

$= \cos^2\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3+2}{4} = \frac{5}{4}$

At $x = \pi$, $f(\pi) = \cos^2\pi + \sin\pi = 1$

Hence, the absolute maximum value is 5/4 and absolute minimum value is 1.

**23.** Given, equation of curve is $y^2 = 4x$.

Let $P(x, y)$ be a point on the curve, which is nearest to point $A(2, -8)$.

Now, distance between the points $A$ and $P$ is given by

$AP = \sqrt{(x-2)^2 + (y+8)^2} = \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2}$

$= \sqrt{\frac{y^4}{16} + 16y + 68}$

Let $z = AP^2 = \frac{y^4}{16} + 16y + 68$

Now, $\frac{dz}{dy} = \frac{y^3}{4} + 16$

For maximum or minimum value of $z$, put

$\frac{dz}{dy} = 0 \Rightarrow \frac{y^3}{4} + 16 = 0$

$\Rightarrow y^3 + 64 = 0 \Rightarrow (y+4)(y^2 - 4y + 16) = 0 \Rightarrow y = -4$

[$\because y^2 - 4y + 16 = 0$ gives imaginary values of $y$]

Now, $\frac{d^2z}{dy^2} = \frac{1}{4}\times 3y^2 = \frac{3}{4}y^2$

For $y = -4$,

$\frac{d^2z}{dy^2} = \frac{3}{4}(-4)^2 = 12 > 0$

Thus, $z$ is minimum when $y = -4$.

Substituting $y = -4$ in equation of the curve $y^2 = 4x$; we obtain $x = 4$.

Hence, the point $(4, -4)$ on the curve $y^2 = 4x$ is nearest to the point $(2, -8)$.

**24.** Similar as Example 2.

[**Ans.** $x = 1$ is point of local maxima, $x = 3$ is point of local minima, $x = 0$ is point of inflection, Local maximum value = 0, Local minimum value = $-28$]

**25.** Similar as Example 14.

**26.** **Hint** Let $r$ be the radius and $h$ be the height of cylinder.

Given, $S = 2\pi r^2 + 2\pi rh \Rightarrow h = \dfrac{S - 2\pi r^2}{2\pi r}$

$\because \quad V = \pi r^2 h = \pi r^2 \left(\dfrac{S - 2\pi r^2}{2\pi r}\right) = \dfrac{rS - 2\pi r^3}{2}$

Now, $\dfrac{dV}{dr} = 0$, we get $h = 2r$

**27.** Similar as Example 14.

**28.** Let the point on $y^2 = 4ax$ be $(x_1, y_1)$.

Then, $\quad y_1^2 = 4ax_1 \quad$ ...(i)

Distance between $(x_1, y_1)$ and $(11a, 0)$ is given by

$$D = \sqrt{(x_1 - 11a)^2 + (y_1 - 0)^2} = \sqrt{(x_1 - 11a)^2 + y_1^2}$$

$$= \sqrt{(x_1 - 11a)^2 + 4ax_1} \quad \text{[from Eq. (i)]}$$

On differentiating both sides w.r.t. $x_1$, we get

$$\frac{dD}{dx_1} = \frac{1}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}}[2(x_1 - 11a) + 4a]$$

$$= \frac{2x_1 - 22a + 4a}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}} = \frac{x_1 - 9a}{\sqrt{(x_1 - 11a)^2 + 4ax_1}}$$

For critical points, put $\dfrac{dD}{dx_1} = 0 \Rightarrow x_1 - 9a = 0 \Rightarrow x_1 = 9a$

If $x_1 = 9a$, then $y_1^2 = 36a^2 \Rightarrow y_1 = \pm 6a$

Hence, required points are $(9a, 6a)$ and $(9a, -6a)$.

$$\text{Now, } \frac{d^2D}{dx_1^2} = \frac{d}{dx_1}\left(\frac{dD}{dx_1}\right) = \frac{d}{dx_1}\left(\frac{x_1 - 9a}{\sqrt{(x_1 - 11a)^2 + 4ax_1}}\right)$$

$$= \frac{\sqrt{(x_1 - 11a)^2 + 4ax_1} - (x_1 - 9a)\dfrac{1}{2}\cdot\dfrac{[2(x_1 - 11a) + 4a]}{\sqrt{(x_1 - 11a)^2 + 4ax_1}}}{(x_1 - 11a)^2 + 4ax_1}$$

At $(9a, 6a)$, $\dfrac{d^2D}{dx_1^2} > 0$

So, at $(9a, 6a)$, $D$ is minimum.

Hence, the required point is $(9a, 6a)$.

**29.** **Hint** Let $A(h, k)$ be any point on the curve $y = x^2 + 7$ and $B(3, 7)$ be the given point.

$\therefore$ Distance between $A$ and $B$ is

$$AB = \sqrt{(h - 3)^2 + (k - 7)^2}$$

Since, the point $A(h, k)$ lies on the curve.

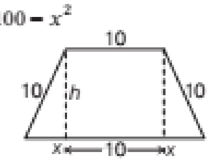
$\therefore \quad k = h^2 + 7 \Rightarrow AB = \sqrt{(h - 3)^2 + (h^2 + 7 - 7)^2}$

or $\quad AB^2 = (h - 3)^2 + (h^2)^2$

Let $AB^2 = f(h)$, then $AB$ is maximum or minimum accordingly as $f(h)$ is maximum or minimum.

$\therefore \quad f(h) = (h + 1)^2 + h^4 \quad$ [**Ans.** $\sqrt{5}$]

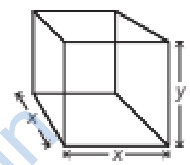
**30.** **Hint** $\quad h = \sqrt{100 - x^2}$




$\because \quad$ Area, $A = \dfrac{1}{2}(20 + 2x)\sqrt{100 - x^2}$

$\because \quad \dfrac{dA}{dx} = 0$ [**Ans.** $75\sqrt{3}$ cm$^2$]

**31.** Let the length of side of the square base of open box be $x$ units and its height be $y$ units.




$\therefore$ Area of the cardboard used $= x^2 + 4xy$

$\Rightarrow \quad c^2 = x^2 + 4xy \Rightarrow y = \dfrac{c^2 - x^2}{4x} \quad$ ...(i)

Now, volume of the box, $V = x^2 y$

$\Rightarrow \quad V = x^2 \cdot \left(\dfrac{c^2 - x^2}{4x}\right) \quad$ [from Eq. (i)]

$\Rightarrow \quad V = \dfrac{1}{4}x(c^2 - x^2) = \dfrac{1}{4}(c^2 x - x^3)$

On differentiating both sides w.r.t. $x$, we get

$\dfrac{dV}{dx} = \dfrac{1}{4}(c^2 - 3x^2) \quad$ ...(ii)

For maximum volume, put $\dfrac{dV}{dx} = 0 \Rightarrow c^2 = 3x^2$

$\Rightarrow \quad x^2 = \dfrac{c^2}{3} \Rightarrow x = \dfrac{c}{\sqrt{3}} \quad$ [using positive sign]

Again, differentiating both sides of Eq. (ii) w.r.t. $x$, we get

$\dfrac{d^2V}{dx^2} = \dfrac{1}{4}(-6x) = \dfrac{-3}{2}x < 0$

At $x = \dfrac{c}{\sqrt{3}}$, $\dfrac{d^2V}{dx^2} = -\dfrac{3}{2}\cdot\left(\dfrac{c}{\sqrt{3}}\right) < 0$

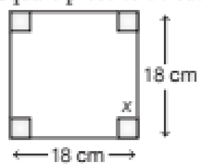
So, the volume $(V)$ is maximum at $x = \dfrac{c}{\sqrt{3}}$

Hence, maximum volume of the box,

$$(V)_{x = \frac{c}{\sqrt{3}}} = \frac{1}{4}\left(c^2 \cdot \frac{c}{\sqrt{3}} - \frac{c^3}{3\sqrt{3}}\right)$$

$$= \frac{1}{4}\cdot\frac{(3c^3 - c^3)}{3\sqrt{3}} = \frac{1}{4}\cdot\frac{2c^3}{3\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \text{ cu units}$$

**32.** Let the side of square piece to be cut $x$ cm.




$\therefore$ Length of box, $l = 18 - 2x$

Breadth of box, $b = 18 - 2x$

Height of box, $h = x$

We know that,

Volume of box, $V = lbh$

$\therefore \quad V = (18 - 2x) \cdot (18 - 2x) \cdot x = 2(9 - x) \cdot 2(9 - x) \cdot x$

$= 4(9 - x)^2 \cdot x \quad$ ...(i)

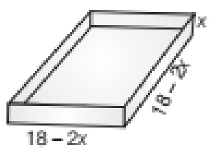
$= 4(81 - 18x + x^2) \cdot x$

$\Rightarrow \quad V = 4(x^3 - 18x^2 + 81x)$

On differentiating $V$ w.r.t. $x$, we get

$\frac{dV}{dx} = 4(3x^2 - 36x + 81) = 4 \cdot 3(x^2 - 12x + 27)$

$= 12(x^2 - 12x + 27) \quad$ ...(ii)




For maximum or minimum volume $V$, put $\frac{dV}{dx} = 0$

$\Rightarrow \quad 12(x^2 - 12x + 27) = 0$

$\Rightarrow \quad x^2 - 12x + 27 = 0 \Rightarrow x(x - 9) - 3(x - 9) = 0$

$\Rightarrow \quad (x - 9)(x - 3) = 0 \Rightarrow x - 9 = 0$ and $x - 3 = 0$

$\Rightarrow \quad x = 9$ and $x = 3$

But $x = 9$ is not possible, as whole tin piece will be cut off into two equal parts.

$\therefore \quad x = 3$

Again, differentiating both sides of Eq. (ii) w.r.t. $x$, we get

$$\frac{d^2V}{dx^2} = 12(2x - 12) = 12 \cdot 2(x - 6) = 24(x - 6)$$

At $x = 3$, $\left[\frac{d^2V}{dx^2}\right]_{x=3} = 24(3 - 6) = 24(-3) = -72 < 0$

So, volume is maximum at $x = 3$ cm.

Maximum volume $= 4(9 - 3)^2 \cdot 3 \quad$ [from Eq. (i)]

$= 4 \cdot 36 \cdot 3 = 432 \text{ cm}^3$

**33.** **Hint** Side of a square base is $x$ and height is $y$.

$V = x^2y$ and cost $= 2x^2 \times 5 + 4xy \times 2.50$ [**Ans.** ₹ 1920]

**34.** Let $x$ be the side of square and $r$ be the radius of circle.

Then, perimeter of a circle $= 2\pi r$

and perimeter of a square $= 4x$

Given, sum of perimeter of circle and square $= k$

$\Rightarrow \quad 2\pi r + 4x = k \Rightarrow x = \frac{k - 2\pi r}{4} \quad$ ...(i)

Now, sum of areas of a circle and a square is

$$A = x^2 + \pi r^2 = \left[\frac{k - 2\pi r}{4}\right]^2 + \pi r^2$$

$\Rightarrow \quad A = \left(\frac{1}{16}\right)(k^2 - 4k\pi r + 4\pi^2 r^2) + \pi r^2$

On differentiating twice w.r.t. $r$, we get

$$\frac{dA}{dr} = \left(\frac{1}{16}\right)(-4k\pi + 8\pi^2 r) + 2\pi r$$

and $\frac{d^2A}{dr^2} = \frac{1}{16}[0 + 8\pi^2] + 2\pi = 2\pi + \frac{\pi^2}{2} > 0$

For maximum or minimum value, put $\frac{dA}{dr} = 0$

$\Rightarrow \quad 2\pi r - \frac{4k\pi}{16} + \frac{8\pi^2 r}{16} = 0 \Rightarrow r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{k\pi}{4}$

$\Rightarrow \quad r = \frac{\left(\frac{k\pi}{4}\right)}{2\pi + \frac{\pi^2}{2}} = \frac{k}{8 + 2\pi} \quad$ ...(ii)

Now, $\left(\frac{d^2A}{dr^2}\right)_{r = \frac{k}{8 + 2\pi}} = +\text{ve}$

$\therefore$ $A$ is least, when $r = \frac{k}{8 + 2\pi}$ and put this value in Eq. (i), we get

$$x = \frac{1}{4}\left(k - 2\pi \times \frac{k}{8 + 2\pi}\right) = \frac{1}{4}\left[\frac{8k + 2\pi k - 2\pi k}{8 + 2\pi}\right]$$

$\Rightarrow \quad x = \frac{2k}{8 + 2\pi} = 2\left(\frac{k}{8 + 2\pi}\right) = 2r \quad$ [using Eq. (ii)]

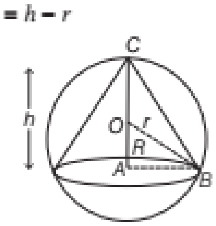
Hence, $A$ (i.e. sum of their areas) is least, when side of the square is double the radius of the circle.

**35.** Similar as Example 12.

$\left[\textbf{Ans. } \frac{144\sqrt{3}}{4\sqrt{3} + 9} \text{ cm}, \frac{324}{4\sqrt{3} + 9} \text{ cm}\right]$

**36.** Let $R$ be the radius and $h$ be the height of cone, which is inscribed in a sphere of radius $r$.

$\therefore \quad OA = h - r$




In $\Delta OAB$, by Pythagoras theorem, we get

$$r^2 = R^2 + (h - r)^2 \Rightarrow r^2 = R^2 + h^2 + r^2 - 2rh$$

$\Rightarrow \quad R^2 = 2rh - h^2 \quad$ ...(i)

The volume of sphere $= \frac{4}{3}\pi r^3$

and the volume of the cone, $V = \frac{1}{3}\pi R^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi h(2rh - h^2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(2rh^2 - h^3) \quad \text{...(ii)}$$

On differentiating both sides of Eq. (ii) w.r.t. $h$, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2) \quad \text{...(iii)}$$

For maximum or minimum value, put $\frac{dV}{dh} = 0$.

$$\Rightarrow \quad 4rh = 3h^2 \Rightarrow 4r = 3h$$

$$\Rightarrow \quad h = \frac{4r}{3} \quad [\because h \neq 0]$$

Again, differentiating both sides of Eq. (iii) w.r.t. $h$, we get

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$$

At $h = \frac{4r}{3}$, $\left[\frac{d^2V}{dh^2}\right]_{h = \frac{4r}{3}} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4r}{3}\right)$

$$= \frac{\pi}{3}(4r - 8r) = -\frac{4r\pi}{3} < 0$$

$\Rightarrow$ $V$ is maximum at $h = \frac{4r}{3}$.

On putting the value of $h$ in Eq. (ii), we get

$$V = \frac{1}{3}\pi\left[2r\left(\frac{4r}{3}\right)^2 - \left(\frac{4r}{3}\right)^3\right]$$

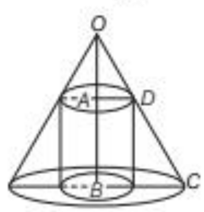
$$= \frac{\pi}{3}\left[\frac{32}{9}r^3 - \frac{64}{27}r^3\right]$$

$$= \frac{\pi}{3}r^3\left[\frac{32}{9} - \frac{64}{27}\right] = \frac{\pi}{3}r^3\left[\frac{96 - 64}{27}\right] = \frac{\pi}{3}r^3\left(\frac{32}{27}\right)$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right) = \frac{8}{27} \times \text{Volume of sphere}$$

Hence, maximum volume of the cone is 8/27 of the volume of the sphere.

**37.** Let $r_1$ be radius of the cone and $h_1$ be its height and let $r$ be the radius and $h$ be the height of the inscribed cylinder.




Clearly, $\Delta OAD$ and $\Delta OBC$ are similar.

$$\therefore \quad \frac{r}{r_1} = \frac{h_1 - h}{h_1} \Rightarrow r = \frac{r_1}{h_1}(h_1 - h)$$

Let $S$ be the curved surface area of the cylinder.

Then, $S = 2\pi rh \Rightarrow S = \frac{2\pi r_1}{h_1}(h_1 - h)h$

On differentiating both sides w.r.t. $h$, we get

$$\frac{dS}{dh} = \frac{2\pi r_1}{h_1}(h_1 - 2h) \text{ and } \frac{d^2S}{dh^2} = -\frac{4\pi r_1}{h_1} < 0$$

Now, put $\frac{dS}{dh} = 0 \Rightarrow h_1 = 2h$

Also, at $h_1 = 2h$, $\frac{d^2S}{dh^2} < 0$

So, $S$ is maximum when $h_1 = 2h$

$$\therefore \quad \frac{r}{r_1} = \frac{2h - h}{2h} = \frac{1}{2} \Rightarrow r_1 = 2r$$

$$\Rightarrow \quad r = \frac{r_1}{2} = \text{Half of radius of cone}$$

**38.** Let $VAB$ be the cone of height $h$, semi-vertical angle $\alpha$ and $x$ be the radius of the base of the cylinder $A'B'DC$ which is inscribed in the cone $VAB$. Then, $OO'$ is the height of the cylinder $= VO - VO' = h - x\cot\alpha$

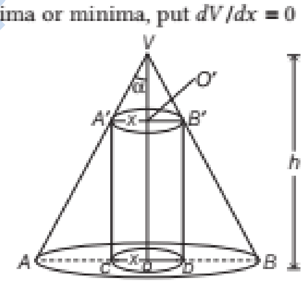
and volume of the cylinder,

$$V = \pi x^2(h - x\cot\alpha) \quad \text{...(i)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dV}{dx} = 2\pi xh - 3\pi x^2\cot\alpha \quad \text{...(ii)}$$

For maxima or minima, put $dV/dx = 0$




$$\Rightarrow 2\pi xh - 3\pi x^2\cot\alpha = 0 \Rightarrow \pi x(2h - 3x\cot\alpha) = 0$$

$$\Rightarrow \quad x = \frac{2h}{3}\tan\alpha \quad [\because x \neq 0]$$

Again, differentiating both sides of Eq. (ii) w.r.t. $x$, we get

$$\frac{d^2V}{dx^2} = (2\pi h - 6\pi x\cot\alpha) = \pi(2h - 6x\cot\alpha)$$

At $x = \frac{2h}{3}\tan\alpha$, $\frac{d^2V}{dx^2} = \pi(2h - 4h) = -2\pi h < 0$

$\Rightarrow$ $V$ is maximum, when $x = \frac{2h}{3}\tan\alpha$.

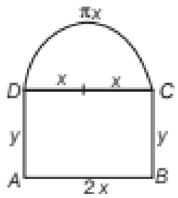
Now, $OO' = h - x\cot\alpha = h - \frac{2h}{3} = \frac{h}{3}$

$\therefore$ Maximum volume of the cylinder is

$$V = \pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(h - \frac{2h}{3}\right) = \frac{4}{27}\pi h^3\tan^2\alpha.$$

**Hence proved.**

**39.** Let length of rectangle = $2x$ m and breadth = $y$ m




Given, perimeter of the window = 10 m

$\because$ Perimeter of rectangle

+ Perimeter of semi-circle = 10 m

$$\Rightarrow \quad 2y + 2x + \frac{1}{2}(2\pi x) = 10$$

$$\Rightarrow \quad 2y = 10 - x(\pi + 2) \quad \text{...(i)}$$

Let $A$ be area of the window, then

$A$ = Area of semi-circle + Area of rectangle

$$= \frac{1}{2}\pi x^2 + 2xy$$

$$\Rightarrow A = \frac{1}{2}(\pi x^2) + x[10 - x(\pi + 2)] \quad \text{[using Eq. (i)]}$$

$$= \frac{1}{2}(\pi x^2) + 10x - x^2\pi - 2x^2$$

$$= 10x - \frac{\pi x^2}{2} - 2x^2$$

On differentiating twice w.r.t. $r$, we get

$$\frac{dA}{dx} = 10 - \pi x - 4x \quad \text{...(ii)}$$

and
$$\frac{d^2A}{dx^2} = -\pi - 4 \quad \text{...(iii)}$$

For maxima or minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 10 - \pi x - 4x = 0 \Rightarrow 10 = (4 + \pi)x$$

$$\Rightarrow \quad x = \frac{10}{4 + \pi}$$

On putting $x = \frac{10}{4 + \pi}$ in Eq. (iii), we get

$$\frac{d^2A}{dx^2} = \text{Negative}$$

Thus, $A$ has local maxima, when $x = \frac{10}{4 + \pi}$ ...(iv)

$\therefore$ Radius of semi-circle, $x = \frac{10}{4 + \pi}$

and one side of rectangle $= 2x = \frac{2 \times 10}{4 + \pi} = \frac{20}{4 + \pi}$

Now, from Eq. (i), we get

$$y = \frac{1}{2}[10 - x(\pi + 2)]$$

$$= \frac{1}{2}\left[10 - \left(\frac{10}{\pi + 4}\right)(\pi + 2)\right] \quad \text{[from Eq. (iv)]}$$

$$= \frac{10\pi + 40 - 10\pi - 20}{2(\pi + 4)}$$

$$= \frac{20}{2(\pi + 4)} = \frac{10}{\pi + 4}$$

So, other side of rectangle $= y = \frac{10}{\pi + 4}$

Light is maximum, when area is maximum.

Hence, dimensions of the window are length $= \frac{20}{\pi + 4}$ m and breadth $= \frac{10}{\pi + 4}$ m.

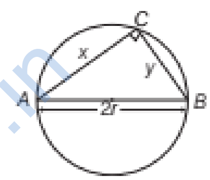
**40.** Solve as Question 39.

$\left[\textbf{Ans.}\ \text{Length} = \frac{2p}{4 + \pi} \text{ and breadth} = \frac{p}{4 + \pi}\right]$

**41.** Let the side of $\Delta ABC$ be $x$ and $y$.

Also, let $r$ be the radius of circle and $\angle C = 90°$.

[$\because$ angle made in semi-circle]




In $\Delta ABC$, we have

$$(AB)^2 = (AC)^2 + (BC)^2 \quad \text{[by Pythagoras theorem]}$$

$$\Rightarrow \quad (2r)^2 = x^2 + y^2 \Rightarrow 4r^2 = x^2 + y^2$$

Area of $\Delta ABC$, $A = \frac{1}{2}xy$

On squaring both sides, we get $A^2 = \frac{1}{4}x^2y^2$

Let $A^2 = S$, then $S = \frac{1}{4}x^2y^2$

On putting the value of $y^2$ from Eq. (i), we get

$$S = \frac{1}{4}x^2(4r^2 - x^2) = \frac{1}{4}(4x^2r^2 - x^4)$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dS}{dx} = \frac{1}{4}[8r^2x - 4x^3]$$

For maximum and minimum, put $\frac{dS}{dx} = 0$

$$\Rightarrow \quad \frac{1}{4}[8r^2x - 4x^3] = 0$$

$$\Rightarrow \quad 8r^2x = 4x^3 \Rightarrow 8r^2 = 4x^2$$

$$\Rightarrow \quad x^2 = 2r^2 \Rightarrow x = \sqrt{2}r$$

Then, from Eq. (i), we get

$$y^2 = 4r^2 - 2r^2 = 2r^2$$

$$\Rightarrow \quad y = \sqrt{2}r$$

i.e. $x = y$, so it is an isosceles triangle.

Also, $\dfrac{d^2S}{dx^2} = \dfrac{d}{dx}\left[\dfrac{1}{4}(8r^2x - 4x^3)\right] = \dfrac{1}{4}(8r^2 - 12x^2)$

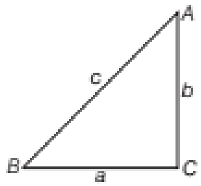
$$= 2r^2 - 3x^2$$

At $x = \sqrt{2}r$, $\dfrac{d^2S}{dx^2} = 2r^2 - 3x^2 < 0$

So, area is minimum.

Hence, area is minimum, when triangle is isosceles.

**42.** Let $a$ and $b$ be the sides of a right angled triangle.




From $\Delta ABC$, we get $c^2 = a^2 + b^2$

$\therefore$ Area of $\Delta ABC$, $A = \dfrac{1}{2}a \cdot b$

$\Rightarrow \quad A = \dfrac{a}{2}\sqrt{c^2 - a^2} \qquad [\because b = \sqrt{c^2 - a^2}]$

On differentiating both sides w.r.t. $a$, we get

$$\frac{dA}{da} = \frac{1}{2} \cdot 1 \cdot \sqrt{c^2 - a^2} + \frac{1}{2} \cdot a \cdot \frac{1}{2} \cdot \frac{(-2a)}{\sqrt{c^2 - a^2}}$$

$$= \frac{1}{2}\left[\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right]$$

For maxima, put $\dfrac{dA}{da} = 0$

$$\Rightarrow \quad \frac{1}{2}\left[\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right] = 0$$

$$\Rightarrow \quad c^2 - a^2 - a^2 = 0$$

$$\Rightarrow \quad c^2 = 2a^2 \Rightarrow a = \frac{c}{\sqrt{2}}$$

Again, differentiating both sides of $\dfrac{dA}{da}$ w.r.t. $a$, we get

$$\frac{d^2A}{da^2} = \frac{1}{2}\left[\frac{-2a}{2\sqrt{c^2 - a^2}} - \frac{\sqrt{c^2 - a^2}\, 2a - a^2 \dfrac{1}{2\sqrt{c^2 - a^2}}(-2a)}{\sqrt{(c^2 - a^2)^2}}\right]$$

$$= \frac{1}{2}\left[\frac{-a}{\sqrt{c^2 - a^2}} - \frac{\dfrac{(c^2 - a^2)\, 2a + a^3}{\sqrt{c^2 - a^2}}}{c^2 - a^2}\right]$$

$$= \frac{-1}{2}a\left[\frac{c^2 - a^2 + 2c^2 - 2a^2 + a^2}{(c^2 - a^2)^{3/2}}\right] = \frac{-1}{2}a\left[\frac{3c^2 - 2a^2}{(c^2 - a^2)^{3/2}}\right]$$

At $a = \dfrac{c}{\sqrt{2}}$, $\dfrac{d^2A}{da^2} = -\dfrac{c}{2\sqrt{2}}\left[\dfrac{3c^2 - 2\left(\dfrac{c^2}{2}\right)}{\left(c^2 - \dfrac{c^2}{2}\right)^{3/2}}\right]$

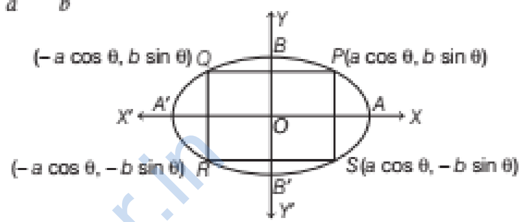
$$= -\frac{c}{2\sqrt{2}}\left[\frac{2c^2 \times 2^{3/2}}{(c^2)^{3/2}}\right] = -\frac{2c^3}{c^3} = -2 < 0$$

So, the area of $\Delta ABC$ is maximum

and $\qquad b = \sqrt{c^2 - a^2} = \sqrt{2a^2 - a^2} = a$

Hence, the area of a right angled triangle of given hypotenuse is maximum, when the triangle is an isosceles triangle.

**43.** Let $PQRS$ be a rectangle which inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.




Length of rectangle, $l = 2a\cos\theta$ and

breadth of rectangle, $b = 2b\sin\theta$

$\because$ Area of rectangle, $A = l \cdot b$

$\therefore \quad A = 2ab \cdot 2\sin\theta\cos\theta = 2ab\sin 2\theta$

On differentiating both sides w.r.t. $\theta$, we get

$$\frac{dA}{d\theta} = 2ab \cdot \cos 2\theta \cdot 2$$

For critical points, put $\dfrac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0$

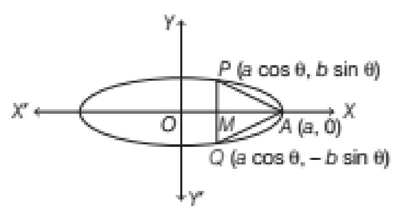
$$\Rightarrow \quad 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Now, $\dfrac{d^2A}{d\theta^2} = -8ab\sin 2\theta$ which is negative for $\theta = \dfrac{\pi}{4}$.

Hence, area is maximum.

Now, maximum area $= 2ab\sin\dfrac{\pi}{2} = 2ab$

**44.** Let the equation of an ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then any point on the ellipse is $P(a\cos\theta, b\sin\theta)$.



From $P$, draw $PM \perp OX$ and produce it to meet the ellipse at $Q$, then $APQ$ is an isosceles triangle, let $S$ be its area, then

$$S = 2 \times \frac{1}{2} \times AM \times MP$$
$$= (OA - OM) \times MP$$
$$= (a - a\cos\theta) \cdot b\sin\theta$$

$$\Rightarrow \quad S = ab(\sin\theta - \sin\theta\cos\theta) = ab\left(\sin\theta - \frac{1}{2}\sin 2\theta\right)$$

On differentiating twice w.r.t. $\theta$, we get

$$\frac{dS}{d\theta} = ab(\cos\theta - \cos 2\theta)$$

and $$\frac{d^2S}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$$

For maxima or minima, put $dS/d\theta = 0$

$\Rightarrow \cos\theta = \cos 2\theta \;\Rightarrow\; 2\theta = 2\pi - \theta$

$\Rightarrow \quad 3\theta = 2\pi \Rightarrow \theta = \dfrac{2\pi}{3}$

At $\theta = \dfrac{2\pi}{3}$, $\dfrac{d^2S}{d\theta^2} = ab\left[-\sin\dfrac{2\pi}{3} + 2\sin\left(2 \times \dfrac{2\pi}{3}\right)\right]$

$$= ab\left[-\sin\left(\pi - \frac{\pi}{3}\right) + 2\sin\left(\pi + \frac{\pi}{3}\right)\right]$$

$$= ab\left(-\sin\frac{\pi}{3} - 2\sin\frac{\pi}{3}\right) \quad \left[\because \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} \text{ and } \sin\left(\pi + \frac{\pi}{3}\right) = \frac{-\sin\pi}{3}\right]$$

$$= ab\left(-\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}ab}{2} < 0$$

$\therefore$ $S$ is maximum, when $\theta = \dfrac{2\pi}{3}$ and maximum value of

$$S = ab\left(\sin\frac{2\pi}{3} - \frac{1}{2} \cdot 2\sin\frac{2\pi}{3}\cos\frac{2\pi}{3}\right)$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$= ab\left[\sin\left(\pi - \frac{\pi}{3}\right) - \sin\left(\pi - \frac{\pi}{3}\right)\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= ab\left[\sin\frac{\pi}{3} - \sin\frac{\pi}{3}\left(-\cos\frac{\pi}{3}\right)\right]$$

$$= ab\left(\sin\frac{\pi}{3} + \sin\frac{\pi}{3} \cdot \cos\frac{\pi}{3}\right)$$

$$= ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2}\right)$$

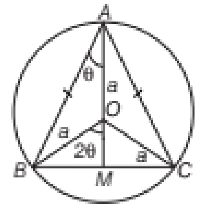
$$= ab\left(\frac{2\sqrt{3} + \sqrt{3}}{4}\right)$$

$$= \frac{3\sqrt{3}}{4} ab \text{ sq units}$$

Hence, the maximum area of isosceles triangle is $\dfrac{3\sqrt{3}}{4} ab$ sq units.

**45.** **Hint** Base $BC = 2a\sin 2\theta$

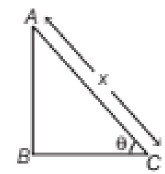
and height $AM = OA + OM = a + a\cos 2\theta$




$\because$ Area of $\Delta ABC$, $\Delta = \dfrac{1}{2} BC \times AM = a^2 \sin 2\theta (1 + \cos 2\theta)$

Now, put $\dfrac{d\Delta}{d\theta} = 0$.

**46.** Let $ABC$ be a right angled triangle and




$$AC + BC = \text{constant} = K \text{ (say)}$$

Let $\angle ACB = \theta$. Then, $BC = x\cos\theta$ and $AB = x\sin\theta$

Let $y$ be the area of $\Delta ABC$.

Then, $y = \dfrac{1}{2} BC \cdot AB = \dfrac{1}{2} \cdot x\cos\theta \cdot x\sin\theta$

$$\Rightarrow \quad y = \frac{1}{2}x^2 \sin\theta\cos\theta \qquad \text{...(i)}$$

According to the given condition, $K = AC + BC$

$$\Rightarrow \quad K = x + x\cos\theta \;\Rightarrow\; x = \frac{K}{1 + \cos\theta} \qquad \text{...(ii)}$$

On putting the value of $x$ in Eq. (i), we get

$$y = \frac{K^2}{2} \cdot \frac{\sin\theta\cos\theta}{(1 + \cos\theta)^2}$$

On differentiating both sides w.r.t. $\theta$, we get

$$\frac{dy}{d\theta} = \frac{K^2}{2} \cdot \frac{\left[\begin{array}{l}(1 + \cos\theta)^2(\cos^2\theta - \sin^2\theta) \\ - \sin\theta\cos\theta\, 2(1 + \cos\theta)(-\sin\theta)\end{array}\right]}{(1 + \cos\theta)^4}$$

$$= \frac{K^2}{2} \cdot \frac{\left[\begin{array}{c}(1 + \cos\theta)[(1 + \cos\theta)(\cos^2\theta - \sin^2\theta) \\ + 2\sin^2\theta\cos\theta]\end{array}\right]}{(1 + \cos\theta)^4}$$

$$= \frac{K^2}{2} \cdot \frac{\left[\begin{array}{c}\cos^2\theta - \sin^2\theta + \cos^3\theta - \cos\theta\sin^2\theta) \\ + 2\sin^2\theta\cos\theta\end{array}\right]}{(1 + \cos\theta)^3}$$

$$= \frac{K^2}{2(1 + \cos\theta)^3} \cdot (2\cos^2\theta - 1 + \cos^3\theta + \cos\theta\sin^2\theta)$$

$[\because \sin^2\theta = 1 - \cos^2\theta]$

$$= \frac{K^2}{2(1+\cos\theta)^3}\cdot[2\cos^2\theta - 1 + \cos\theta(\cos^2\theta + \sin^2\theta)]$$

$$= \frac{K^2}{2(1+\cos\theta)^3}\cdot(2\cos^2\theta + \cos\theta - 1)$$

$$[\because \cos^2\theta + \sin^2\theta = 1]$$

For $0 < \theta < \frac{\pi}{2}, \frac{K^2}{2(1+\cos\theta)^3} > 0$

$\therefore$ Sign scheme of $\frac{dy}{d\theta}$ will depend on $2\cos^2\theta + \cos\theta - 1$.

Now, $\quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad [\because \cos\theta \neq -1]$

$\Rightarrow \quad \theta = \frac{\pi}{3} \qquad [\because 0 < \theta < 90^\circ]$

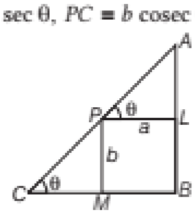
$\therefore \quad \frac{d^2y}{d\theta^2} = \frac{(-k\sin\theta)(2-\cos\theta)}{(1+\cos\theta)^3} < 0$

Hence, area of the triangle is maximum at $\theta = \frac{\pi}{3}$.

**47.** Let $P$ be a point on the hypotenuse $AC$ of right angled $\Delta ABC$. Such that $PL \perp AB = a$ and $PM \perp BC = b$

Let $\angle APL = \angle ACB = \theta$ (say)

Then, $AP = a\sec\theta$, $PC = b\,\text{cosec}\,\theta$




Let $l$ be the length of the hypotenuse, then

$l = AP + PC$

$\Rightarrow \quad l = a\sec\theta + b\,\text{cosec}\,\theta, 0 < \theta < \frac{\pi}{2}$

On differentiating both sides w.r.t. $\theta$, we get

$$\frac{dl}{d\theta} = a\sec\theta\tan\theta - b\,\text{cosec}\,\theta\cot\theta \qquad \text{...(i)}$$

For maxima or minima, put $\frac{dl}{d\theta} = 0$

$\Rightarrow a\sec\theta\tan\theta = b\,\text{cosec}\,\theta\cot\theta$

$\Rightarrow \quad \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta} \Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{1/3}$

Again, differentiating both sides of Eq. (i) w.r.t. $\theta$, we get

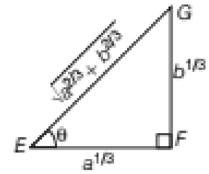
$$\frac{d^2l}{d\theta^2} = a(\sec\theta\times\sec^2\theta + \tan\theta\times\sec\theta\tan\theta)$$

$$-b[\text{cosec}\,\theta(-\text{cosec}^2\theta) + \cot\theta(-\text{cosec}\,\theta\cot\theta)]$$

$$= a\sec\theta(\sec^2\theta + \tan^2\theta)$$

$$+ b\,\text{cosec}\,\theta(\text{cosec}^2\theta + \cot^2\theta)$$

For $0 < \theta < \frac{\pi}{2}$, all trigonometric ratios are positive.




Also, $2a > 0$ and $b > 0$.

$\therefore \quad \frac{d^2l}{d\theta^2}$ is positive.

and least value of

$l = a\sec\theta + b\,\text{cosec}\,\theta$

$$= a\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + b\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$$

$$= \sqrt{a^{2/3}+b^{2/3}}(a^{2/3}+b^{2/3}) = (a^{2/3}+b^{2/3})^{3/2}$$

$$\left[\because \text{in } \Delta EFG, \tan\theta = \frac{b^{1/3}}{a^{1/3}}, \sec\theta = \frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}\right.$$

$$\left.\text{and } \text{cosec}\,\theta = \frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}\right]$$

**Hence proved.**

**48.** Let the length and breadth of the tank be $x$ and $y$ m, respectively.

Then, volume $= 75\text{ m}^3$

$\Rightarrow \quad 3xy = 75 \qquad [\because \text{depth of tank} = 3\text{ m}]$

$\Rightarrow \quad y = \frac{25}{x}$

Let $C$ be the cost of the tank.

Then, $C = 100xy + 50(3\times 2x + 3\times 2y)$

$= 100xy + 300x + 300y$

$= 100x\times\frac{25}{x} + 300x + 300\times\frac{25}{x} \qquad \left[\because y = \frac{25}{x}\right]$

$\Rightarrow \quad C = 2500 + 300x + \frac{7500}{x}$

On differentiating twice w.r.t. $x$, we get

$$\frac{dC}{dx} = 300 - \frac{7500}{x^2} \text{ and } \frac{d^2C}{dx^2} = \frac{15000}{x^3} > 0$$

For minimum value, put $\frac{dC}{dx} = 0$

$\Rightarrow \quad 300 - \frac{7500}{x^2} = 0 \Rightarrow x^2 = 25$

$\Rightarrow \quad x = 5 \qquad [\because \text{length cannot be negative}]$

At $x = 5, \frac{d^2C}{dx^2} = \frac{15000}{5^3} = 120 > 0$

So, $C$ is minimum.

When $x = 5$, then $C = 2500 + 1500 + 1500 = 5500$

Hence, the cost of least expensive tank is ₹ 5500.

# SUMMARY

- The **rate of change** of $y$ with respect to $x$ at point $x = x_0$ is given by $\left(\frac{dy}{dx}\right)_{x=x_0}$ or $f'(x_0)$.
- Suppose $y = f(t)$ and $x = g(t)$. Then, **rate of change** of $y$ with respect to $x$ is given by
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ provided } \frac{dx}{dt} \neq 0 \text{ or } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
- A function $f$ is called an **increasing** function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2), \forall x_1, x_2 \in I$.
- A function $f$ is called a **strictly increasing** function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \forall x_1, x_2 \in I$.
- A function $f$ is called a **decreasing** function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2), \forall x_1, x_2 \in I$.
- A function $f$ is called a **strictly decreasing** function in $I$, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \forall x_1, x_2 \in I$.
- $f$ be continuous on $[a, b]$ and differentiable on the open interval $(a, b)$. Then,
  (i) $f$ is increasing in $[a, b]$, if $f'(x) \geq 0$ for each $x \in (a, b)$.
  (ii) $f$ is decreasing in $[a, b]$, if $f'(x) \leq 0$ for each $x \in (a, b)$.
  (iii) $f$ is strictly increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$.
  (iv) $f$ is strictly decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$.
  (v) $f$ is constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$.
- **Local Maxima and Local Minima** Let $f$ be a real valued function and let $c$ be an interior point in the domain of $f$, then
  (i) $c$ is called a point of **local maxima**, if there is a $h > 0$ such that $f(c) > f(x), \forall x$ in $(c - h, c + h)$.
  Here, value $f(c)$ is called the **local maximum value** of $f$.
  (ii) $c$ is called a point of local minima, if there is a $h > 0$ such that $f(c) < f(x), \forall x$ in $(c - h, c + h)$.
  Here, value $f(c)$ is called the **local minimum value** of $f$.
- A point $c$ in the domain of a function $f$ at which either $f'(c) = 0$ or $f$ is not differentiable is called a **critical point** of $f$.
- **First Derivative Test**
  (i) If $f'(x)$ change sign from positive to negative as $x$ increases through $c$, then $c$ is a point of local maxima and $f(c)$ is local maximum value.
  (ii) If $f'(x)$ change sign from negative to positive as $x$ increases through point $c$, then $c$ is a point of local minima and $f(c)$ is local minimum value.
  (iii) If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Infact, such a point is called **point of inflection**.
- **Second Derivative Test** Let a function $f$ be twice differentiable at $c$. Then,
  (i) $x = c$ is a point of local maxima, if $f'(c) = 0$ and $f''(c) < 0$.
  The value $f(c)$ is local maximum value of $f$.
  (ii) $x = c$ is a point of local minima, if $f'(c) = 0$ and $f''(c) > 0$.
  The value $f(c)$ is local minimum value of $f$.
  (iii) the test fails, if $f'(c) = 0$ and $f''(c) = 0$.



# CHAPTER PRACTICE

## OBJECTIVE TYPE QUESTIONS

**1** The function $y = x^2e^{-x}$ is decreasing in the interval [CBSE 2021 (TermI)]

(a) $(0, 2)$ (b) $(2, \infty)$
(c) $(-\infty, 0)$ (d) $(-\infty, 0) \cup (2, \infty)$

**2** For $0 < \theta < \frac{\pi}{2}$, the value of $\theta$, if it increases twice as

(c) both maximum and minimum values
(d) neither maximum nor minimum value

**7** The area of a trapezium is defined by function $f$ and given by $f(x) = (10 + x)\sqrt{100 - x^2}$, then the area when it is maximised is [CBSE 2021 (TermI)] fast as its sine, is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) None of these

**3** The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is **[CBSE 2021 (TermI)]**

(a) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$

(b) Strictly decreasing in $(-2, 3)$

(c) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$

(d) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$

**4** Let $g(x) = 2f\left(\frac{x}{2}\right) + f(2 - x)$ and $f''(x) < 0$ for all $x \in (0, 2)$. Then, $g(x)$ is

(a) increasing on $(4/3, 2)$ and increasing on $(0, 4/3)$

(b) decreasing on $(0, 4/3)$ and increasing on $(4/3, 2)$

(c) increasing on $(0, 4/3)$ and decreasing on $(4/3, 2)$

(d) None of the above

**5** The value of $b$ for which the function $f(x) = x + \cos x + b$ is strictly decreasing over $R$ is **[CBSE 2021 (TermI)]**

(a) $b < 1$ (b) No value of $b$ exists

(c) $b \le 1$ (d) $b \ge 1$

**6** A function $f : R \to R$ is defined as $f(x) = x^3 + 1$. Then, the function has **[CBSE 2021 (TermI)]**

(a) no minimum value

(b) no maximum value

(a) $75 \text{ cm}^2$ (b) $7\sqrt{3} \text{ cm}^2$

(c) $75\sqrt{3} \text{ cm}^2$ (d) $5 \text{ cm}^2$

**8** The area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

(a) $ab$ sq units (b) $\frac{ab}{2}$ sq units

(c) $2ab$ sq units (d) $3ab$ sq units

**9** The smallest value of polynomial $x^3 - 18x^2 + 96x$ in $[0, 9]$ is

(a) 126 (b) 0

(c) 135 (d) 160

**10** The maximum value of $[x(x-1)+1]^{\frac{1}{3}}, 0 \le x \le 1$ is **[CBSE 2021 (TermI)]**

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $\sqrt[3]{\frac{1}{3}}$

## VERY SHORT ANSWER Type Questions

**11** Find the rate of increase in the surface area of a cube with respect to its edge $x$, when $x = 5$ cm.

**12** The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.

**13** The radius of a circle is increasing at the rate of 0.9 cm/s. What is the rate of increase of its circumference?

**14** The total revenue received from the sale of $x$ units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue when $x = 5$, where by marginal revenue, we mean the rate of change of total revenue with respect to the number of items sold at an instant. **[NCERT]**

**15** Show that $f(x) = e^{1/x}$ is a strictly decreasing function for all $x > 0$.

**16** Show that $f(x) = x - \sin x$ is increasing for all $x \in R$.

**17** Show that $f(x) = \left(x - \frac{1}{x}\right)$ is increasing for all $x \in R, x \ne 0$.

**18** Show that the function given by $f(x) = e^{2x}$ is strictly increasing on $R$. **[NCERT]**

**19** Show that the function $f(x) = \cos^2 x$ is a decreasing function on $\left(0, \frac{\pi}{2}\right)$.

**26** At what points of the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decrease at the same rate at which the abscissa increase?

**27** Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $R$. **[All India 2017]**

**28** Find the values of $k$ for which $f(x) = kx^3 - 9kx^2 + 9x + 3$ is increasing on $R$.

## SHORT ANSWER Type II Questions

**29** The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing, when the side of the triangle is 20 cm? **[Delhi 2015]**

**30** Water is dripping out from a conical funnel at a uniform rate of 4 $\text{cm}^3$/s through a tiny hole at the vertex in the bottom. When the slant height of the water is 3 cm. Find rate of decrease of the slant height of the water-cone. Given, the vertical angle of the funnel is 120°.

**20** If the radius of a circle increasing from 5 cm to 5.1 cm, then find the increase area.

## SHORT ANSWER Type I Questions

**21** A stone is dropped into a quiet lake and waves moves in circles at a speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing? **[NCERT]**

**22** An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing, when the edge is 10 cm long?

**23** The volume of a sphere is increasing at the rate of 8 $cm^3/s$. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm. **[All India 2017]**

**24** The surface area of a spherical bubble is increasing at the rate of 2 $cm^2/s$. Find the rate at which the volume of the bubbles is increasing at the instant, when its radius is 6 cm.

**25** A balloon which always remains spherical is being inflated by pumping in gas at the rate of 800 $cm^3/s$. Find the rate at which the radius of the balloon is increasing, when the radius is 20 cm.

**37** Find the maximum profit that a company can make, if the profit function is given by $P(x) = 41 + 24x - 18x^2$. **[NCERT]**

**38** The sum of two numbers is 24. Find the numbers, so that their product is maximum. **[NCERT]**

**39** A telephone company in a town has 500 subscribers on its list and collect fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1 per one subscriber will discontinue the service. Find what increase will bring maximum profit?

## LONG ANSWER Type Questions

**40** Find the point on the curve $y^2 = 2x$, which is at a minimum distance from point (1, 4). **[All India 2011]**

**41** Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

**42** Show that a cylinder of a given volume, which is open at the top has minimum total surface area, when its height is equal to the radius of its base. **[Foreign 2014; Delhi 2011]**

**43** A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the

**31** If $x$ and $y$ are the sides of two squares such that $y = x - x^2$, then find the rate of change of the area of second square with respect to the area of first quadrant. **[NCERT Exemplar]**

**32** Find the intervals in which the function given by $f(x) = \sin 3x, x \in \left[0, \frac{\pi}{2}\right]$ is

(i) increasing (ii) decreasing.

**33** Find the intervals of the function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly decreasing. **[NCERT Exemplar]**

**34** Determine the interval in which the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is strictly increasing and strictly decreasing.

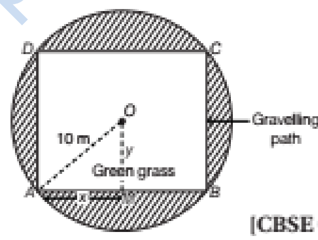
**Directions** (Q. Nos. 35-36) *Find the interval(s) in which the following functions are*

(i) *increasing.*

(ii) *decreasing.*

**35** $f(x) = \log(1 + x) - \frac{x}{1 + x}, x \neq -1$

**36** $f(x) = (x + 2) e^{-x}$ **[Delhi 2010C]**




**[CBSE Question Bank]**

*Based on the above information, answer the following*

(i) $2x$ and $2y$ represents the length and breadth of the rectangular part, then the relation between the variables is

(a) $x^2 - y^2 = 10$ (b) $x^2 + y^2 = 10$
(c) $x^2 - y^2 = 100$ (d) $x^2 + y^2 = 100$

(ii) The area of the green grass $A$ expressed as a function of $x$ is

(a) $2x\sqrt{100 - x^2}$ (b) $4x\sqrt{100 - x^2}$
(c) $2x\sqrt{100 + x^2}$ (d) $4x\sqrt{100 + x^2}$

(iii) The maximum value of area $A$ is

(a) 100 $m^2$ (b) 200 $m^2$
(c) 400 $m^2$ (d) 1600 $m^2$

(iv) The value of length of rectangle, if $A$ is maximum, is

(a) $10\sqrt{2}$ m (b) $20\sqrt{2}$ m
(c) 20 m (d) $5\sqrt{2}$ m

(v) The area of gravelling path is

(a) $100(\pi + 2) m^2$ (b) $100(\pi - 2) m^2$
(c) $200(\pi + 2) m^2$ (d) $200(\pi - 2) m^2$

**47.** Let a cone is inscribed in sphere of radius $R$.

soldier and the jet?

**44** Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ with its vertex at one end of the major axis.

**45** The demand function of an output is $x=106-2p$, where $x$ is the number of units and $p$ be the price per unit and the average cost per unit is $5+\frac{x}{50}$. If the total revenue is $px$, then determine the number of units for maximum profit.

## CASE BASED Questions

**46.** An architect designs a garden in a society. The garden is in the shape of a rectangle inscribed in a circle of radius 10m as shown in given figure.

(iii) Maximum value of volume $v$ is

(a) $\frac{32\pi}{81}R^3$ (b) $\frac{8}{27}\pi R^3$ (c) $\frac{32}{27}\pi R^3$ (d) $\frac{8}{81}\pi R^3$

(iv) The values of $r, h$ when $v$ is maximum, are

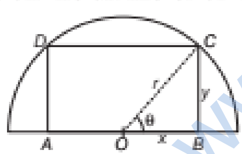
(a) $\frac{2\sqrt{2}}{3}R, \frac{4R}{3}$ (b) $\frac{4R}{3}, \frac{2\sqrt{2}}{3}R$

(c) $\frac{\sqrt{2}}{3}R, \frac{2R}{3}$ (d) $\frac{2R}{3}, \frac{\sqrt{2}}{3}R$

(v) The ratio of volume of cone and volume of sphere, when volume of cone is maximum, is

(a) $\frac{2}{27}$ (b) $\frac{8}{27}$ (c) $\frac{2}{8}$ (d) $\frac{1}{27}$

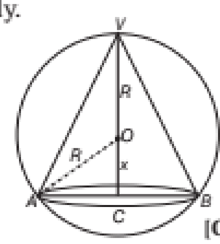
**48.** A building has a gate in the form of a rectangle is inscribed in a semi-circle of radius $r$ with one of its sides on the diameter of the semi-circle.




[CBSE Question Bank]

*On the basis of above information, answer the following questions.*

The height and radius of cone are $h$ and $r$ respectively.




[CBSE Question Bank]

*Based on above information, answer the following questions*

(i) The relation between $r$ and $R$ in terms of $x$ is

(a) $r=\sqrt{R^2+x^2}$ (b) $r=\sqrt{R^2-x^2}$

(c) $r=R+x$ (d) $r=R-x$

(ii) The volume $v$ of the cone expressed as a function of $x$ is

(a) $\frac{1}{3}\pi(R+x)(R-x)^2$ (b) $\frac{1}{3}\pi(R+x)^2(R-x)$

(c) $\frac{1}{3}\pi(R+x)^3$ (d) $\frac{1}{3}\pi(R-x)^3$

(i) The values of $x$ and $y$ in terms of $\theta$ and $r$ are

(a) $(r\sin\theta, r\cos\theta)$ (b) $(r\cos\theta, r\sin\theta)$

(c) $(r\sec\theta, r\tan\theta)$ (d) $(r\tan\theta, r\sec\theta)$

(ii) If area of rectangle $A$ is expressed in terms of $\theta$ and $r$, then $A=$

(a) $r^2\sin 2\theta$ (b) $r^2\cos 2\theta$

(c) $\frac{1}{2}r^2\sin 2\theta$ (d) $\frac{1}{2}r^2\cos 2\theta$

(iii) When $A$ is maximum, the value of $\theta$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$

(iv) For maximum value of $A$, the dimensions of rectangle are

(a) $(\sqrt{2}r, \sqrt{2}r)$ (b) $\left(\frac{1}{\sqrt{2}}r, \frac{1}{\sqrt{2}}r\right)$

(c) $\left(\sqrt{2}r, \frac{1}{\sqrt{2}}r\right)$ (d) $\left(\frac{1}{\sqrt{2}}r, \sqrt{2}r\right)$

(v) Maximum value of $A$ is

(a) $2r^2$ (b) $r^2$

(c) $\frac{1}{2}r^2$ (d) $\frac{1}{4}r^2$

# ANSWERS

**1.** (d) **2.** (b) **3.** (b) **4.** (c) **5.** (b) **6.** (d)

**7.** (c) **8.** (c) **9.** (b) **10.** (c) **11.** 60 $cm^2$ **12.** 0.8 cm/s

**13.** $1.8\pi$ cm/s **14.** ₹66 **20.** $\pi$ $cm^2$ **21.** $80\pi$ $cm^2/s$ **22.** 900 $cm^3/s$ **23.** $\frac{4}{3}$ $cm^2/s$

**24.** 6 $cm^3/s$ **25.** $\frac{1}{2\pi}$ cm/s **26.** $\left(3, \frac{16}{3}\right)$ and $\left(-3, \frac{-16}{3}\right)$ **28.** $k\in\left(0, \frac{1}{3}\right)$ **29.** $20\sqrt{3}$ $cm^2/s$

**30.** $\frac{32}{27}$ cm/s **31.** $2x^2-3x+1$ **32.** (i) $\left[0, \frac{\pi}{6}\right]$ (ii) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ **33.** $\left(\frac{\pi}{2}, \pi\right)$

**34.** Strictly increasing in $(-\infty, 2) \cup (3, \infty)$ and strictly decreasing in $(2, 3)$.

**35.** Increasing in $[0, \infty)$ and decreasing in $(-\infty, -1) \cup (-1, 0)$. **36.** Increasing in $(-\infty, -1]$ and decreasing in $[-1, \infty)$.

**37.** ₹ 49 **38.** 12, 12 **39.** The company should increase the subscription fee by ₹ 100.

**40.** (2, 2) **43.** $\sqrt{5}$ units **44.** $9\sqrt{3}$ sq units **45.** $p$ is maximum, when $x = \frac{600}{13}$.

**46.** (i) → (d), (ii) → (b), (iii) → (b), (iv) → (a), (v) → (b) **47.** (i) → (b), (ii) → (b), (iii) → (a), (iv) → (a), (v) → (b)

**48.** (i) → (b), (ii) → (a), (iii) → (b), (iv) → (c), (v) → (b)