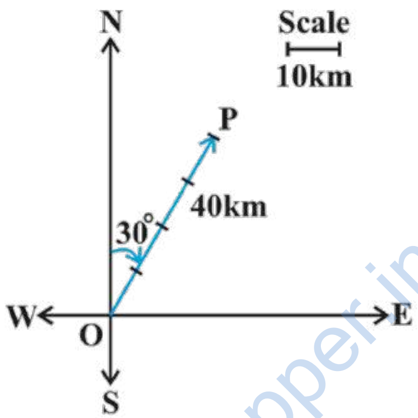
Chapter 10 Vector Algebra

EXERCISE 10.1

Question 1:
Represent graphically a displacement of 40km, 30° east of north.

Solution:



$\overrightarrow{OP}$ represents the displacement of 40km, 30° north-east.

Question 2:
Classify the following measures as scalars and vectors.

(i) 10 kg (ii) 2 meters north-east (iii) 40°
(iv) 40 watt (v) 10^{-19} coulomb (vi) $20m/s^2$

Solution:

(i) 10kg is a scalar.
(ii) 2 meters north-west is a vector.
(iii) 40° is a scalar.
(iv) 40 watts is a scalar.
(v) 10^{-19} Coulomb is a scalar.
(vi) $20m/s^2$ is a vector

Question 3:
Classify the following as scalar and vector quantities.

(i) time period (ii) distance (iii) force
(iv) velocity (v) work done.

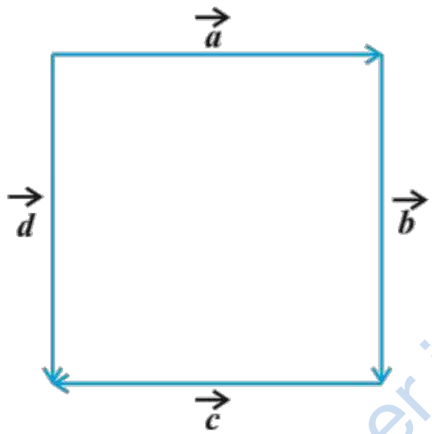
Solution:

(i) Time period is a scalar.
(ii) Distance is a scalar.

(iii) Force is a vector.
(iv) Velocity is a vector.
(v) Work done is a scalar.

Question 4:

In figure, identify the following vectors.



(i) Coinitial (ii) Equal (iii) Collinear but not equal.

Solution:

(i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial.
(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal.
(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal.

Question 5:

Answer the following as true or false.

(i) $\vec{a}$ and $-\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude.
(iii) Two vectors having same magnitude are collinear.
(iv) Two collinear vectors having the same magnitude are equal.

Solution:

(i) True.
(ii) False.
(iii) False.
(iv) False

EXERCISE 10.2

Question 1:
Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \qquad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Solution:

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2} = \sqrt{4 + 49 + 9} = \sqrt{62}$$

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

Question 2:
Write two different vectors having same magnitude.

Solution:

Let $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

But $\vec{a} \neq \vec{b}$

Question 3:
Write two different vectors having same direction.

Solution:

Let $\vec{p} = (\vec{i} + \vec{j} + \vec{k})$ and $\vec{q} = (2\vec{i} + 2\vec{j} + 2\vec{k})$

The DCs of $\vec{p}$ are

$$l=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$
$$m=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$
$$n=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

The DCs of $\vec{q}$ are

$$l=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}$$
$$m=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}$$
$$n=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}$$

But $\vec{p}\neq\vec{q}$

Question 4:

Find the values of x and y so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal.

Solution:

It is given that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal.

Therefore,

$$2\hat{i}+3\hat{j}=x\hat{i}+y\hat{j}$$

On comparing the components of both sides

$$\Rightarrow x=2$$
$$\Rightarrow y=3$$

Question 5:

Find the scalar and vector components of the vector with initial point $(2,1)$ and terminal point $(-5,7)$.

Solution:

Let the points be $P(2,1)$ and $Q(-5,7)$

$$\overrightarrow{PQ}=(-5-2)\hat{i}+(7-1)\hat{j}$$
$$=-7\hat{i}+6\hat{j}$$

So, the scalar components are -7 and 6, and
the vector components are $-7\hat{i}$ and $6\hat{j}$.

Question 6:

Find the sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=-2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}+7\hat{k}$

Solution:

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=-2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}+7\hat{k}$.

Therefore,
$$\vec{a}+\vec{b}+\vec{c}=(1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}$$
$$=0\hat{i}-4\hat{j}-\hat{k}$$
$$=-4\hat{j}-\hat{k}$$

Question 7:

Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$.

Solution:

We have $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$

Hence,
$$|\vec{a}|=\sqrt{1^2+1^2+2^2}$$
$$=\sqrt{1+1+4}$$
$$=\sqrt{6}$$

Therefore,
$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$$
$$=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

Question 8:

Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points $(1,2,3)$ and $(4,5,6)$ respectively.

Solution:

We have the given points $P(1,2,3)$ and $Q(4,5,6)$

Hence,

$$\begin{aligned}\overrightarrow{PQ} &= (4-1)\hat{i}+(5-2)\hat{j}+(6-3)\hat{k}\\ &= 3\hat{i}+3\hat{j}+3\hat{k}\\ \left|\overrightarrow{PQ}\right| &= \sqrt{3^2+3^2+3^2}\\ &= \sqrt{9+9+9}\\ &= \sqrt{27}\\ &= 3\sqrt{3}\end{aligned}$$

So, unit vector is

$$\begin{aligned}\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} &= \frac{3\hat{i}+3\hat{j}+3\hat{k}}{3\sqrt{3}}\\ &= \frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\end{aligned}$$

Question 9:

For given vectors, $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$, find the unit vector in the direction of the vector $\vec{a}+\vec{b}$.

Solution:

The given vectors are $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$

Therefore,

$$\begin{aligned}\vec{a}+\vec{b} &= (2-1)\hat{i}+(-1+1)\hat{j}+(2-1)\hat{k}\\ &= 1\hat{i}+0\hat{j}+1\hat{k}\\ &= \hat{i}+\hat{k}\\ \left|\vec{a}+\vec{b}\right| &= \sqrt{1^2+1^2} = \sqrt{2}\end{aligned}$$

Thus, unit vector is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{\hat{i}+\hat{k}}{\sqrt{2}}$$
$$=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$$

Question 10:

Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units.

Solution:

Let $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$

Hence,

$$|\vec{a}|=\sqrt{5^2+(-1)^2+(2)^2}$$
$$=\sqrt{25+1+4}$$
$$=\sqrt{30}$$

Therefore,

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}$$

Thus, a vector parallel to $5\hat{i}-\hat{j}+2\hat{k}$ with magnitude 8 units is

$$8\hat{a}=8\left(\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}\right)$$
$$=\frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}$$

Question 11:

Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

Solution:

We have $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}$

Now,

$$\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}$$
$$=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)$$
$$=-2\vec{a}$$

Since, $\vec{b} = \lambda \vec{a}$

Therefore, $\lambda = -2$

So, the vectors are collinear.

Question 12:

Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

Solution:

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

Therefore,

$$\begin{aligned} |\vec{a}| &= \sqrt{1^2 + 2^2 + 3^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned}$$

Thus, the DCs of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Question 13:

Find the direction of the cosines of the vectors joining the points $A(1,2,-3)$ and $B(-1,-2,1)$ directions from A to B.

Solution:

The given points are $A(1,2,-3)$ and $B(-1,-2,1)$.

Therefore,

$$\begin{aligned} \overrightarrow{AB} &= (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k} \\ &= -2\hat{i} - 4\hat{j} + 4\hat{k} \\ |\overrightarrow{AB}| &= \sqrt{(-2)^2 + (-4)^2 + 4^2} \\ &= \sqrt{4+16+16} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Thus, the DCs of $\overrightarrow{AB}$ are $\left(-\frac{2}{6},-\frac{4}{6},\frac{4}{6}\right)=\left(-\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$

Question 14:

Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axis OX, OY and OZ.

Solution:

Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$

Therefore,

$$\begin{aligned}|\vec{a}|&=\sqrt{1^2+1^2+1^2}\\&=\sqrt{3}\end{aligned}$$

Thus, the DCs of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Now, let α,β and γ be the angles formed by $\vec{a}$ with the positive directions of x,y and z axes respectively

Then,

$$\cos\alpha=\frac{1}{\sqrt{3}},\cos\beta=\frac{1}{\sqrt{3}},\cos\gamma=\frac{1}{\sqrt{3}}$$

Hence, the vector is equally inclined to OX, OY and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively, in the ratio $2:1$.

(i) Internally

(ii) Externally

Solution:

Position vectors of P and Q are given as:

$$\overrightarrow{OP}=\hat{i}+2\hat{j}-\hat{k} \text{ and } \overrightarrow{OQ}=-\hat{i}+\hat{j}+\hat{k}$$

(i) The position vector of R which divides the line joining two points P and Q internally in the ratio $2:1$ is

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}$$

$$=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$

$$=\frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$$

$$=-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$$

(ii) The position vector of R which divides the line joining two points P and Q externally in the ratio $2:1$ is

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}$$

$$=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)}{1}$$

$$=-3\hat{i}+3\hat{k}$$

Question 16:

Find the position vector of the mid-point of the vector joining the points $P(2,3,4)$ and $Q(4,1,-2)$.

Solution:

The position vector of the mid-point R is

$$\overrightarrow{OR}=\frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2}$$

$$=\frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$

$$=\frac{6\hat{i}+4\hat{j}+2\hat{k}}{2}$$

$$=3\hat{i}+2\hat{j}+\hat{k}$$

Question 17:

Show that the points A,B and C with position vectors, $\vec{a}=3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}-5\hat{k}$, respectively form the vertices of a right angled triangle.

Solution:

Position vectors of points A, B, and C are respectively given as:

$$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$$

Therefore,

$$\overrightarrow{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k}$$
$$= -\hat{i} + 3\hat{j} + 5\hat{k}$$
$$\overrightarrow{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k}$$
$$= -\hat{i} + 2\hat{j} - 6\hat{k}$$
$$\overrightarrow{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k}$$
$$= 2\hat{i} - \hat{j} + \hat{k}$$

Now,

$$\left|\overrightarrow{AB}\right|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$
$$\left|\overrightarrow{BC}\right|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$
$$\left|\overrightarrow{CA}\right|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

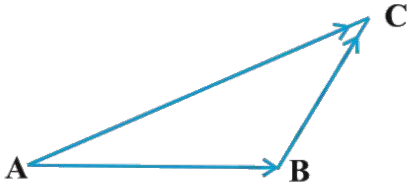
Also,

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 = 35 + 6$$
$$= 41$$
$$= \left|\overrightarrow{BC}\right|^2$$

Thus, ABC is a right-angled triangle.

Question 18:

In triangle ABC which of the following is not true.



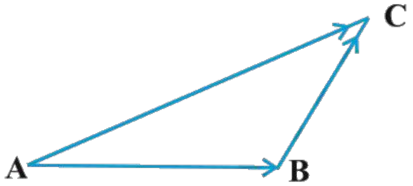
(A) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

(B) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = 0$

(C) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = 0$

(D) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = 0$

Solution:



On applying the triangle law of addition in the given triangle, we have:

$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC} \qquad \ldots(1)$$
$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$$
$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0 \qquad \ldots(2)$$

Hence, the equation given in option A is true.

Now, from equation (2)

$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$$
$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=0$$

Hence, the equation given in option B is true.

Also,

$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$$
$$\Rightarrow \overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=0$$

Hence, the equation given in option D is true

Now, consider the equation given in option C,

$$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=0$$
$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA} \qquad \ldots(3)$$

From equations (1) and (2)

$$\Rightarrow \overrightarrow{AC}=\overrightarrow{CA}$$
$$\Rightarrow \overrightarrow{AC}=-\overrightarrow{AC}$$
$$\Rightarrow \overrightarrow{AC}+\overrightarrow{AC}=0$$
$$\Rightarrow 2\overrightarrow{AC}=0$$
$$\Rightarrow \overrightarrow{AC}=0$$

Which is not true. So, the equation given in option C is incorrect.

Thus, the correct option is C.

Question 19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect?

(A) $\vec{b} = \lambda \vec{a}$, for some scalar λ

(B) $\vec{a} = \pm \vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

If $\vec{a}$ and $\vec{b}$ are collinear vectors, they are parallel.

Therefore, for some scalar λ

$$\vec{b} = \lambda \vec{a}$$

If $\lambda = \pm 1$, then $\vec{a} = \pm \vec{b}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then,

$$\Rightarrow \vec{b} = \lambda \vec{a}$$
$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$
$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

Comparing the components of both the sides

$$\Rightarrow \vec{b_1} = \lambda a_1$$
$$\Rightarrow \vec{b_2} = \lambda a_2$$
$$\Rightarrow \vec{b_3} = \lambda a_3$$

Therefore,

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

However, $\vec{a}$ and $\vec{b}$ may have different directions.

Hence, that statement given in D is incorrect.

Thus, the correct option is D.

EXERCISE 10.3

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2, respectively have $\vec{a}.\vec{b}=\sqrt{6}$

Solution:

It is given that

$$|\vec{a}|=\sqrt{3}$$
$$|\vec{b}|=2$$
$$\vec{a}.\vec{b}=\sqrt{6}$$

Therefore,

$$\Rightarrow \sqrt{6}=\sqrt{3}\times 2\cos\theta$$
$$\Rightarrow \cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$
$$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$$
$$\Rightarrow \theta=\frac{\pi}{4}$$

Question 2:

Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

Solution:

Let $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$.

Hence,

$$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$
$$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$
$$\vec{a}.\vec{b}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}-2\hat{j}+\hat{k}\right)$$
$$=1\times 3+(-2)(-2)+3\times 1$$
$$=3+4+3$$
$$=10$$

Therefore,

$$\Rightarrow 10 = \sqrt{14}\sqrt{14}\cos\theta$$
$$\Rightarrow \cos\theta = \frac{10}{14}$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

Solution:

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{1}{\sqrt{1+1}}\{(1)(1)+(-1)1\}$$
$$= \frac{1}{\sqrt{2}}(1-1)$$
$$= 0$$

Question 4:

Find the projection of vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

Solution:

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{(1)(7)+3(-1)+7(8)\}$$
$$= \frac{1}{\sqrt{49+1+64}}(7-3+56)$$
$$= \frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector which are manually perpendicular to each other.

$\frac{1}{7}\left(\hat{2i}+\hat{3j}+\hat{6k}\right), \frac{1}{7}\left(\hat{3i}-\hat{6j}+\hat{2k}\right), \frac{1}{7}\left(\hat{6i}+\hat{2j}-\hat{3k}\right)$

Solution:

Let

$$\vec{a}=\frac{1}{7}\left(\hat{2i}+\hat{3j}+\hat{6k}\right)=\frac{2}{7}i+\frac{3}{7}j+\frac{6}{7}k$$
$$\vec{b}=\frac{1}{7}\left(\hat{3i}-\hat{6j}+\hat{2k}\right)=\frac{3}{7}i-\frac{6}{7}j+\frac{2}{7}k$$
$$\vec{c}=\frac{1}{7}\left(\hat{6i}+\hat{2j}-\hat{3k}\right)=\frac{6}{7}i+\frac{2}{7}j-\frac{3}{7}k$$

Now,

$$\left|\vec{a}\right|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$
$$\left|\vec{b}\right|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$
$$\left|\vec{c}\right|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

So, each of the vector is a unit vector.

Hence,

$$\vec{a}.\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(-\frac{6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$
$$\vec{b}.\vec{c}=\frac{3}{7}\times\frac{6}{7}+\frac{2}{7}\times\left(-\frac{6}{7}\right)+\left(-\frac{3}{7}\right)\times\frac{2}{7}=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$
$$\vec{c}.\vec{a}=\frac{2}{7}\times\frac{6}{7}+\frac{3}{7}\times\frac{2}{7}+\frac{6}{7}\times\left(-\frac{3}{7}\right)=\frac{12}{49}+\frac{6}{49}+\frac{18}{49}=0$$

So, the vectors are mutually perpendicular to each other.

Question 6:

Find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$, if $\left(\vec{a}+\vec{b}\right)\left(\vec{a}-\vec{b}\right)=8$ and $\left|\vec{a}\right|=8\left|\vec{b}\right|$

Solution:

It is given that $\left(\vec{a}+\vec{b}\right)\left(\vec{a}-\vec{b}\right)=8$ and $\left|\vec{a}\right|=8\left|\vec{b}\right|$

Therefore,

$$\Rightarrow (\vec{a}+\vec{b})(\vec{a}-\vec{b}) = 8$$
$$\Rightarrow \vec{a}.\vec{a} - \vec{a}\vec{b} + \vec{b}\vec{a} - \vec{b}\vec{b} = 8$$
$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$$
$$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8$$
$$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$$
$$\Rightarrow 63|\vec{b}|^2 = 9$$
$$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$$
$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}}$$
$$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

Now,

$$|\vec{a}| = 8|\vec{b}|$$
$$= \frac{8\times 2\sqrt{2}}{3\sqrt{7}}$$
$$= \frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $(3\vec{a}-5\vec{b})(2\vec{a}+7\vec{b})$

Solution:

$$(3\vec{a}-5\vec{b})(2\vec{a}+7\vec{b}) = 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$$
$$= 6\vec{a}\vec{a} + 21\vec{a}\vec{b} - 10\vec{a}\vec{b} - 35\vec{b}\vec{b}$$
$$= 6|\vec{a}|^2 + 11\vec{a}\vec{b} - 35|\vec{b}|^2$$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that angle between them is $60°$ and their scalar product is $\frac{1}{2}$

Solution:

Let θ be the angle between $\vec{a}$ and $\vec{b}$

It is given that $|\vec{a}| = |\vec{b}|$, $\vec{a}.\vec{b} = \frac{1}{2}$ and $\theta = 60°$

Therefore,

$$\Rightarrow \frac{1}{2} = |\vec{a}||\vec{b}|\cos 60°$$
$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$
$$\Rightarrow |\vec{a}|^2 = 1$$
$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 12$

Solution:

$$\Rightarrow (\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 12$$
$$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}\vec{x} - \vec{a}\vec{a} = 12$$
$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$
$$\Rightarrow |\vec{x}|^2 - 1 = 12$$
$$\Rightarrow |\vec{x}|^2 = 13$$
$$\Rightarrow |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Solution:

We have $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$

Then,

$$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right)$$
$$= (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

Now,

$$\Rightarrow \left(\vec{a}+\lambda\vec{b}\right).\vec{c}=0$$
$$\Rightarrow \left[(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right].\left(3\hat{i}+\hat{j}\right)=0$$
$$\Rightarrow 3(2-\lambda)+(2+2\lambda)+0(3+\lambda)=0$$
$$\Rightarrow 6-3\lambda+2+2\lambda=0$$
$$\Rightarrow -\lambda+8=0$$
$$\Rightarrow \lambda=8$$

Question 11:

Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any non-zero vectors $\vec{a}$ and $\vec{b}$.

Solution:

$$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)=|\vec{a}|^2\,\vec{b}.\vec{b}-|\vec{a}||\vec{b}|\vec{b}.\vec{a}+|\vec{b}||\vec{a}|\vec{a}.\vec{b}-|\vec{b}|^2\,\vec{a}.\vec{a}$$
$$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$$
$$=0$$

Question 12:

If $\vec{a}\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded above the vector $\vec{b}$?

Solution:

We have $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

Hence,

$$\Rightarrow |\vec{a}|^2=0$$
$$\Rightarrow |\vec{a}|=0$$

Therefore, $\vec{a}$ is the zero vector

Thus, any vector $\vec{b}$ can satisfy $\vec{a}.\vec{b}=0$

Question 13:

If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

Solution:

We have $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$

Therefore,

$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$$

$$0=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$

$$0=1+1+1+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$

$$\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\frac{-3}{2}$$

Question 14:

If either vector $\vec{a}=0$ or $\vec{b}=0$, then $\vec{a}.\vec{b}=0$. But the converse need not be true. Justify the answer with an example.

Solution:

Let $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$

Therefore,

$$\begin{aligned}\vec{a}.\vec{b}&=2(3)+4(3)+3(-6)\\&=6+12-18\\&=0\end{aligned}$$

Now,

$$\left|\vec{a}\right|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$$

$$\Rightarrow \vec{a}\neq 0$$

$$\left|\vec{b}\right|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$$

$$\Rightarrow \vec{b}\neq 0$$

So, the converse of the statement need not to be true.

Question 15:

If the vertices A, B, C of a triangle ABC are $(1,2,3),(-1,0,0),(0,1,2)$ respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Solution:

Vertices of the triangle are $A(1,2,3)$, $B(-1,0,0)$ and $C(0,1,2)$.

Hence,

$$\overrightarrow{BA} = \{1-(1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k}$$
$$= 2\hat{i} + 2\hat{j} + 3\hat{k}$$
$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k}$$
$$= \hat{i} + \hat{j} + 2\hat{k}$$
$$\overrightarrow{BA}.\overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right)\left(\hat{i} + \hat{j} + 2\hat{k}\right)$$
$$= 2\times1 + 2\times1 + 3\times2$$
$$= 2+2+6$$
$$= 10$$
$$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2}$$
$$= \sqrt{4+4+9}$$
$$= \sqrt{17}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2}$$
$$= \sqrt{6}$$
$$\overrightarrow{BA}.\overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\angle ABC)$$

Therefore,

$$\Rightarrow 10 = \sqrt{17} \times \sqrt{6}(\cos\angle ABC)$$
$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$
$$\Rightarrow (\angle ABC) = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Question 16:

Show that the points $A(1,2,7)$, $B(2,6,3)$ and $C(3,10-1)$ are collinear.

Solution:

The given points are $A(1,2,7)$, $B(2,6,3)$ and $C(3,10-1)$.

Hence,

$$\overrightarrow{AB}=(2-1)\hat{i}+(6-2)\hat{j}+(3-7)\hat{k}=\hat{i}+4\hat{j}-4\hat{k}$$
$$\overrightarrow{BC}=(3-2)\hat{i}+(10-6)\hat{j}+(-1-3)\hat{k}=\hat{i}+4\hat{j}-4\hat{k}$$
$$\overrightarrow{AC}=(3-1)\hat{i}+(10-2)\hat{j}+(-1-7)\hat{k}=2\hat{i}+8\hat{j}-8\hat{k}$$
$$\left|\overrightarrow{AB}\right|=\sqrt{1^2+4^2+(-4)^2}=\sqrt{1+16+16}=\sqrt{33}$$
$$\left|\overrightarrow{BC}\right|=\sqrt{1^2+4^2+(-4)^2}=\sqrt{1+16+16}=\sqrt{33}$$
$$\left|\overrightarrow{AC}\right|=\sqrt{2^2+8^2+8^2}=\sqrt{4+64+64}=2\sqrt{33}$$

Therefore,

$$\begin{aligned}\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|&=\sqrt{33}+\sqrt{33}\\&=2\sqrt{33}\\&=\left|\overrightarrow{AC}\right|\end{aligned}$$

Hence, the points are collinear.

Question 17:

Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Solution:

Let $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k}, \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

Hence,

$$\overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$$
$$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$$
$$\overrightarrow{AC}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$$
$$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$$
$$\left|\overrightarrow{BC}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$
$$\left|\overrightarrow{AC}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$$

Therefore,

$$\begin{aligned}\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AC}\right|^2&=6+35\\&=41\\&=\left|\overrightarrow{AB}\right|^2\end{aligned}$$

Thus, ΔABC is a right-angled triangle.

Question 18:

If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is a unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \dfrac{1}{|\lambda|}$

Solution:

$\Rightarrow |\lambda\vec{a}| = 1$

$\Rightarrow |\lambda||\vec{a}| = 1$

$\Rightarrow |\vec{a}| = \dfrac{1}{|\lambda|}$

$\Rightarrow a = \dfrac{1}{|\lambda|}$

Hence the correct option is D.

EXERCISE 10.4

Question 1:

Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

We have, $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$
Hence,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)$$

$$=19\hat{j}+19\hat{k}$$

Therefore,

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(19)^2+(19)^2}$$

$$=\sqrt{2\times(19)^2}$$

$$=19\sqrt{2}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Solution:

We have $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.
Hence,

$$\vec{a}+\vec{b}=4\hat{i}+4\hat{j}$$

$$\vec{a}-\vec{b}=2\hat{i}+4\hat{k}$$

Therefore,

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}$$
$$=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)$$
$$=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{16^2+(-16)^2+(-8)^2}=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$
$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}$$
$$=8\times3=24$$

So, the unit vector is

$$\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$
$$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}$$
$$=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

Question 3:

If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.

Solution:

Let the unit vector $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Then, $\left|\vec{a}\right|=1$

Now,

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|} \Rightarrow a_1 = \frac{1}{2}$$

$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|} \Rightarrow a_2 = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{a_3}{|\vec{a}|} \Rightarrow a_3 = \cos\theta$$

Therefore,

$$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Hence,

$$a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

So, $\theta = \frac{\pi}{3}$ and components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

Question 4:

Show that $\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$

Solution:

$$LHS = \left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$
$$= \left(\vec{a}-\vec{b}\right)\times\vec{a}+\left(\vec{a}-\vec{b}\right)\times\vec{b}$$
$$= \vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b}$$
$$= 0+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-0$$
$$= 2\vec{a}\times\vec{b}$$
$$= RHS$$

Question 5:

Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=0$

Solution:

We have $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=0$

Therefore,

$$\Rightarrow\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=0$$
$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$$
$$\Rightarrow\hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$$6\mu-27\lambda=0$$
$$2\mu-27=0$$
$$2\lambda-6=0$$

Now,

$$2\lambda-6=0\Rightarrow\lambda=3$$
$$2\mu-27=0\Rightarrow\mu=\frac{27}{2}$$

Question 6:

Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=0$. What can you conclude about $\vec{a}$ and $\vec{b}$?

Solution:

When $\vec{a}.\vec{b}=0$

Either $\left|\vec{a}\right| = 0$ or $\left|\vec{b}\right| = 0$

Or $\vec{a} \perp \vec{b}$ (if $\left|\vec{a}\right| \neq 0$ and $\left|\vec{b}\right| \neq 0$)

When $\vec{a} \times \vec{b} = 0$

Either $\left|\vec{a}\right| = 0$ or $\left|\vec{b}\right| = 0$

Or $\vec{a} \parallel \vec{b}$ (if $\left|\vec{a}\right| \neq 0$ and $\left|\vec{b}\right| \neq 0$)

Since, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

So, $\vec{a} = 0$ or $\vec{b} = 0$.

Question 7:

Let the vectors $\vec{a}, \vec{b}, \vec{c}$ given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Solution:

We have

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Then,

$$\left(\vec{b} + \vec{c}\right) = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

Now,

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} = \begin{pmatrix} \hat{i}\left[a_2(b_3 + c_3) - a_3(b_2 + c_2)\right] - \hat{j}\left[a_1(b_3 + c_3) - a_3(b_1 + c_1)\right] \\ + \hat{k}\left[a_1(b_2 + c_2) - a_2(b_1 + c_1)\right] \end{pmatrix}$$

$$= \begin{pmatrix} \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1\right] \\ + \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right] \end{pmatrix}$$

Also,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$
$$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[a_3b_1-a_1b_3\right]+\hat{k}\left[a_2b_2-a_2b_1\right]$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$
$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_2c_2-a_2c_1\right]$$

Therefore,

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\begin{pmatrix}\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[a_3b_1-a_1b_3\right]+\hat{k}\left[a_2b_2-a_2b_1\right]\\ +\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_2c_2-a_2c_1\right]\end{pmatrix}$$
$$=\begin{pmatrix}\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]\\ +\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]\end{pmatrix}$$

Thus,

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence proved.

Question 8:

If either $\vec{a}=0$ or $\vec{b}=0$, then $\vec{a}\times\vec{b}=0$. Is the converse true? Justify your answer with an example.

Solution:

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$

Therefore,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}$$
$$=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)$$
$$=0$$

Now,

$$\left|\vec{a}\right|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

Thus,

$$\vec{a} \neq 0$$

Also,

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

Thus,

$$\vec{b} \neq 0$$

Hence, converse of the statement need not to be true.

Question 9:

Find the area of triangle with vertices $A(1,1,2)$ $B(2,3,5)$ and $C(1,5,5)$.

Solution:

Vertices of the triangle are $A(1,1,2)$ $B(2,3,5)$ and $C(1,5,5)$

Hence,

$$\begin{aligned}\overrightarrow{AB} &= (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} \\ &= \hat{i} + 2\hat{j} + 3\hat{k} \\ \overrightarrow{BC} &= (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} \\ &= -\hat{i} + 2\hat{j}\end{aligned}$$

Therefore,

Area of the triangle $ar(\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

Now,

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{BC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} \\ &= \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) \\ &= -6\hat{i} - 3\hat{j} + 4\hat{k} \\ \left|\overrightarrow{AB} \times \overrightarrow{BC}\right| &= \sqrt{(-6)^2 + (-3)^2 + 4^2} \\ &= \sqrt{36 + 9 + 16} \\ &= \sqrt{61}\end{aligned}$$

Therefore,

$$ar(\Delta ABC) = \frac{1}{2}\sqrt{61}$$
$$= \frac{\sqrt{61}}{2}$$

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution:

We have $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

Hence,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$
$$= \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2)$$
$$= 20\hat{i} + 5\hat{j} - 5\hat{k}$$
$$\left|\vec{a} \times \vec{b}\right| = \sqrt{20^2 + 5^2 + 5^2}$$
$$= \sqrt{400 + 25 + 25}$$
$$= 15\sqrt{2}$$

Thus, the area of parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $\left|\vec{a}\right| = 3$ and $\left|\vec{b}\right| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

We have $\left|\vec{a}\right| = 3$, $\left|\vec{b}\right| = \frac{\sqrt{2}}{3}$ and $\left|\vec{a} \times \vec{b}\right| = 1$

Therefore,

$$\Rightarrow \left| |\vec{a}||\vec{b}|\sin\theta \right| = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Question 12:

Area of the rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution:

We have vertices $A\left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$, $B\left(\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$, $C\left(\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$ and $D\left(-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$.

Therefore,

$$\overrightarrow{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overrightarrow{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

Now,

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \hat{k}(-2)$$

$$= -2\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-2)^2}$$

$$= 2$$

So, area of the rectangle is 2 square units.

MISCELLANEOUS EXERCISE

Question 1:

Write down a unit vector in XY-plane, making an angle of 30° with the positive direction x-axis.

Solution:

Unit vector is $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$, where θ is angle with positive x-axis.
Therefore,

$$\begin{aligned}\vec{r} &= \cos 30^\circ\hat{i} + \sin 30^\circ\hat{j}\\ &= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\end{aligned}$$

Question 2:

Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Solution:

We have $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Therefore,

$$\begin{aligned}\overrightarrow{PQ} &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\\ \left|\overrightarrow{PQ}\right| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\end{aligned}$$

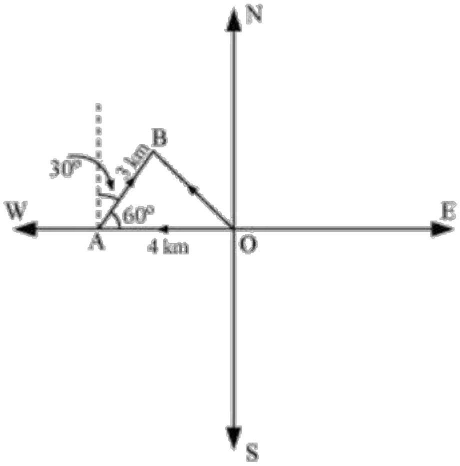
Hence, the scalar components of the vector is $\{(x_2 - x_1) + (y_2 - y_1) + (z_2 - z_1)\}$ and magnitude of the vector is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Question 3:

A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution:

Let O and B be the initial and final positions of the girl, respectively.
Then, the girl's position can be shown by the below diagram:



We have:

$$\overrightarrow{OA} = -4\hat{i}$$
$$\overrightarrow{AB} = \hat{i}\left|\overrightarrow{AB}\right|\cos 60° + \hat{j}\left|\overrightarrow{AB}\right|\sin 60°$$
$$= \hat{i}3\times\frac{1}{2} + \hat{j}3\times\frac{\sqrt{3}}{2}$$
$$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

By the triangle law of addition for vector,

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$
$$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$$
$$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$
$$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$
$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

Hence, the girl's displacement from her initial point of departure is $\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$.

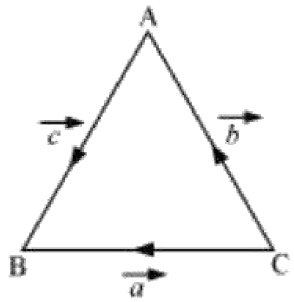
Question 4:

If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Solution:

In ΔABC

$\overrightarrow{CB} = \vec{a}, \overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$



By triangle law of addition for vectors

$$\vec{a} = \vec{b} + \vec{c}$$

By triangle inequality law of lengths

$$|\vec{a}| < |\vec{b}| + |\vec{c}|$$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$

Question 5:

Find the value of x for which $x\left(\hat{i} + \hat{j} + \hat{k}\right)$ is a unit vector.

Solution:

We have a unit vector $x\left(\hat{i} + \hat{j} + \hat{k}\right)$
Therefore,

$$\Rightarrow \left|x\left(\hat{i} + \hat{j} + \hat{k}\right)\right| = 1$$

$$\Rightarrow \sqrt{x^2 + x^2 + x^2} = 1$$

$$\Rightarrow \sqrt{3x^2} = 1$$

$$\Rightarrow \sqrt{3}x = 1$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

Question 6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

Solution:

We have $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

Hence,

$$\begin{aligned}\vec{c}&=\vec{a}+\vec{b}\\&=(2+1)\hat{i}+(3-2)\hat{j}+(-1+1)\hat{k}\\&=3\hat{i}+\hat{j}\\|\vec{c}|&=\sqrt{3^2+1^2}\\&=\sqrt{9+1}\\&=\sqrt{10}\end{aligned}$$

Therefore,

$$\hat{c}=\frac{\vec{c}}{|\vec{c}|}=\frac{(3\hat{i}+\hat{j})}{\sqrt{10}}$$

So, a vector of magnitude 5 and parallel to the resultant of $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned}\pm(\hat{c})&=\pm5\left(\frac{1}{\sqrt{10}}(3\hat{i}+\hat{j})\right)\\&=\pm\frac{3\sqrt{10}}{2}\hat{i}\pm\frac{\sqrt{10}}{2}\hat{j}\end{aligned}$$

Question 7:

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$ find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Solution:

We have $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$

Therefore,

$$\begin{aligned}2\vec{a}-\vec{b}+3\vec{c} &= 2(\hat{i}+\hat{j}+\hat{k})-(2\hat{i}-\hat{j}+3\hat{k})+3(\hat{i}-2\hat{j}+\hat{k})\\ &= 2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}\\ &= 3\hat{i}-3\hat{j}+2\hat{k}\end{aligned}$$

$$\begin{aligned}\left|2\vec{a}-\vec{b}+3\vec{c}\right| &= \sqrt{3^2+(-3)^2+2^2}\\ &= \sqrt{9+9+4}\\ &= \sqrt{22}\end{aligned}$$

So, the required unit vector is

$$\begin{aligned}\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|} &= \frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}\\ &= \frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}\end{aligned}$$

Question 8:

Show that the points $A(1,-2,-8), B(5,0,-2)$ and $C(11,3,7)$ are collinear and find the ratio in which B divides AC.

Solution:

We have points $A(1,-2,-8), B(5,0,-2)$ and $C(11,3,7)$

Therefore,

$$\begin{aligned}\overrightarrow{AB} &= (5-1)\hat{i}+(0+2)\hat{j}+(-2+8)\hat{k} = 4\hat{i}+2\hat{j}+6\hat{k}\\ \overrightarrow{BC} &= (11-5)\hat{i}+(3-0)\hat{j}+(7+2)\hat{k} = 6\hat{i}+3\hat{j}+9\hat{k}\\ \overrightarrow{AC} &= (11-1)\hat{i}+(3+2)\hat{j}+(7+8)\hat{k} = 10\hat{i}+5\hat{j}+15\hat{k}\\ \left|\overrightarrow{AB}\right| &= \sqrt{4^2+2^2+6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}\\ \left|\overrightarrow{BC}\right| &= \sqrt{6^2+3^2+9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}\\ \left|\overrightarrow{AC}\right| &= \sqrt{10^2+5^2+15^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}\end{aligned}$$

Now,

$$\begin{aligned}\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right| &= 2\sqrt{14}+3\sqrt{14}\\ &= 5\sqrt{14}\\ &= \left|\overrightarrow{AC}\right|\end{aligned}$$

Thus, the points are collinear.

Let B divides AC in the ratio $\lambda:1$

Therefore,

$$\overrightarrow{OB}=\frac{\lambda\overrightarrow{OC}+\overrightarrow{OA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i}-2\hat{k}=\frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow(\lambda+1)\left(5\hat{i}-2\hat{k}\right)=11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k}=(11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

On equating the corresponding components, we get

$$\Rightarrow 5(\lambda+1)=(11\lambda+1)$$

$$\Rightarrow 5\lambda+5=11\lambda+1$$

$$\Rightarrow 6\lambda=4$$

$$\Rightarrow \lambda=\frac{2}{3}$$

Thus, the ratio is $2:3$.

Question 9:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\left(2\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-3\vec{b}\right)$ externally in the ratio $1:2$. Also show that P is midpoint of the line segment RQ.

Solution:

We have $\overrightarrow{OP}=2\vec{a}+\vec{b},\overrightarrow{OQ}=\vec{a}-3\vec{b}$

It is given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2.

Then, on using the section formula, we get:

$$\overrightarrow{OR}=\frac{2\left(2\vec{a}+\vec{b}\right)-\left(\vec{a}-2\vec{b}\right)}{2-1}$$

$$=\frac{4\vec{a}+2\vec{b}-\vec{a}-3\vec{b}}{1}$$

$$=3\vec{a}+5\vec{b}$$

Hence, the position vector of R is $3\vec{a}+5\vec{b}$

Thus, the position vector of midpoint of $RQ = \dfrac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$\begin{aligned}\frac{\overrightarrow{OQ}+\overrightarrow{OR}}{2} &= \frac{\left(\vec{a}-3\vec{b}\right)+\left(3\vec{a}+5\vec{b}\right)}{2} \\ &= 2\vec{a}+\vec{b} \\ &= \overrightarrow{OP}\end{aligned}$$

Thus, P is the midpoint of line segment RQ.

Question 10:

The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

Diagonal of a parallelogram is $\vec{a}+\vec{b}$

$$\begin{aligned}\vec{a}+\vec{b} &= (2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k} \\ &= 3\hat{i}-6\hat{j}+2\hat{k}\end{aligned}$$

So, the unit vector parallel to the diagonal is

$$\begin{aligned}\frac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|} &= \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}} \\ &= \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}} \\ &= \frac{3\hat{i}-6\hat{j}+2\hat{k}}{7} \\ &= \frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right)\end{aligned}$$

Area of the parallelogram is $\left|\vec{a}+\vec{b}\right|$

Now,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}$$

$$=\hat{i}(12+10)-\hat{j}(-6-5)+\hat{k}(-4+4)$$

$$=22\hat{i}+11\hat{j}$$

$$=11\left(2\hat{i}+\hat{j}\right)$$

$$\left|\vec{a}+\vec{b}\right|=11\sqrt{2^2+1^2}=11\sqrt{5}$$

So, area of parallelogram is $11\sqrt{5}$ square units.

Question 11:

Show that the direction cosines of a vector equally inclined to the axis OX, OY and OZ are $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Solution:

Let a vector be equally inclined to OX, OY and OZ at an angle α

So, the DCs of the vectors are $\cos\alpha$, $\cos\alpha$ and $\cos\alpha$.

Therefore,

$$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$

$$\Rightarrow 3\cos^2\alpha=1$$

$$\Rightarrow \cos^2\alpha=\frac{1}{3}$$

$$\Rightarrow \cos\alpha=\pm\frac{1}{\sqrt{3}}$$

Thus, the DCs of the vector are $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Question 12:

Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d}=15$.

Solution:

Let $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$

Since, $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have

$$\vec{d}.\vec{a}=0$$
$$\Rightarrow d_1+4d_2+2d_3=0 \qquad \ldots(1)$$

And

$$\vec{d}.\vec{b}=0$$
$$\Rightarrow 3d_1-2d_2+7d_3=0 \qquad \ldots(2)$$

Also, it is given that

$$\vec{c}.\vec{d}=15$$
$$\Rightarrow 2d_1-d_2+4d_3=15 \qquad \ldots(3)$$

On solving equations $(1),(2)$ and (3), we get

$$d_1=\frac{160}{3}, d_2=-\frac{5}{3}, d_3=-\frac{70}{3}$$

Therefore,

$$\vec{d}=\frac{160}{3}\hat{i}-\frac{5}{3}\hat{j}-\frac{70}{3}\hat{k}$$
$$=\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$$

Question 13:

The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

Solution:

$$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Therefore, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of $\hat{i}+\hat{j}+\hat{k}$ with this unit vector is 1.

$$\Rightarrow\left(\hat{i}+\hat{j}+\hat{k}\right).\left(\frac{(2+\lambda)\hat{i}+\hat{6j}-\hat{2k}}{\sqrt{\lambda^2+4\lambda+44}}\right)=1$$

$$\Rightarrow\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow\sqrt{\lambda^2+4\lambda+44}=\lambda+6$$

$$\Rightarrow\lambda^2+4\lambda+44=(\lambda+6)^2$$

$$\Rightarrow\lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$

$$\Rightarrow 8\lambda=8$$

$$\Rightarrow\lambda=1$$

Question 14:

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is inclined to $\vec{a},\vec{b},\vec{c}$.

Solution:

Since, $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes
Therefore,

$$\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$$

And

$$\left|\vec{a}\right|=\left|\vec{b}\right|=\left|\vec{c}\right|$$

Let $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b},\vec{c}$ at angles $\theta_1,\theta_2,\theta_3$ respectively.

$$\cos\theta_1=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right).\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|}=\frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|}=\frac{\left|\vec{a}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{a}\right|}=\frac{\left|\vec{a}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

$$\cos\theta_2=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right).\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|}=\frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|}=\frac{\left|\vec{b}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{b}\right|}=\frac{\left|\vec{b}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

$$\cos\theta_3=\frac{\left(\vec{a}+\vec{b}+\vec{c}\right).\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}=\frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}=\frac{\left|\vec{c}\right|^2}{\left|\vec{a}+\vec{b}+\vec{c}\right|\left|\vec{c}\right|}=\frac{\left|\vec{c}\right|}{\left|\vec{a}+\vec{b}+\vec{c}\right|}$$

Since $\left|\vec{a}\right|=\left|\vec{b}\right|=\left|\vec{c}\right|,\cos\theta_1=\cos\theta_2=\cos\theta_3$

Thus, $\theta_1=\theta_2=\theta_3$

Question 15:

Prove that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular, given $\vec{a} \neq 0, \vec{b} \neq 0$.

Solution:

$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow 2\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}.\vec{b} = 0$

Thus, $\vec{a}$ and $\vec{b}$ are perpendicular.

Question 16:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$

(B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$

(D) $0 \leq \theta \leq \pi$

Solution:

$\vec{a}.\vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

$\Rightarrow \cos\theta \geq 0 \qquad \left[\because |\vec{a}| \geq 0 \text{ and } |\vec{b}| \geq 0\right]$

$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$

Hence $\vec{a}.\vec{b} \geq 0$ if $0 \leq \theta \leq \frac{\pi}{2}$

Thus, the correct option is B.

Question 17:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the right angle between them. Then $\vec{a}+\vec{b}$ is a unit vector

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

Solution:

We have $\vec{a}$ and $\vec{b}$, two unit vectors and θ is the angle between them.

Then,

$$|\vec{a}| = |\vec{b}| = 1$$

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}| = 1$

Therefore,

$$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 1$$

$$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$$

$$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$$

$$\Rightarrow 1 + 2(1)(1)\cos\theta + 1 = 1$$

$$\Rightarrow \cos\theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \frac{2\pi}{2}$$

Hence, $\vec{a}+\vec{b}$ is a unit vector if $\theta = \frac{2\pi}{2}$

Thus, the correct option is D.

Question 18:

The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}(\hat{i}\times\hat{j})$ is

(A) 0 (B) -1 (C) 1 (D) 3

Solution:

$$\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}(\hat{i}\times\hat{j}) = \hat{i}.\hat{i} + \hat{j}.(-\hat{j}) + \hat{k}.\hat{k}$$

$$= 1 - 1 + 1$$

$$= 1$$

Thus, the correct option is C.

Question 19:

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}| = |\vec{a}\times\vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) n

Solution:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

Now,

$$\Rightarrow |\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$$
$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$$
$$\Rightarrow \cos\theta = \sin\theta$$
$$\Rightarrow \tan\theta = 1$$
$$\Rightarrow \theta = \frac{\pi}{4}$$

So, $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when $\theta = \frac{\pi}{4}$

Thus, the correct option is B.