Chapter 11 Three dimensional Geometry

EXERCISE 11.1

Question 1:

If a line makes angles $90°, 135°, 45°$ with x, y and $z-$axes respectively, find its direction cosines.

Solution:

Let direction cosines of the line be l, m and n.
Hence,

$$l = \cos 90° = 0$$

$$m = \cos 135° = \cos(90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45° = \frac{1}{\sqrt{2}}$$

Thus, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$

Question 2:

Find the direction cosines l, m and n of a line which makes equal angles with the coordinates axes.

Solution:

Let the direction cosines of the line make an angle α with each of the coordinates axes.
Hence,

$$l = \cos\alpha$$

$$m = \cos\alpha$$

$$n = \cos\alpha$$

Since, $l^2 + m^2 + n^2 = 1$

Hence,

$$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow 3\cos^2\alpha = 1$$

$$\Rightarrow \cos^2\alpha = \frac{1}{3}$$

$$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes, are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$ and $\pm\frac{1}{\sqrt{3}}$.

Question 3:

If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines?

Solution:

If a line has direction ratios $-18, 12, -4$, then its direction cosines are

$$l = \frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}} = \frac{-18}{\sqrt{484}} = \frac{-18}{22} = \frac{-9}{11}$$

$$m = \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}} = \frac{12}{\sqrt{484}} = \frac{12}{22} = \frac{6}{11}$$

$$n = \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}} = \frac{-4}{\sqrt{484}} = \frac{-4}{22} = \frac{-2}{11}$$

Hence, the direction cosines are $\frac{-9}{11}, \frac{6}{11}$ and $\frac{-2}{11}$.

Question 4:

Show that the points $(2, 3, 4), (-1, -2, 1), (5, 8, 7)$ are collinear.

Solution:

Given points are $A(2, 3, 4), B(-1, -2, 1)$ and $C(5, 8, 7)$.

As we know that the direction cosines of points, (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by $(x_2 - x_1), (y_2 - y_1)$ and $(z_2 - z_1)$.

Therefore, the direction ratios of AB are

$$(-1-2), (-2-3) \text{ and } (1-4)$$
$$\Rightarrow -3, -5 \text{ and } -3$$

The direction ratios of BC are

$$[5-(-1)], [8-(-2)] \text{ and } (7-1)$$
$$\Rightarrow 6, 10 \text{ and } 6$$

It can be seen that the direction ratios of BC are -2 times that AB i.e., they are proportional.

Hence, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

Question 5:

Find the direction cosines of the sides of the triangle whose vertices are $(3,5,-4),(-1,1,2)$ and $(-5,-5,-2)$.

Solution:

Vertices of the triangle are $A(3,5,-4), B(-1,1,2)$ and $C(-5,-5,-2)$
The direction ratios of the side AB are

$$(-1-3),(1-5) \text{ and } [2-(-4)]$$
$$\Rightarrow -4,-4 \text{ and } 6$$

Hence, the direction cosines of AB are

$$l_1 = \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}} = \frac{-4}{\sqrt{68}} = \frac{-4}{2\sqrt{17}} = \frac{-2}{\sqrt{17}}$$
$$m_1 = \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}} = \frac{-4}{\sqrt{68}} = \frac{-4}{2\sqrt{17}} = \frac{-2}{\sqrt{17}}$$
$$n_1 = \frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}} = \frac{6}{\sqrt{68}} = \frac{6}{2\sqrt{17}} = \frac{3}{\sqrt{17}}$$

The direction ratios of BC are

$$[-5(-1)],(-5-1) \text{ and } (-2-2)$$
$$\Rightarrow -4,-6 \text{ and } -4$$

Hence, the direction cosines of BC are

$$l_2 = \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}} = \frac{-4}{\sqrt{68}} = \frac{-4}{2\sqrt{17}} = \frac{-2}{\sqrt{17}}$$
$$m_2 = \frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}} = \frac{-6}{\sqrt{68}} = \frac{-6}{2\sqrt{17}} = \frac{-3}{\sqrt{17}}$$
$$n_2 = \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}} = \frac{-4}{\sqrt{68}} = \frac{-4}{2\sqrt{17}} = \frac{-2}{\sqrt{17}}$$

The direction ratios of CA are

$$(-5-3),(-5-5) \text{ and } [-2-(-4)]$$
$$\Rightarrow -8,-10 \text{ and } 2$$

Hence, the direction cosines of AC are

$$l_3 = \frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}} = \frac{-8}{\sqrt{168}} = \frac{-8}{2\sqrt{42}} = \frac{-4}{\sqrt{42}}$$
$$m_3 = \frac{-10}{\sqrt{(-8)^2+(10)^2+(2)^2}} = \frac{-10}{\sqrt{168}} = \frac{-10}{2\sqrt{42}} = \frac{-5}{\sqrt{42}}$$
$$n_3 = \frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}} = \frac{2}{\sqrt{168}} = \frac{2}{2\sqrt{42}} = \frac{1}{\sqrt{42}}$$

Thus, the direction cosines of the sides of the triangle are

$$\left(\frac{-2}{\sqrt{17}},\frac{-2}{\sqrt{17}},\frac{3}{\sqrt{17}}\right),\left(\frac{-2}{\sqrt{17}},\frac{-3}{\sqrt{17}},\frac{-2}{\sqrt{17}}\right) \text{ and } \left(\frac{-4}{\sqrt{42}},\frac{-5}{\sqrt{42}},\frac{1}{\sqrt{42}}\right)$$

EXERCISE 11.2

Question 1:

Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Solution:

Two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we get

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13} \\ &= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} \\ &= 0 \end{aligned}$$

Hence, the lines are perpendicular.

For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we get

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13} \\ &= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\ &= 0 \end{aligned}$$

Hence, the lines are perpendicular.

For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we get

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \left(\frac{3}{13}\right) \times \left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right) \times \left(\frac{-3}{13}\right) + \left(\frac{12}{13}\right) \times \left(\frac{-4}{13}\right) \\ &= \frac{36}{169} + \frac{12}{169} - \frac{48}{169} \\ &= 0 \end{aligned}$$

Hence, the lines are perpendicular.

So, the all three lines are mutually perpendicular.

Question 2:

Show that the line through the points $(1,-1,2),(3,4,-2)$ is perpendicular to the line through the points $(0,3,2)$ and $(3,5,6)$.

Solution:

Let AB be the line joining the points $(1,-1,2)$ and $(3,4,-2)$; and CD be the line through the points $(0,3,2)$ and $(3,5,6)$

Hence,

$$\begin{aligned} a_1 &= (3-1) = 2 \\ b_1 &= [4-(-1)] = 5 \\ c_1 &= (-2-2) = -4 \end{aligned}$$

$$\begin{aligned} a_2 &= (3-0) = 3 \\ b_2 &= (5-3) = 2 \\ c_2 &= (6-2) = 4 \end{aligned}$$

If, $AB \perp CD$; $\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

Here,

$$\begin{aligned} a_1a_2 + b_1b_2 + c_1c_2 &= 2\times3 + 5\times2 + (-4)\times4 \\ &= 6 + 10 - 16 \\ &= 0 \end{aligned}$$

Hence, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points $(4,7,8),(2,3,4)$ is parallel to the line through the points $(-1,-2,1),(1,2,5)$.

Solution:

Let AB be the line through the points $(4,7,8)$ and $(2,3,4)$; CD be the line through the points $(-1,-2,1)$ and $(1,2,5)$.

Hence,

$$a_1 = (2-4) = -2$$
$$b_1 = (3-7) = -4$$
$$c_1 = (4-8) = -4$$

$$a_2 = [1-(-1)] = 2$$
$$b_2 = [2-(-2)] = 4$$
$$c_2 = (5-1) = 4$$

If, $AB \perp CD$; $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = 0$

Here,

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$
$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$
$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$
$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Hence, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through point $(1,2,3)$ and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$.

Solution:

It is given that the line passes through the point $A(1,2,3)$.

Therefore, the position vector through $A(1,2,3)$ is

$$\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$$
$$\vec{b} = 3\hat{i}+2\hat{j}-2\hat{k}$$

So, line passes through point $A(1,2,3)$ and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a}+\lambda\vec{b}$, where □ is a real number.

Hence,

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the point with positive vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$

Solution:

It is given that

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$$
$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

Since, the vector equation of the line is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is some real number.

Hence,

$$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$$

Since, $\vec{r}$ is the position vector of any point (x, y, z) on the line

Therefore,

$$x\hat{i} - y\hat{j} + z\hat{k} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$$
$$= (2+\lambda)\hat{i} + (-1+2\lambda)\hat{j} + (4-\lambda)\hat{k}$$

Eliminating □, we get the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

Thus, the equation of the line in vector form is $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$ and cartesian form is $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$

Question 6:

Find the Cartesian equation of the line which passes through the point $(-2, 4, -5)$ and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

Solution:

It is given that the required line passes through the point $(-2, 4, -5)$ and is parallel to

$$\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$$

Therefore, its direction ratios are $3k, 5k$ and $6k$, where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios a, b, c is given by $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Hence, the equation of the required line is

$$\Rightarrow \frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

Thus, the cartesian equation of the line is $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$.

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Solution:

It is given that the Cartesian equation of the line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$

Hence,

The given line passes through the point $(5, -4, 6)$

Therefoe,

The position vector of the point is $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are $3, 7$ and 2

This means that the line is in the direction of the vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

As we known that the line through positive vector $\bar{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b};\ \lambda \in R$

Hence,

$$\Rightarrow \vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

This is the required equation of the given line in vector form.

Question 8:

Find the vector and the Cartesian equation of the lines that passes through the origin and $(5,-2,3)$.

Solution:

The required line passes through the origin.

Therefore, its position vector is $\vec{a}=0 \qquad \ldots(1)$

The direction ratios of the line passing through origin and $(5,-2,3)$ are

$$(5-0)=5$$
$$(-2-0)=-2$$
$$(3-0)=3$$

Hence, the line is parallel to the vector given by the equation, $\vec{b}=5\hat{i}-2\hat{j}+3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is,

$$\Rightarrow \vec{r}=\vec{a}+\lambda\vec{b};\ \lambda \in R$$
$$\Rightarrow \vec{r}=0+\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$$
$$\Rightarrow \vec{r}=\lambda\left(5\hat{i}-2\hat{j}+3\hat{k}\right)$$

The equation of the line through the point (x_1, y_1, z_1), and direction ratios a,b,c is given by, $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Hence, the equation of the required line in the Cartesian form is

$$\Rightarrow \frac{x-0}{5}=\frac{y-0}{-2}=\frac{z-0}{3}$$
$$\Rightarrow \frac{x}{5}=\frac{y}{2}=\frac{z}{3}$$

Question 9:

Find the vector and the cartesian equations of the line that passes through the points $(3,-2,-5),(3,-2,6)$.

Solution:

Let the line passing through the points, $P(3,-2,-5)$ and $Q(3,-2,6)$ be PQ. Since PQ passes through $P(3,-2,-5)$, its position vector is given by

$$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

The direction ratios of PQ are given by

$$\begin{aligned} (3-3) &= 0 \\ (-2+2) &= 0 \\ (6+5) &= 11 \end{aligned}$$

The equation of the vector in the direction of PQ is

$$\begin{aligned} \vec{b} &= 0.\hat{i} - 0.\hat{j} + 11\hat{k} \\ &= 11\hat{k} \end{aligned}$$

The equation of PQ in vector form is given by,

$$\begin{aligned} \vec{r} &= \vec{a} + \lambda b,\ \ \lambda \in R \\ &= \left(3\hat{i} - 2\hat{j} + 5\hat{k}\right) + 11\lambda\hat{k} \end{aligned}$$

The equation of PQ in Cartesian form is

$$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$
$$\Rightarrow \frac{x-3}{5} = \frac{y+2}{2} = \frac{z+5}{3}$$

Question 10:

Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - 2\hat{k}\right)$ and $\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

Solution:

Let □ be the angle between the given lines.

Then the angle between the given pairs of lines is given by

$$\cos\theta = \left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|.\left|\vec{b_2}\right|}\right|$$

(i) The given lines are parallel to the vectors, $\vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

Therefore,

$$\left|\vec{b_1}\right| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$$

$$\left|\vec{b_2}\right| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

$$\vec{b_1}.\vec{b_2} = (3i + 2j + 6k).(i + 2j + 2k)$$

$$= 3\times 1 + 2\times 2 + 6\times 2$$

$$= 3 + 4 + 12$$

$$= 19$$

Hence,

$$\cos\theta = \left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|.\left|\vec{b_2}\right|}\right|$$

$$\cos\theta = \left|\frac{19}{7\times 3}\right| = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b_1} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

Therefore,

$$\left|\vec{b_1}\right| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$\left|\vec{b_2}\right| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\vec{b_1}.\vec{b_2} = (\hat{i} - \hat{j} - 2\hat{k}).(3\hat{i} - 5\hat{j} - 4\hat{k})$$

$$= 1\times 3 - 1\times(-5) - 2\times(-4)$$

$$= 3 + 5 + 8$$

$$= 16$$

Hence,

$$\cos\theta = \left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}|.|\vec{b_2}|}\right|$$

$$\cos\theta = \left|\frac{16}{\left(\sqrt{6}\right).\left(5\sqrt{2}\right)}\right| = \left|\frac{16}{\sqrt{2}.\sqrt{3}.5\sqrt{2}}\right| = \left|\frac{16}{10\sqrt{3}}\right|$$

$$\cos\theta = \frac{8}{5\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 11:

Find the angle between the following pair of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Solution:

(i) Let $\vec{b_1}$ and $\vec{b_2}$ be the vectors parallel to the line pair of lines $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$ respectively.

Hence, $\vec{b_1} = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b_2} = -i + 8\hat{j} + 4\hat{k}$

Therefore,

$$|\vec{b_1}| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$$

$$|\vec{b_2}| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$$

$$\vec{b_1}.\vec{b_2} = \left(2\hat{i} + 5\hat{j} - 3\hat{k}\right).\left(-\hat{i} + 8\hat{j} + 4\hat{k}\right)$$

$$= 2\times(-1) + 5\times 8 + (-3)\times 4$$

$$= -2 + 40 - 12$$

$$= 26$$

The angle □ between the given pair of lines is given by the relation,

$$\theta os \;=\left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|.\left|\vec{b_2}\right|}\right|$$

$$\theta os \;=\left|\frac{26}{\sqrt{38\times 9}}\right|=\frac{26}{9\sqrt{38}}$$

$$\theta=\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\vec{b_1}$ and $\vec{b_2}$ be the vectors parallel to the given pair of lines $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$, respectively.

Hence, $\vec{b_1}=2\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b_2}=4i+\hat{j}+8\hat{k}$

Therefore,

$$\left|\vec{b_1}\right|=\sqrt{(2)^2+(2)^2+(1)^2}=\sqrt{9}=3$$

$$\left|\vec{b_2}\right|=\sqrt{(4)^2+(1)^2+(8)^2}=\sqrt{81}=9$$

$$\begin{aligned}\vec{b_1}.\vec{b_2}&=\left(2\hat{i}+2\hat{j}+\hat{k}\right).\left(4\hat{i}+\hat{j}+8\hat{k}\right)\\&=2\times4+2\times1+1\times8\\&=8+2+8\\&=18\end{aligned}$$

If □ is the angle between the pair of lines, then $\theta os \;=\left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|.\left|\vec{b_2}\right|}\right|$

$$\theta os \;=\left|\frac{\vec{b_1}.\vec{b_2}}{\left|\vec{b_1}\right|.\left|\vec{b_2}\right|}\right|$$

$$\theta os \;=\left|\frac{18}{3\times 9}\right|=\frac{2}{3}$$

$$\theta=\cos^{-1}\left(\frac{2}{3}\right)$$

Question 12:

Find the values of p so the line $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles.

Solution:

The given equations can be written in the standard form as $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$

The direction ratios of the lines are given by

$$a_1=-3, b_1=\frac{2p}{7} \text{ and } c_1=2$$

$$a_2=\frac{-3p}{7}, b_2=1 \text{ and } c_2=-5$$

Since, both the lines are perpendicular to each other,

Therefore,

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$(-3)\times\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\times1+2\times(-5)=0$$

$$\frac{9p}{7}+\frac{2p}{7}-10=0$$

$$\frac{11}{7}p=10$$

$$11p=10\times7$$

$$p=\frac{70}{11}$$

Hence the value of $p=\frac{70}{11}$

Question 13:

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

Solution:

The equations of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

Here,

$$a_1=7, b_1=-5 \text{ and } c_1=1$$

$$a_2=1, b_2=2 \text{ and } c_2=3$$

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Since,

$$\begin{aligned} 7\times1+(-5)\times2+1\times3 &= 7-10+3 \\ &= 0 \end{aligned}$$

Hence, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$\vec{r} = \left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r} = 2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

Solution:

Given lines are $\vec{r} = \left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and $\vec{r} = 2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

Hence,

$$\vec{a_1} = \left(\hat{i}+2\hat{j}+\hat{k}\right) \text{ and } \vec{b_1} = \left(\hat{i}-\hat{j}+\hat{k}\right)$$
$$\vec{a_2} = \left(2\hat{i}-\hat{j}-\hat{k}\right) \text{ and } \vec{b_2} = \left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

Shortest distance between the lines $\vec{r} = \vec{a_1}+\lambda\vec{b_1}$ and $\vec{r} = \vec{a_2}+\mu\vec{b_2}$ is given by,

$$d = \left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right| \quad \ldots(1)$$

Here,

$$\vec{a_2}-\vec{a_1} = \left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right) = \hat{i}-3\hat{j}-2\hat{k}$$

$$\begin{aligned} \vec{b_1}\times\vec{b_2} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-2-1)\hat{i}-(2-2)\hat{j}+(1+2)\hat{k} \\ &= -3\hat{i}+3\hat{k} \end{aligned}$$

$$\begin{aligned} \left|\vec{b_1}\times\vec{b_2}\right| &= \sqrt{(-3)^2+(3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

Putting all the values in equation (1), we get

$$d=\left|\frac{\left(-3\hat{i}+3\hat{k}\right).\left(\hat{i}-3\hat{j}-2\hat{k}\right)}{3\sqrt{2}}\right|$$

$$=\left|\frac{-3.1+3(-2)}{3\sqrt{2}}\right|$$

$$=\left|\frac{-9}{3\sqrt{2}}\right|$$

$$=\frac{3}{\sqrt{2}}$$

$$=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{3\sqrt{2}}{2}$$

Hence, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 15:

Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$.

Solution:

The given lines are $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

The shortest distance between the two lines,

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by,

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}} \qquad \ldots(1)$$

Here,

$x_1=-1, y_1=-1, z_1=-1$ and $x_2=3, y_2=5, z_2=7$

$a_1=7, b_1=-6, c_1=1$ and $a_2=1, b_2=-2, c_2=1$

Hence,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-6+2)-6(1+7)+8(-14+6)$$
$$= -16-36-64$$
$$= -116$$

Also,

$$\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2} = \sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$$
$$= \sqrt{16+36+64}$$
$$= \sqrt{116}$$

Putting all the values in equation (1), we get

$$d = \frac{-116}{\sqrt{116}}$$
$$= -\sqrt{116}$$
$$= -2\sqrt{29}$$
$$|d| = 2\sqrt{29}$$

Therefore, the distance between the given lines is $2\sqrt{29}$ units.

Question 16:

Find the shortest distance between the lines whose vector equations are
$\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r} = 4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$.

Solution:

The given lines are $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r} = 4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$

Hence,

$\vec{a_1} = (\hat{i}+2\hat{j}+3\hat{k})$ and $\vec{b_1} = (\hat{i}-3\hat{j}+2\hat{k})$

$\vec{a_2} = (4\hat{i}+5\hat{j}+6\hat{k})$ and $\vec{b_2} = (2\hat{i}+3\hat{j}+\hat{k})$

Shortest distance between the lines $\vec{r} = \vec{a_1}+\lambda\vec{b_1}$ and $\vec{r} = \vec{a_2}+\mu\vec{b_2}$ is given by,

$$d = \left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right| \qquad \ldots(1)$$

Here,

$$\vec{a_2}-\vec{a_1}=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}$$

$$=(-3-6)\hat{i}-(1-4)\hat{j}+(3+6)\hat{k}$$

$$=-9\hat{i}+3\hat{j}+9\hat{k}$$

$$\left|\vec{b_1}\times\vec{b_2}\right|=\sqrt{(-9)^2+(3)^2+(9)^2}$$

$$=\sqrt{81+9+81}$$

$$=\sqrt{171}$$

$$=3\sqrt{19}$$

Putting all the values in equation (1), we get

$$d=\left|\frac{\left(-9\hat{i}+3\hat{j}+9\hat{k}\right).\left(3\hat{i}+3\hat{j}+3\hat{k}\right)}{3\sqrt{19}}\right|$$

$$=\left|\frac{-9\times3+3\times3+9\times3}{3\sqrt{19}}\right|$$

$$=\left|\frac{-27+9+27}{3\sqrt{19}}\right|$$

$$=\left|\frac{9}{3\sqrt{19}}\right|$$

$$=\frac{3}{\sqrt{19}}$$

Hence, the shortest distance between the two lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$.

Solution:

The given lines are $\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

i.e., $\vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}+\hat{j}-2\hat{k}\right)$ and $\vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+s\left(\hat{i}+2\hat{j}-2\hat{k}\right)$

Hence,

$\vec{a_1} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$ and $\vec{b_1} = \left(-\hat{i} + \hat{j} - 2\hat{k}\right)$
$\vec{a_2} = \left(\hat{i} - \hat{j} - \hat{k}\right)$ and $\vec{b_2} = \left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

Shortest distance between the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by,

$$d = \left| \frac{\left(\vec{b_1} \times \vec{b_2}\right) . \left(\vec{a_2} - \vec{a_1}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|} \right| \qquad \ldots(1)$$

Here,

$$\vec{a_2} - \vec{a_1} = \left(\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = \hat{j} - 4\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k}$$

$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{(2)^2 + (-4)^2 + (-3)^2}$$

$$= \sqrt{4+16+9}$$

$$= \sqrt{29}$$

Putting all the values in equation (1), we get

$$d = \left| \frac{\left(2\hat{i} - 4\hat{j} - 3\hat{k}\right) . \left(\hat{j} - 4\hat{k}\right)}{\sqrt{29}} \right|$$

$$= \left| \frac{-4 \times 1 - 3 \times (-4)}{\sqrt{29}} \right|$$

$$= \left| \frac{-4+12}{\sqrt{29}} \right|$$

$$= \frac{8}{\sqrt{29}}$$

Hence, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

EXERCISE 11.3

Question 1:
In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z=2$

(b) $x+y+z=1$

(c) $2x+3y-z=5$

(d) $5y+8=0$

Solution:

(a) The equation of the plane is $z=2$ or $0x+0y+z=2$ ...(1)

The direction ratios of normal are $0, 0$ and 1.

Therefore,

$$\sqrt{0^2+0^2+1^2}=1$$

Dividing both sides of equation (1) by 1, we obtain

$$0.x+0.y+1.z=2$$

This is of the form $lx+my+nz=d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Hence, the direction cosines are $0, 0$ and 1 and the distance of the plane form the origin is 2 units.

(b) $x+y+z=1$ ...(1)

The direction ratios of normal are $1, 1$ and 1.

Therefore,

$$\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we get

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}}$$

This equation is one of the form $lx+my+nz=d$, where l, m, n are direction cosines of normal to the plane and d is the distance of normal from the origin.

Hence, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ and the distance of normal form the origin is $\frac{1}{\sqrt{3}}$ units.

(c) $2x+3y-z=5 \qquad \ldots(1)$

The direction ratios of normal are $2, 3$ and -1.
Therefore,

$$\sqrt{(2)^2+(3)^2+(-1)^2}=\sqrt{14}$$

Dividing both sides of equation (1) by $\sqrt{14}$, we get

$$\frac{2}{\sqrt{14}}x+\frac{3}{\sqrt{14}}y-\frac{1}{\sqrt{14}}z=\frac{5}{\sqrt{14}}$$

This equation is one of the form $lx+my+nz=d$, where l, m, n are direction cosines of normal to the plane and d is the distance of normal from the origin.

Hence, the direction cosines of the normal are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ and $\frac{-1}{\sqrt{14}}$ and the distance of normal form the origin is $\frac{5}{\sqrt{14}}$ units.

(d) $5y+8=0$

$\Rightarrow 0x+5y+0z=8 \qquad \ldots(1)$

The direction ratios of normal are $0, -5$ and 0.
Therefore,

$$\sqrt{0^2+(-5)^2+0^2}=5$$

Dividing both sides of equation (1) by 5, we get

$$0x+y+0z=\frac{8}{5}$$

This equation is one of the form $lx+my+nz=d$, where l, m, n are direction cosines of normal to the plane and d is the distance of normal from the origin.

Hence, the direction cosines of the normal to the plane are $0, 1$ and 0 and the distance of normal form the origin is $\frac{8}{5}$ units.

Question 2:

Find the vector equation of a plane which is at the distance of 7 units from the origin and normal to the vector $3\hat{i}+5\hat{j}-6\hat{k}$.

Solution:

The normal vector is, $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

The equation of the plane with position vector $\vec{r}$ is given by, $\vec{r}.\hat{n} = d$

Hence,

$$\vec{r}.\left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}\right) = 7$$

Question 3:

Find the Cartesian equation of the following planes:

(a) $\vec{r}.\left(\hat{i} + \hat{j} - \hat{k}\right) = 2$

(b) $\vec{r}.\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right) = 1$

(c) $\vec{r}.\left[(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}\right] = 15$

Solution:

(a) Given equation of the plane is

$\vec{r}.\left(\hat{i} + \hat{j} - \hat{k}\right) = 2 \qquad \ldots(1)$

For any arbitrary point, $P(x, y, z)$ on the plane, position vector $\vec{r}$ is given by,

$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$

Putting the values of $\vec{r}$ in equation (1), we get

$$\left(x\hat{i} + y\hat{j} - z\hat{k}\right).\left(\hat{i} + \hat{j} - \hat{k}\right) = 2$$

$$\Rightarrow x + y - z = 2$$

(b) $\vec{r}.\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right) = 1 \qquad \ldots(1)$

For any arbitrary point $P(x, y, z)$ on the plane, position vector $\vec{r}$ is given by, $\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$

Putting the values of $\vec{r}$ in equation (1), we get

$$\left(x\hat{i} + y\hat{j} - z\hat{k}\right).\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right) = 1$$

$$\Rightarrow 2x + 3y - 4z = 1$$

(c) $\vec{r}.\left[(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}\right] = 15 \qquad \ldots(1)$

For any arbitrary point, $P(x, y, z)$ on the plane, position vector $\vec{r}$ is given by,

$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$

Putting the values of $\vec{r}$ in equation (1), we get

$$(x\hat{i} + y\hat{j} - z\hat{k}).\left[(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}\right] = 15$$

$$\Rightarrow (s-2t)x + (3-t)y + (2s+t)z = 15$$

Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) $2x + 3y + 4z - 12 = 0$

(b) $3y + 4z - 6 = 0$

(c) $x + y + z = 1$

(d) $5y + 8 = 0$

Solution:

(a) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1)

$$2x + 3y + 4z - 12 = 0 \qquad \ldots(1)$$

The direction ratios of normal are $2, 3$ and 4

Therefore,

$$\sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

Dividing both sides of equation (1) by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}} \qquad \ldots(2)$$

This equation is one of the form $lx + my + nz = d$, where l, m, n are direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd)

Hence, the coordinates of the foot of the perpendicular are

$$\left(\frac{2}{\sqrt{29}} \times \frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}} \times \frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}} \times \frac{12}{\sqrt{29}}\right)$$

$$\Rightarrow \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$$

(b) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1)

$$3y + 4z - 6 = 0 \qquad \ldots(1)$$

The direction ratios of the normal are $0, 3$ and 4.
Therefore,

$$\sqrt{0^2+3^2+4^2}=5$$

Dividing both sides of equation (1) by 5, we get

$$0x+\frac{3}{5}y+\frac{4}{5}z=\frac{6}{5}$$

This equation is one of the form $lx+my+nz=d$, where l, m, n are direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd)
Hence, the coordinates of the foot of the perpendicular are

$$\left(0\times\frac{6}{5},\frac{3}{5}\times\frac{6}{5},\frac{4}{5}\times\frac{6}{5}\right)$$

$$\Rightarrow\left(0,\frac{18}{25},\frac{24}{25}\right)$$

(c) Let the coordinates of the foot of perpendicular P from the origin to the plane be

$$x+y+z=1 \qquad \ldots(1)$$

The direction ratios of the normal are $1, 1$ and 1.
Therefore,

$$\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we get,

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}}$$

This equation is one of the form $lx+my+nz=d$, where l, m, n are direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld, md, nd)

Hence, the coordinates of the foot of the perpendicular are

$$\left(0\times\frac{6}{5},\frac{3}{5}\times\frac{6}{5},\frac{4}{5}\times\frac{6}{5}\right)$$

$$\Rightarrow\left(0,\frac{18}{25},\frac{24}{25}\right)$$

(d) Let the coordinates of the foot of perpendicular P from the origin to the plane be

$$5y+8=0$$
$$\Rightarrow 0x-5y+0z=8 \qquad \dots(1)$$

The direction ratios of the normal are $0,-5$ and 0.

Therefore,

$$\sqrt{0^2+(-5)^2+0}=5$$

Dividing both sides of equation (1) by 5, we obtain

$$0x-y+0z=\frac{8}{5}$$

This equation is one of the form $lx+my+nz=d$, where l,m,n are direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by (ld,md,nd)

Hence, the coordinates of the foot of the perpendicular are

$$\left(0\times\frac{8}{5},-1\times\frac{8}{5},0\times\frac{8}{5}\right)$$
$$\Rightarrow\left(0,-\frac{8}{5},0\right)$$

Question 5:

Find the vector and Cartesian equation of the planes

(a) that passes through the point $(1,0,-2)$ and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$

(b) that passes through the point $(1,4,6)$ and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$.

Solution:

(a) The position vector of point $(1,0,-2)$ is $\vec{a}=\hat{i}-2\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}+\hat{j}-\hat{k}$

The vector equation of the plane is given by,

$$\Rightarrow(\vec{r}-\vec{a}).\vec{N}=0$$
$$\Rightarrow\left[\vec{r}-\left(\vec{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0 \qquad \dots(1)$$

Since, $\vec{r}$ is the positive vector of any point $P(x,y,z)$ in the plane.

Hence,

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Thus, equation (1) becomes

$$\Rightarrow \left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$
$$\Rightarrow \left[(x-1)\hat{i}+y\hat{j}+(z+2)\hat{k}\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$
$$\Rightarrow (x-1)+y-(z+2)=0$$
$$\Rightarrow x+y-z-3=0$$
$$\Rightarrow x+y-z=3$$

(b) The position vector of point $(1,4,6)$ is $\vec{a}=\hat{i}+4\hat{j}+6\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}-2\hat{j}+\hat{k}$

The vector equation of the plane is given by,

$$\Rightarrow \left(\vec{r}-\vec{a}\right).\vec{N}=0$$
$$\Rightarrow \left[\vec{r}-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0 \qquad \ldots(1)$$

Since, $\vec{r}$ is the positive vector of any point $P(x,y,z)$ in the plane.

Hence,

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Thus, equation (1) becomes

$$\Rightarrow \left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$
$$\Rightarrow \left[(x-1)\hat{i}+(y-4)\hat{j}+(z-6)\hat{k}\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$
$$\Rightarrow (x-1)-2(y-4)+(z-6)=0$$
$$\Rightarrow x-2y+z+1=0$$

Question 6:

Find the equations of the planes that passes through the points.

(a) $(1,1,-1),(6,4,-5),(-4,2,3)$

(b) $(1,1,0),(1,2,1),(-2,2,-1)$

Solution:

(a) The given points are $A(1,1,-1),B(6,4,-5)$ and $C(-4,2,3)$.

$$\begin{pmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{pmatrix} = (12-10)-(18-20)-(-12+16)$$
$$= 2+2-4$$
$$= 0$$

Since, the points are collinear, there will be infinite number of planes passing through the given points.

(b) The given points are $A(1,1,0), B(1,2,1)$ and $C(-2,2,-1)$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{pmatrix} = (-2-2)-(2+2)$$
$$= -8$$
$$\neq 0$$

Thus, a plane will pass through the points.

The equation of the plane through the points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is given by

$$\Rightarrow \begin{pmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{pmatrix} = 0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z = 0$$
$$\Rightarrow -2x+2-3y+3+3z = 0$$
$$\Rightarrow -2x-3y+3z+5 = 0$$
$$\Rightarrow 2x+3y-3z = 5$$

Question 7:

Find the intercepts cut off by the plane $2x+y-z=5$

Solution:

$2x+y-z=5$

Dividing both sides of equation by 5, we get

$$\Rightarrow \frac{2}{5}x+\frac{y}{5}-\frac{z}{5}=1$$

$$\Rightarrow \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1$$

The equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where a,b and c are intercepts cut off by the plane at x,y and $z-$axes respectively.

Hence, for the given equation,

$$a=\frac{5}{2}, b=5 \text{ and } c=-5$$

Thus, the intercepts cut off by plane are $\frac{5}{2},5$ and -5.

Question 8:

Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

Solution:

The equation of the plane ZOX is $Y=0$

Any plane parallel to it is of the form, $y=a$

Since the y-intercept of the plane is 3,

Therefore, $a=3$

Hence, the equation of the required plane is $y=3$.

Question 9:

Find the equation of the plane through the intersection of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$ and the point $(2,2,1)$.

Solution:

The equation of the given plane through the intersection of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$ is given by

$$(3x-y+2z-4)+\alpha(x+y+z-2)=0;\ \alpha\in R \qquad \ldots(1)$$

This plane passes through the point $(2,2,1)$.

Hence, this point will satisfy equation

$$\Rightarrow (3\times 2-2+2\times 1-4)+\alpha(2+2+1-2)=0$$
$$\Rightarrow 2+3\alpha=0$$
$$\Rightarrow \alpha=\frac{-2}{3}$$

Putting $\alpha=\frac{-2}{3}$ in equation (1), we get

$$\Rightarrow (3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$$
$$\Rightarrow 3(3x-y+2z-4)-2(x+y+z-2)=0$$
$$\Rightarrow 9x-3y+6z-12-2x-2y-2z+4=0$$
$$\Rightarrow 7x-5y+4z-8=0$$

Question 10:

Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})=7, \vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})=9$ and through the point $(2,1,3)$.

Solution:

The equations of the planes are $\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})=7$ and $\vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})=9$

Hence,

$$\Rightarrow \vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})-7=0 \qquad \ldots(1)$$
$$\Rightarrow \vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})-9=0 \qquad \ldots(2)$$

Equation of the required plane is given by

$$\left[\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})-7\right]+\lambda\left[\vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})-9\right]=0;\ \lambda\in R$$

Therefore,

$$\Rightarrow \vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})-7+\lambda\vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})-9\lambda=0$$
$$\Rightarrow \vec{r}.\left[(2\hat{i}+2\hat{j}-3\hat{k})+\lambda(2\hat{i}+5\hat{j}+3\hat{k})\right]=9\lambda+7$$
$$\Rightarrow \vec{r}.\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7 \qquad \ldots(3)$$

The plane passes through the point $(2,1,3)$

Hence, its position vector is given by, $\vec{r}=2\hat{i}+\hat{j}+3\hat{k}$

Putting in equation (3), we get

$$\Rightarrow \left(2\hat{i}+\hat{j}+3\hat{k}\right).\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(3\lambda-3)\hat{k}\right]=9\lambda+7$$
$$\Rightarrow 2(2+2\lambda)+(2+5\lambda)+3(3\lambda-3)=9\lambda+7$$
$$\Rightarrow 4+4\lambda+2+5\lambda+9\lambda-9-9\lambda-7=0$$
$$\Rightarrow 9\lambda-10=0$$
$$\Rightarrow \lambda=\frac{10}{9}$$

Putting $\lambda=\frac{10}{9}$ in equation (3), we get

$$\Rightarrow \vec{r}.\left(\frac{38}{9}\hat{i}+\frac{68}{9}\hat{j}+\frac{3}{9}\hat{k}\right)=17$$
$$\Rightarrow \vec{r}.\left(38\hat{i}+68\hat{j}+3\hat{k}\right)=153$$

Question 11:

Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$.

Solution:

The equation of the plane through the intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ is

$$\Rightarrow (x+y+z-1)+\lambda(2x+3y+4z-5)=0$$
$$\Rightarrow (2\lambda+1)x+(3\lambda+1)y+(4\lambda+1)z-(5\lambda+1)=0 \qquad \ldots(1)$$

The plane in equation (1) is perpendicular to the plane $x-y+z=0$

Since the planes are perpendicular, $a_1a_2+b_1b_2+c_1c_2=0$

Here,

$a_1=(2\lambda+1), b_1=(3\lambda+1)$ and $c_1=(4\lambda+1)$

$a_2=1, b_2=-1$ and $c_2=1$

Hence,

$$\Rightarrow (2\lambda+1)\times 1+(3\lambda+1)\times(-1)+(4\lambda+1)\times 1=0$$
$$\Rightarrow 2\lambda+1-3\lambda-1+4\lambda+1=0$$
$$\Rightarrow 3\lambda+1=0$$
$$\Rightarrow \lambda=\frac{-1}{3}$$

Putting $\lambda=\frac{-1}{3}$ in equation (1), we get

$$\Rightarrow \frac{1}{3}x+\frac{1}{3}z+\frac{2}{3}=0$$
$$\Rightarrow x-z+2=0$$

Question 12:

Find the angle between the planes whose vector equations are $\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$.

Solution:

The equations of the given planes are $\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$
If $\vec{n_1}$ and $\vec{n_2}$ are normal to the planes, $\vec{r}.\vec{n_1}=d_1$ and $\vec{r}.\vec{n_2}=d_2$

Then the angle between them θ is given by,

$$\theta=\cos^{-1}\frac{\left|\vec{n_1}.\vec{n_2}\right|}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|} \qquad \ldots(1)$$

Here,

$$\vec{n_1}=2\hat{i}+2\hat{j}-3\hat{k} \text{ and } \vec{n_2}=3\hat{i}-3\hat{j}+5\hat{k}$$

Hence,

$$\vec{n_1}.\vec{n_2}=\left(2\hat{i}+2\hat{j}-3\hat{k}\right).\left(3\hat{i}-3\hat{j}+5\hat{k}\right)$$
$$=2\times3+2\times(-3)+(-3)\times5$$
$$=-15$$
$$\left|\vec{n_1}\right|=\sqrt{(2)^2+(2)^2+(-3)^2}=\sqrt{17}$$
$$\left|\vec{n_2}\right|=\sqrt{(3)^2+(-3)^2+(5)^2}=\sqrt{43}$$

Substituting these values in equation (1), we obtain

$$\theta=\cos^{-1}\frac{|-15|}{\left|\sqrt{17}\right|.\left|\sqrt{43}\right|}$$
$$=\cos^{-1}\frac{15}{\sqrt{731}}$$

Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x+5y+6z+30=0$ and $3x-y-10z+4=0$

(b) $2x+y+3z-2=0$ and $x-2y+5=0$

(c) $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

(d) $2x-y+3z-1=0$ and $2x-y+3z+3=0$

(e) $4x+8y+z-8=0$ and $y+z-4=0$

Solution:

The directions ratios of normal to be the plane $L_1 : a_1x+b_1y+c_1z=0$ are a_1,b_1,c_1 and $L_2 : a_2x+b_2y+c_2z=0$ are a_2,b_2,c_2

If,

$$L_1 \parallel L_2; \Rightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

$$L_1 \perp L_2; \Rightarrow a_1a_2+b_1b_2+c_1c_2=0$$

The angle between L_1 and L_2 is given by

$$\theta=\cos^{-1}\left|\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

(a) The equations of the planes are $7x+5y+6z+30=0$ and $3x-y-10z+4=0$

Here,

$$a_1=7, b_1=5 \text{ and } c_1=6$$

$$a_2=3, b_2=-1 \text{ and } c_2=-10$$

Hence,

$$\begin{aligned} a_1a_2+b_1b_2+c_1c_2 &= 7\times3+5\times(-1)+6\times(-10) \\ &= -44 \\ &\neq 0 \end{aligned}$$

Therefore, the given planes are not perpendicular.

Also,

$$\frac{a_1}{a_2}=\frac{7}{3}$$

$$\frac{b_1}{b_2}=\frac{5}{-1}=-5$$

$$\frac{c_1}{c_2}=\frac{6}{-10}=-\frac{3}{5}$$

It can be seen that, $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$\theta = \cos^{-1}\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$= \cos^{-1}\left|\frac{7\times3+5\times(-1)+6\times(-10)}{\sqrt{(7)^2+(5)^2+(6)^2}.\sqrt{(3)^2+(-1)^2+(-10)^2}}\right|$$

$$= \cos^{-1}\left|\frac{21-5-60}{\sqrt{110}.\sqrt{110}}\right|$$

$$= \cos^{-1}\left|\frac{-44}{110}\right|$$

$$= \cos^{-1}\frac{2}{5}$$

(b) The equations of the planes are $2x+y+3z-2=0$ and $x-2y+5=0$

Here,

$a_1 = 2, b_1 = 1$ and $c_1 = 3$

$a_2 = 1, b_2 = -2$ and $c_2 = 0$

Hence,

$$\begin{aligned} a_1a_2+b_1b_2+c_1c_2 &= 2\times1+1\times(-2)+3\times0 \\ &= 2-2+0 \\ &= 0 \end{aligned}$$

Thus, the given planes are perpendicular to each other.

(c) The equations of the planes are $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

Here,

$a_1 = 2, b_1 = -2$ and $c_1 = 4$

$a_2 = 3, b_2 = -3$ and $c_2 = 6$

Hence,

$$\begin{aligned} a_1a_2+b_1b_2+c_1c_2 &= 2\times3+(-2)\times(-3)+4\times6 \\ &= 6+6+24 \\ &= 36 \\ &\neq 0 \end{aligned}$$

Thus, the given planes are not perpendicular to each other.

Also,

$$\frac{a_1}{a_2}=\frac{2}{3}$$

$$\frac{b_1}{b_2}=\frac{-2}{-3}=\frac{2}{3}$$

$$\frac{c_1}{c_2}=\frac{4}{6}=\frac{2}{3}$$

It can be seen that, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Hence, the given planes are parallel to each other.

(d) The equations of the planes are $2x-y+3z-1=0$ and $2x-y+3z+3=0$
Here,

$a_1=2, b_1=-1$ and $c_1=3$

$a_2=2, b_2=-1$ and $c_2=3$

Hence,

$$\frac{a_1}{a_2}=\frac{2}{2}=1$$

$$\frac{b_1}{b_2}=\frac{-1}{-1}=1$$

$$\frac{c_1}{c_2}=\frac{3}{3}=1$$

Therefore, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

Hence, the given lines are parallel to each other.

(e) The equations of the given planes are $4x+8y+z-8=0$ and $y+z-4=0$
Here,

$a_1=4, b_1=8$ and $c_1=1$

$a_2=0, b_2=1$ and $c_2=1$

Hence,

$$\begin{aligned}a_1a_2+b_1b_2+c_1c_2&=4\times 0+8\times 1+1\\&=9\\&\neq 0\end{aligned}$$

Thus, the given lines are not perpendicular to each other.

Also,

$$\frac{a_1}{a_2}=\frac{4}{0}$$
$$\frac{b_1}{b_2}=\frac{8}{1}=8$$
$$\frac{c_1}{c_2}=\frac{1}{1}=1$$

It can be seen that, $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

Thus, the given lines not parallel to each other.

The angle between the planes is given by,

$$\theta=\cos^{-1}\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$
$$=\cos^{-1}\left|\frac{4\times0+8\times1+1\times1}{\sqrt{(4)^2+(8)^2+(1)^2}\times\sqrt{(0)^2+(1)^2+(1)^2}}\right|$$
$$=\cos^{-1}\left|\frac{9}{9\sqrt{2}}\right|$$
$$=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$=45°$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

	Point	Plane
(a)	$(0,0,0)$	$3x-4y+12z=3$
(b)	$(3,-2,1)$	$2x-y+2z+3=0$
(c)	$(2,3,-5)$	$x+2y-2z=9$
(d)	$(-6,0,0)$	$2x-3y+6z-2=0$

Solution:

The distance between a point, $P(x_1,y_1,z_1)$ and a plane $Ax+By+Cz+D=0$ is given by,

$$d=\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$$

(a) The given point is $(0,0,0)$ and the plane is $3x-4y+12z-3=0$

Therefore,

$$d=\left|\frac{3\times 0+(-4)\times 0+12\times 0+(-3)}{\sqrt{(-3)^2+(-4)^2+(12)^2}}\right|$$
$$=\left|\frac{-3}{\sqrt{169}}\right|$$
$$=\frac{3}{13}$$

(b) The given point is $(3,-2,1)$ and the plane is $2x-y+2z+3=0$
Therefore,

$$d=\left|\frac{2\times 3+(-1)\times(-2)+2\times 1+3}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right|$$
$$=\left|\frac{13}{\sqrt{9}}\right|$$
$$=\frac{13}{3}$$

(c) The given point is $(2,3,-5)$ and the plane is $x+2y-2z-9=0$
Therefore,

$$d=\left|\frac{1\times 2+2\times 3+(-2)\times(-5)+(-9)}{\sqrt{(1)^2+(2)^2+(-2)^2}}\right|$$
$$=\left|\frac{9}{\sqrt{9}}\right|$$
$$=\frac{9}{3}$$
$$=3$$

(d) The given point is $(-6,0,0)$ and the plane is $2x-3y+6z-2=0$
Therefore,

$$d=\left|\frac{2\times(-6)+(-3)\times 0+6\times 0+(-2)}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right|$$
$$=\left|\frac{-14}{\sqrt{49}}\right|$$
$$=\frac{14}{7}$$
$$=2$$

MISCELLANEOUS EXERCISE

Question 1:

Show that the line joining the origin to the point $(2,1,1)$ is perpendicular to the line determined by the points $(3,5,-1),(4,3,-1)$.

Solution:

Let OA be the line joining the origin $O(0,0,0)$ and the point $A(2,1,1)$.
Also, let BC be the line joining the points, $B(3,5,-1)$ and $C(4,3,-1)$.

The direction ratios of OA are 2, 1 and 1 and of BC are $(4-3)=1,(3-5)=-2$ and $(-1+1)=0$

If, $OA \perp BC \Rightarrow a_1a_2+b_1b_2+c_1c_2=0$
Here,

$$\begin{aligned}a_1a_2+b_1b_2+c_1c_2 &= 2\times1+1(-2)+1\times0\\ &= 2-2\\ &= 0\end{aligned}$$

Thus, $OA \perp BC$ proved.

Question 2:

If l_1,m_1,n_1 and l_2,m_2,n_2 are the direction cosines of two mutually perpendicular lines. Show that the direction cosines of the perpendicular to both of these are $m_1n_2-m_2n_1,\ n_1l_2-n_2l_1,\ l_1m_2-l_2m_1$.

Solution:

$$l_1l_2+m_1m_2+n_1n_2=0 \qquad \ldots(1)$$
$$l_1^2+m_1^2+n_1^2=1 \qquad \ldots(2)$$
$$l_2^2+m_2^2+n_2^2=1 \qquad \ldots(3)$$

Let l,m,n be the direction cosines of the line which is perpendicular to the line with direction cosines l_1,m_1,n_1 and l_2,m_2,n_2.

Therefore,

$$ll_1 + mm_1 + nn_1 = 0$$
$$ll_2 + mm_2 + nn_2 = 0$$
$$\Rightarrow \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$
$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2}$$
$$\Rightarrow \frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} \qquad \ldots(4)$$

Since, l, m, n are direction cosines of the line.
Hence,

$$l^2 + m^2 + n^2 = 1 \qquad \ldots(5)$$

As we know that,

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2) = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

Putting the values from (1), (2) and (3), we get

$$\Rightarrow 1.1 - 0 = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$
$$\Rightarrow (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \qquad \ldots(6)$$

Putting the values from equation (5) and (6) in equation (4), we get

$$\frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} = 1$$

Hence,

$$\frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} = 1$$

Therefore,

$$l = m_1n_2 - m_2n_1$$
$$m = n_1l_2 - n_2l_1$$
$$n = l_1m_2 - l_2m_1$$

Hence, the direction cosines of the required line are $m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1$ proved.

Question 3:

Find the angle between the lines whose direction ratios are a, b, c and $b-c, c-a, a-b$.

Solution:

The angle θ between the lines with direction cosines a, b, c and $(b-c),(c-a),(a-b)$ is given by,

$$\cos\theta = \left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}.\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$\theta = \cos^{-1}\left|\frac{ab-ac+bc-ab+ac-bc}{\sqrt{a^2+b^2+c^2}.\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$= \cos^{-1} 0$$

$$= 90°$$

Thus, the required angle is $90°$

Question 4:

Find the equation of a line parallel to x-axis and passing through the origin.

Solution:

The line parallel to x-axis and passing through the origin is x-axis itself.
Let A be a point on x-axis.

Therefore, the coordinates of A are given by $(a,0,0)$, where $a \in R$

Hence, the direction ratios of OA are $a, 0, 0$

The equation of OA is given by,

$$\Rightarrow \frac{x-a}{0} = \frac{y-0}{0} = \frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$$

Hence, the equation of line parallel to x-axis and passing origin is $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

Question 5:

If the coordinates of the points A, B, C, D be $(1,2,3),(4,5,7),(-4,3,-6)$ and $(2,9,2)$ respectively, then find the angle between the lines AB and CD.

Solution:

The coordinates of A, B, C and D are $(1,2,3),(4,5,7),(-4,3,-6)$ and $(2,9,2)$ respectively.
Hence,

$$a_1 = (4-1) = 3 \qquad a_2 = [2-(-4)] = 6$$
$$b_1 = (5-2) = 3 \qquad b_2 = (9-3) = 6$$
$$c_1 = (7-3) = 4 \qquad c_2 = [2-(-6)] = 8$$

Therefore,

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$$

Hence, $AB \parallel CD$

Thus, the angle between AB and CD is either $0°$ or $180°$.

Question 6:

If the line $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution:

Here,

$$a_1 = -3 \qquad a_2 = 3k$$
$$b_1 = 2k \qquad b_2 = 1$$
$$c_1 = 2 \qquad c_2 = -5$$

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Therefore,

$$\Rightarrow -3(3k) + 2k \times 1 + 2(-5) = 0$$
$$\Rightarrow -9k + 2k - 10 = 0$$
$$\Rightarrow 7k = -10$$
$$\Rightarrow k = \frac{-10}{7}$$

Hence, for $k = -\frac{10}{7}$, the given lines are perpendicular to each other.

Question 7:

Find the vector equation of the plane passing through $(1, 2, 3)$ and perpendicular to the plane $\vec{r}.(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

Solution:

Here,

$$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right) \text{ and } \vec{N}=\left(\hat{i}+2\hat{j}-5\hat{k}\right)$$

The equation of a line passing through a point and perpendicular to the given plane is given by

$$\vec{l}=\vec{r}+\lambda\vec{N};\ \lambda\in R$$

Hence,

$$\Rightarrow \vec{l}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-5\hat{k}\right)$$

Question 8:

Find the equation of the plane passing through (a,b,c) and parallel to the plane $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=2$

Solution:

Any plane parallel to the plane, $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=2$, is of the form

$$\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda \qquad \ldots(1)$$

Since, the plane passes through the point (a,b,c).

Therefore, the position vector $\vec{r}$ of this point is $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$

Hence, equation (1) becomes

$$\Rightarrow \left(a\hat{i}+b\hat{j}+c\hat{k}\right).\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda$$
$$\Rightarrow a+b+c=\lambda$$

Putting $\lambda=a+b+c$ in equation (1), we get

$$\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=a+b+c \qquad \ldots(2)$$

This is vector equation of the required plane.

Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in equation (2), we get

$$\Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}+\hat{j}+\hat{k}\right)=a+b+c$$
$$\Rightarrow x+y+z=a+b+c$$

Question 9:

Find the shortest distance between lines $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$ and $\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$.

Solution:

The given lines are

$$\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right) \quad \ldots(1)$$

$$\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right) \quad \ldots(2)$$

The shortest distance between two lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ is given by

$$d=\left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right| \quad \ldots(3)$$

Comparing, $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ to equation (1) and (2), we get

$$\vec{a_1}=6\hat{i}+2\hat{j}+2\hat{k} \text{ and } \vec{a_2}=-4\hat{i}-\hat{k}$$

$$\vec{b_1}=\hat{i}-2\hat{j}+2\hat{k} \text{ and } \vec{b_2}=3\hat{i}-2\hat{j}-2\hat{k}$$

Therefore,

$$\vec{a_2}-\vec{a_1}=\left(-4\hat{i}-\hat{k}\right)-\left(6\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$=-10\hat{i}-2\hat{j}-3\hat{k}$$

$$\vec{b_1}\times\vec{b_2}=\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{pmatrix}$$

$$=(4+4)\hat{i}-(-2-6)\hat{j}+(-2+6)\hat{k}$$

$$=8\hat{i}+8\hat{j}+4\hat{k}$$

Putting all these values in equation (1), we get

$$d=\left|\frac{\left(8\hat{i}+8\hat{j}+4\hat{k}\right).\left(-10\hat{i}-2\hat{j}-3\hat{k}\right)}{\left|\left(8\hat{i}+8\hat{j}+4\hat{k}\right)\right|}\right|$$
$$=\left|\frac{-80-16-12}{\sqrt{(8)^2+(8)^2+(4)^2}}\right|$$
$$=\left|\frac{-108}{\sqrt{144}}\right|$$
$$=\frac{108}{12}$$
$$=9$$

Hence, the shortest distance between the two given lines is 9 units.

Question 10:

Find the coordinates of the point where the line through $(5,1,6)$ and $(3,4,1)$ crosses the YZplane.

Solution:

The equation of the line passing through the points, (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

The line passing through the points $(5,1,6)$ and $(3,4,1)$ is given by,

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6}\Rightarrow\frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=k(say)$$
$$\Rightarrow x=5-2k,y=3k+1,z=6k-5$$

Any point on the line is of the form $(5-2k,3k+1,6-5k)$

Any point on the line passes through YZ-plane

$$\Rightarrow 5-2k=0$$
$$\Rightarrow k=\frac{5}{2}$$
$$\Rightarrow 3k+1=3\times\left(\frac{5}{2}\right)+1=\frac{17}{2}$$
$$\Rightarrow 6-5k=6-5\times\left(-\frac{5}{2}\right)=-\frac{13}{2}$$

Hence, the required point is $\left(0,\frac{17}{2},\frac{-13}{2}\right)$

Question 11:

Find the coordinates of the point where the line through $(5,1,6)$ and $(3,4,1)$ crosses the ZX-plane.

Solution:

The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

The line passing through the points $(5,1,6)$ and $(3,4,1)$ is given by,

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6}\Rightarrow\frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=k(say)$$
$$\Rightarrow x=5-2k, y=3k+1, z=6k-5$$

Any point on the line is of the form $(5-2k, 3k+1, 6-5k)$

Any point on the line passes through ZX-plane

$$\Rightarrow 3k+1=0$$
$$\Rightarrow k=-\frac{1}{3}$$
$$\Rightarrow 5-2k=5-2\left(-\frac{1}{3}\right)=\frac{17}{3}$$
$$\Rightarrow 6-5k=6-5\left(-\frac{1}{3}\right)=\frac{23}{2}$$

Hence, the required point is $\left(\frac{17}{3},0,\frac{23}{2}\right)$

Question 12:

Find the coordinates of the point where the line through $(3,-4,-5)$ and $(2,-3,1)$ crosses the plane $2x+y+z=7$.

Solution:

The equation of the line through the point (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Since the line passes through the points $(3,-4,-5)$ and $(2,-3,1)$ its equation is given by,

$$\Rightarrow \frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}$$
$$\Rightarrow \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=k(say)$$
$$\Rightarrow x=3-k, y=k-4, z=6k-5$$

Thus, any point on the line is of the form $(3-k, k-4, 6k-5)$
This point lies on the plane, $2x+y+z=7$
$$\Rightarrow 2(3-k)+(k-4)+(6k-5)=7$$
$$\Rightarrow 5k-3=7$$
$$\Rightarrow k=2$$

Hence, the coordinates of the required point are
$$\Rightarrow (3-2, 2-4, 6\times 2-5)$$
$$\Rightarrow (1,-2,7)$$

Question 13:

Find the equation of the plane passing through the points $(-1,3,2)$ and perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$.

Solution:

The equation of the plane passing through the point $(-1,3,2)$ is
$$a(x+1)+b(y-3)+c(z-2)=0 \qquad \ldots(1)$$
where a,b,c are direction ratios of normal to the plane.

We know that two planes,
$a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

Since, plane (1) is perpendicular to the plane, $x+2y+3z=5$
Therefore,
$$\Rightarrow a.1+b.2+c.3=0$$
$$\Rightarrow a+2b+3c=0 \qquad \ldots(2)$$

Also, plane (1) is perpendicular to the plane, $3x+3y+z=0$
$$\Rightarrow a.3+b.3+c.1=0$$
$$\Rightarrow 3a+3b+c=0 \qquad \ldots(3)$$

From equation (2) and (3), we get

$$\Rightarrow \frac{a}{2\times1-3\times3}=\frac{b}{3\times3-1\times1}=\frac{c}{1\times3-2\times3}$$
$$\Rightarrow \frac{a}{-7}=\frac{b}{8}=\frac{c}{3}=k\ (say)$$
$$\Rightarrow a=-7k, b=8k, c=-3k$$

Putting the values of a, b and c in equation (1), we get

$$\Rightarrow -7k(x+1)+8k(y-3)-3k(z-2)=0$$
$$\Rightarrow (-7x-7)+(8y-24)-3z+6=0$$
$$\Rightarrow -7x+8y-3z-25=0$$
$$\Rightarrow 7x-8y+3z+25=0$$

Question 14:

If the points $(1,1,p)$ and $(-3,0,1)$ be equidistant from the plane $\vec{r}.(3\hat{i}+4\hat{j}-12\hat{k})+13=0$ then find the value of p.

Solution:

Here,

$$\vec{a_1}=\hat{i}+\hat{j}+p\hat{k}$$
$$\vec{a_2}=-3\hat{i}+\hat{k}$$

The equation of the given plane is $\vec{r}.(3\hat{i}+4\hat{j}-12\hat{k})+13=0$

The perpendicular distance between a point whose vector is $\bar{a}$ and the plane $\vec{r}.\vec{N}=d$ is given by

$$D=\frac{\left|\vec{a}.\vec{N}-d\right|}{\left|\vec{N}\right|}$$

Here,

$\vec{N}=3\hat{i}+4\hat{j}-12\hat{k}$ and $d=-13$

Hence, the distance between the point $(1,1,p)$ and the given plane is

$$\Rightarrow D_1=\frac{\left|(\hat{i}+\hat{j}+p\hat{k}).(3\hat{i}+4\hat{j}-12\hat{k})-(-13)\right|}{\left|3\hat{i}+4\hat{j}-12\hat{k}\right|}$$
$$\Rightarrow D_1=\frac{|3+4-12p+13|}{\sqrt{3^2+4^2+(-12)^2}}$$
$$\Rightarrow D_1=\frac{|20-12p|}{13} \qquad \ldots(1)$$

Similarly, the distance between the point $(-3,0,1)$ and the given plane is

$$\Rightarrow D_2 = \frac{\left|\left(-3\hat{i}+\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)-(-13)\right|}{\left|3\hat{i}+4\hat{j}-12\hat{k}\right|}$$

$$\Rightarrow D_2 = \frac{|-9-12+13|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$\Rightarrow D_2 = \frac{8}{13} \qquad \ldots(2)$$

From the given condition, $D_1 = D_2$

$$\Rightarrow \frac{|20-12p|}{13} = \frac{8}{13}$$

$$\Rightarrow |20-12p| = 8$$

$$\Rightarrow 20-12p = 8 \text{ or } -(20-12p) = 8$$

$$\Rightarrow 12p = 12 \text{ or } 12p = 28$$

$$\Rightarrow p = 1 \text{ or } p = \frac{7}{3}$$

Thus, the value of $p=1$ or $p=\frac{7}{3}$.

Question 15:

Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ and parallel to x-axis.

Solution:

The given planes are $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$

The equation of any plane passing through the line of intersection of these planes is given by

$$\left[\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)-1\right]+\lambda\left[\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4\right]=0$$

$$\vec{r}.\left[(2\lambda+1)\hat{i}+(3\lambda+1)\hat{j}+(1-\lambda)\hat{k}\right]+(4\lambda-1)=0 \qquad \ldots(1)$$

Here,

$$a_1=(2\lambda+1), b_1=(3\lambda+1) \text{ and } c_1=(1-\lambda)$$

Since, the required plane is parallel to x-axis.

Therefore, its normal is perpendicular to x-axis.

The direction ratios of x-axis are 1, 0 and 0.

Therefore,

$$a_2 = 1, b_2 = 0 \text{ and } c_2 = 0$$

Hence,

$$\Rightarrow 1.(2\lambda+1) + 0(3\lambda+1) + 0(1-\lambda) = 0$$
$$\Rightarrow 2\lambda + 1 = 0$$
$$\Rightarrow \lambda = -\frac{1}{2}$$

Putting, $\lambda = -\frac{1}{2}$ in equation (1), we get

$$\Rightarrow \vec{r}\left[-\frac{1}{2}\hat{j} + \frac{3}{2}\hat{k}\right] + (-3) = 0$$
$$\Rightarrow \vec{r}\left(\hat{j} - 3\hat{k}\right) + 6 = 0$$

Thus, its Catersian equation is $y - 3z + 6 = 0$

Question 16:

If O be the origin and the coordinates of P be $(1, 2, -3)$, then find the equation of the plane passing through P and perpendicular to OP.

Solution:

The given points are $O(0,0,0)$ and $P(1,2,-3)$
The direction ratios of OP are

$$a = (1-0) = 1$$
$$b = (2-0) = 2$$
$$c = (-3-0) = -3$$

The equation of the plane passing through the point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where, a, b and c are the direction ratios of normal.

Here, the direction ratios of normal are $1, 2$ and -3 and the point P is $(1, 2, -3)$.

Hence, the equation of the required plane is

$$\Rightarrow 1(x-1) + 2(y-2) - 3(z+3) = 0$$
$$\Rightarrow x + 2y - 3z - 14 = 0$$

Question 17:

Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0, \vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$.

Solution:

The equations of the given planes are

$$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0 \qquad \ldots(1)$$

$$\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0 \qquad \ldots(2)$$

The equation of the required plane is,

$$\left[\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5\right]=0$$

$$\vec{r}\left[(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\right]+(5\lambda-4)=0 \qquad \ldots(3)$$

The plane in equation (3) is perpendicular to the plane, $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

Therefore,

$$\Rightarrow 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$$

$$\Rightarrow 19\lambda-7=0$$

$$\Rightarrow \lambda=\frac{7}{19}$$

Putting $\lambda=\frac{7}{19}$ in equation (3), we get

$$\Rightarrow \vec{r}.\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right]\frac{-41}{19}=0$$

$$\Rightarrow \vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0 \qquad \ldots(4)$$

The Cartesian equation of this plane is given by

$$\Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0$$

$$\Rightarrow 33x+45y+50z-41=0$$

Question 18:

Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$.

Solution:

The equation of the given line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right) \qquad \ldots(1)$$

The equation of the given plane is

$$\vec{r}.\left(\hat{i} - \hat{j} + \hat{k}\right) = 5 \qquad \ldots(2)$$

Putting the value of r from equation (1) in equation (2), we get

$$\Rightarrow \left[2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)\right].\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$$

$$\Rightarrow \left[(3\lambda + 2)\hat{i} + (4\lambda - 1)\hat{j} + (2\lambda + 2)\hat{k}\right].\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$$

$$\Rightarrow (3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5$$

$$\Rightarrow \lambda = 0$$

Putting this value in equation (1), we get the equation of the line as $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$

This means that the position vector of the point of intersection of the line and plane is $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$

This shows that the point of intersection of the given line and plane is given by the coordinates $(2, -1, 2)$ and $(-1, -5, -10)$.

The required distance between the points $(2, -1, 2)$ and $(-1, -5, -10)$ is

$$d = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{9 + 16 + 144}$$

$$= \sqrt{169}$$

$$= 13$$

Question 19:

Find the vector equation of the line passing through $(1, 2, 3)$ and parallel to the planes $\vec{r}.\left(\hat{i} - \hat{j} + 2\hat{k}\right) = 5$ and $\vec{r}.\left(3\hat{i} + \hat{j} + \hat{k}\right) = 6$

Solution:

Let the required line be parallel to vector $\vec{b}$ given by,

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

The postion vector of the point $(1, 2, 3)$ is

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

The equation of line passing through $(1, 2, 3)$ and parallel to $\vec{b}$ is given by,

$$\Rightarrow \vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \vec{r}.\left(\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \qquad \ldots(1)$$

The equations of the given planes are

$$\vec{r}.\left(\hat{i} - \hat{j} + 2\hat{k}\right) = 5 \qquad \ldots(2)$$

$$\vec{r}.\left(3\hat{i} + \hat{j} + \hat{k}\right) = 6 \qquad \ldots(3)$$

The line in equation (1) and plane in equation (2) are parallel.
Therefore, the normal to the plane of equation (2) and the given line are perpendicular.

$$\Rightarrow \left(\hat{i} - \hat{j} + 2\hat{k}\right).\lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = 0$$

$$\Rightarrow \lambda\left(b_1 - b_2 + 2b_3\right) = 0$$

$$\Rightarrow b_1 - b_2 + 2b_3 = 0 \qquad \ldots(4)$$

Similarly,

$$\Rightarrow \left(3\hat{i} + \hat{j} + \hat{k}\right).\lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = 0$$

$$\Rightarrow \lambda\left(3b_1\hat{i} + b_2 + b_3\right) = 0$$

$$\Rightarrow 3b_1 + b_2 + b_3 = 0 \qquad \ldots(5)$$

From equation (4) and (5), we obtain

$$\Rightarrow \frac{b_1}{(-1)\times 1 - 1\times 2} = \frac{b_2}{2\times 3 - 1\times 1} = \frac{b_3}{1\times 1 - 3(-1)}$$

$$\Rightarrow \frac{b_1}{-3} = \frac{b_2}{5} = \frac{b_3}{4}$$

Thus,
The direction ratios of $\vec{b}$ are $-3, 5$ and 4.

Hence,

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$= -3\hat{i} + 5\hat{j} + 4\hat{k}$$

Putting, the value of $\vec{b}$ in eqation (1), we get

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-3\hat{i} + 5\hat{j} + 4\hat{k}\right)$$

Question 20:

Find the vector equation of the line passing through the point $(1, 2, -4)$ and perpendicular to the two lines: $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Solution:

Here,

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$$

The equation of the line passing through $(1,2,-4)$ and parallel to vector $\vec{b}$ is given by

$$\Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$$
$$\Rightarrow \vec{r} = \left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \qquad \ldots(1)$$

The equations of the lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \qquad \ldots(2)$$
$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \qquad \ldots(3)$$

Since, lines of the equations (1) and (2) are perpendicular to each other

$$\Rightarrow 3b_1 - 16b_2 + 7b_3 = 0 \qquad \ldots(4)$$

Also,

Lines (1) and (3) are perpendicular to each other

$$\Rightarrow 3b_1 + 8b_2 - 5b_3 = 0 \qquad \ldots(5)$$

From equations (4) and (5), we obtain

$$\Rightarrow \frac{b_1}{(-16)\times(-5) - 8\times 7} = \frac{b_2}{7\times 3 - 3\times(-5)} = \frac{b_3}{3\times 8 - 3\times(-16)}$$
$$\Rightarrow \frac{b_1}{24} = \frac{b_2}{36} = \frac{b_3}{72} \Rightarrow$$
$$\frac{b_1}{2} = \frac{b_2}{3} = \frac{b_3}{6}$$

Hence,

The direction ratios of $\vec{b}$ are $2,3$ and 6

Therefore,

$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Putting $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ in equation (1), we get

$$\Rightarrow \vec{r} = \left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

Question 21:

Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$.

Solution:

The equation of the plane having intercepts a, b, c with x, y, z axes respectively is given by,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

The distance p of the plane from the origin is given by,

$$p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a^2}\right)+\left(\frac{1}{b^2}\right)+\left(\frac{1}{c^2}\right)}}\right|$$

$$=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$p^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Hence, $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$ proved.

Question 22:

Distance between the two planes: $2x+3y+4z=4$ and $4x+6y+8z=12$ is

(A) 2 units (B) 4 units (C) 8 units (D) $\frac{2}{\sqrt{29}}$ units

Solution:

The equations of the planes are

$$\Rightarrow 2x+3y+4z=4 \qquad \ldots(1)$$

$$\Rightarrow 4x+6y+8z=12$$

$$\Rightarrow 2x+3y+4z=4 \qquad \ldots(2)$$

Since given planes are parallel, and we know that the distance between two parallel planes $ax+by+cz=d_1$ and $ax+by+cz=d_2$ is given by,

$$D=\left|\frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}}\right|$$
$$=\left|\frac{6-4}{\sqrt{(2)^2+(3)^2+(4)^2}}\right|$$
$$=\frac{2}{\sqrt{29}}$$

Hence, the distance between the given plane is $\frac{2}{\sqrt{29}}$ units.
Therefore, the correct answer is D.

Question 23:

The planes: $2x-y+4z=5$ and $5x-2.5y+10z=6$ are

(A) Perpendicular (B) Parallel (C) intersect y-axis (D) passes through $\left(0,0,\frac{5}{4}\right)$

Solution:

The equations of the planes are

$$2x-y+4z=5 \qquad \ldots(1)$$
$$5x-2.5y+10z=6 \qquad \ldots(2)$$

Here,

$$\frac{a_1}{a_2}=\frac{2}{5}$$
$$\frac{b_1}{b_2}=\frac{-1}{-2.5}=\frac{2}{5}$$
$$\frac{c_1}{c_2}=\frac{4}{10}=\frac{2}{5}$$

Therefore,

$$\Rightarrow \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Hence, the given planes are parallel.

Therefore, the correct answer is B.