9

Sequence and Series

Short Answer Type Questions

Q. 1 The first term of an AP is a and the sum of the first p terms is zero, show that the sum of its next q terms is $\dfrac{-a(p+q)q}{p-1}$.

Sol. Let the common difference of an AP is d.
According to the question,

$$S_p = 0$$

$$\Rightarrow \quad \frac{p}{2}[2a + (p-1)d] = 0 \qquad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$

$$\Rightarrow \quad 2a + (p-1)d = 0$$

$$\therefore \quad d = \frac{-2a}{p-1}$$

Now, sum of next q terms $= S_{p+q} - S_p = S_{p+q} - 0$

$$= \frac{p+q}{2}[2a + (p+q-1)d]$$

$$= \frac{p+q}{2}[2a + (p-1)d + qd]$$

$$= \frac{p+q}{2}\left[2a + (p-1)\cdot\frac{-2a}{p-1} + \frac{q(-2a)}{p-1}\right]$$

$$= \frac{p+q}{2}\left[2a + (-2a) - \frac{2aq}{p-1}\right]$$

$$= \frac{p+q}{2}\left[\frac{-2aq}{p-1}\right]$$

$$= \frac{-a(p+q)q}{(p-1)}$$

Q. 2 A man saved ₹ 66000 in 20 yr. In each succeeding year after the first year, he saved ₹ 200 more than what he saved in the previous year. How much did he save in the first year?

Sol. Let saved in first year ₹ a. Since, each succeeding year an increment ₹ 200 has made. So,it forms an AP whose

First term = a, common difference (d) = 200 and n = 20 yr

$\therefore \quad S_{20} = \frac{20}{2}[2a + (20-1)d] \qquad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$

$\Rightarrow \quad 66000 = 10\,[2a + 19d]$

$\Rightarrow \quad 66000 = 20a + 190d$

$\Rightarrow \quad 66000 = 20a + 190 \times 200$

$\Rightarrow \quad 20a = 66000 - 38000$

$\Rightarrow \quad 20a = 28000$

$\therefore \quad a = \frac{28000}{20} = 1400$

Hence, he saved ₹ 1400 in the first year.

Q. 3 A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹ 320 in the very next month and each month thereafter.

(i) Find his salary for the tenth month.

(ii) What is his total earnings during the first year?

Sol. Since, the man get a fixed increment of ₹ 320 each month.

Therefore, this forms an AP whose First term = 5200 and Common difference (d) = 320

(i) Salary for tenth month *i.e.*, for $n = 10$,

$a_{10} = a + (n-1)d$

$\Rightarrow \quad a_{10} = 5200 + (10-1) \times 320$

$\Rightarrow \quad a_{10} = 5200 + 9 \times 320$

$\therefore \quad a_{10} = 5200 + 2880$

$\therefore \quad a_{10} = 8080$

(ii) Total earning during the first year.

In a year there are 12 month *i.e.*, $n = 12$,

$S_{12} = \frac{12}{2}[2 \times 5200 + (12-1)320]$

$= 6\,[10400 + 11 \times 320]$

$= 6[10400 + 3520] = 6 \times 13920 = 83520$

Q. 4 If the p th and qth terms of a GP are q and p respectively, then show that its $(p+q)$th term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

Sol. Let the first term and common ratio of GP be a and r, respectively.

According to the question, pth term = q

$\Rightarrow \quad a \cdot r^{p-1} = q \qquad \ldots$(i)

and $\quad q$ th term = p

$\Rightarrow \quad ar^{q-1} = p \qquad \ldots$(ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{ar^{p-1}}{ar^{q-1}} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-1-q+1} = \frac{q}{p}$$

$$\Rightarrow \quad r^{p-q} = \frac{q}{p} \quad \Rightarrow \quad r = \left(\frac{q}{p}\right)^{\frac{1}{p-q}}$$

On substituting the value of r in Eq. (i), we get

$$a\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}} = q \quad \Rightarrow \quad a = \frac{q}{\left(\frac{q}{p}\right)^{\frac{p-1}{p-q}}} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

$$\therefore \quad (p+q)\text{th term, } T_{p+q} = a \cdot r^{p+q-1} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot (r)^{p+q-1}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \left[\left(\frac{q}{p}\right)^{\frac{1}{p-q}}\right]^{p+q-1} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \left(\frac{p}{q}\right)^{\frac{-(p+q-1)}{p-q}} = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q} - \frac{(p+q-1)}{p-q}}$$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1-p-q+1}{p-q}} = q \cdot \left(\frac{p}{q}\right)^{\frac{-q}{p-q}}$$

$$a = q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}}$$

Now, $(p+q)$th term *i.e.*, $a_{p+q} = ar^{p+q-1}$

$$= q \cdot \left(\frac{p}{q}\right)^{\frac{p-1}{p-q}} \cdot \left(\frac{q}{p}\right)^{\frac{p+q-1}{p-q}}$$

$$= q \cdot \frac{q^{\frac{p+q-1-p+1}{p-q}}}{p^{\frac{p+q-1-p+1}{p-q}}} = q \cdot \left(\frac{q^{\frac{q}{p-q}}}{p^{\frac{q}{p-q}}}\right)$$

Q. 5 A carpenter was hired to build 192 window frames. The first day he made five frames and each day, thereafter he made two more frames than he made the day before. How many days did it take him to finish the job?

Sol. Here, $a = 5$ and $d = 2$

Let he finished the job in n days.

Then,
$$S_n = 192$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\Rightarrow \quad 192 = \frac{n}{2}[2 \times 5 + (n-1)2]$$

$$\Rightarrow \quad 192 = \frac{n}{2}[10 + 2n - 2]$$

$$\Rightarrow \quad 192 = \frac{n}{2}[8 + 2n]$$
$$\Rightarrow \quad 192 = 4n + n^2$$
$$\Rightarrow \quad n^2 + 4n - 192 = 0$$
$$\Rightarrow \quad (n - 12)(n + 16) = 0$$
$$\Rightarrow \quad n = 12, -16 \qquad [\because n \neq -16]$$
$$\therefore \quad n = 12$$

Q. 6 **The sum of interior angles of a triangle is 180°. Show that the sum of the interior angles of polygons with 3, 4, 5, 6, ... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon.**

Sol. We know that, sum of interior angles of a polygon of side $n = (2n - 4) \times 90° = (n - 2) \times 180°$

Sum of interior angles of a polygon with sides 3 is 180.

Sum of interior angles of polygon with side 4 = $(4 - 2) \times 180° = 360°$

Similarly, sum of interior angles of polygon with side 5, 6, 7... are 540°, 720°, 900°,...

The series will be 180°, 360° 540°, 720°, 900°,...

Here, $a = 180°$

and $d = 360° - 180° = 180°$

Since, common difference is same between two consecutive terms of the series.

So, it form an AP.

We have to find the sum of interior angles of a 21 sides polygon.

It means, we have to find the 19th term of the above series.

$$\therefore \quad a_{19} = a + (19 - 1)d$$
$$= 180 + 18 \times 180 = 3420$$

Q. 7 **A side of an equilateral triangle is 20 cm long. A second equilateral triangle is inscribed in it by joining the mid-points of the sides of the first triangle. The process is continued as shown in the accompanying diagram. Find the perimeter of the sixth inscribed equilateral triangle.**

Sol. Side of equilateral $\Delta ABC = 20$ cm. By joining the mid-points of this triangle, we get another equilateral triangle of side equal to half of the length of side of ΔABC.

Continuing in this way, we get a set of equilateral triangles with side equal to half of the side of the previous triangle.

$\therefore$ Perimeter of first triangle $= 20 \times 3 = 60$ cm

Perimeter of second triangle $= 10 \times 3 = 30$ cm

Perimeter of third triangle $= 5 \times 3 = 15$ cm

Now, the series will be 60, 30, 15,...

Here, $a = 60$

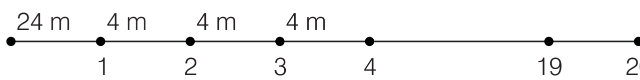
$$\therefore \quad r = \frac{30}{60} = \frac{1}{2} \qquad \left[\because \frac{\text{second term}}{\text{first term}} = r\right]$$

We have, to find perimeter of sixth inscribed triangle. It is the sixth term of the series.

$$\therefore \quad a_6 = ar^{6-1} \qquad [\because a_n = ar^{n-1}]$$
$$= 60 \times \left(\frac{1}{2}\right)^5 = \frac{60}{32} = \frac{15}{8} \text{ cm}$$

Q. 8 In a potato race 20 potatoes are placed in a line at intervals of 4 m with the first potato 24 m from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes?

Sol. According to the given information, we have following diagram.



Distance travelled to bring first potato $= 24 + 24 = 2 \times 24 = 48$ m

Distance travelled to bring second potato $= 2(24 + 4) = 2 \times 28 = 56$ m

Distance travelled to bring third potato $= 2(24 + 4 + 4) = 2 \times 32 = 64$ m

Then, the series of distances are 48, 56, 64,...

Here, $a = 48,$

$d = 56 - 48 = 8$

and $n = 20$

To find the total distance that he run in bringing back all potatoes, we have to find the sum of 20 terms of the above series.

$\therefore \quad S_{20} = \frac{20}{2}[2 \times 48 + 19 \times 8] \qquad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$

$= 10[96 + 152]$

$= 10 \times 248 = 2480$ m

Q. 9 In a cricket tournament 16 school teams participated. A sum of ₹ 8000 is to be awarded among themselves as prize money. If the last placed team is awarded ₹ 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

Sol. Let the first place team got ₹ a.

Since, award money increases by the same amount for successive finishing places. Therefore series is an AP.

Let the constant amount be d.

Here, $l = 275, n = 16$ and $S_{16} = 8000$

$\therefore \quad l = a + (n -)d$

$\Rightarrow \quad l = a + (16 - 1)(-d)$

[we take common difference (–ve) because series is decreasing]

$\Rightarrow \quad 275 = a - 15d \qquad \text{...(i)}$

and $\quad S_{16} = \frac{16}{2}[2a + (n-1)\cdot(-d)]$

$\Rightarrow \quad 8000 = 8[2a + (16-1)(-d)]$

$\Rightarrow \quad 8000 = 8[2a - 15d]$

$\Rightarrow \quad 1000 = 2a - 15d \qquad \text{...(ii)}$

On subtracting Eq. (i) from Eq. (ii), we get

$(2a - 15d) - (a - 15d) = 1000 - 275$

$\Rightarrow \quad 2a - 15d - a + 15d = 725$

$\therefore \quad a = 725$

Hence, first place team receive ₹ 725.

Q. 10 If $a_1, a_2, a_3, \ldots, a_n$ are in AP, where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

Sol. Since, $a_1, a_2, a_3, \ldots, a_n$ are in AP.

$\Rightarrow$ $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$ [common difference]

If $a_2 - a_1 = d$, then $(\sqrt{a_2})^2 - (\sqrt{a_1})^2 = d$

$\Rightarrow$ $(\sqrt{a_2} - \sqrt{a_1})(\sqrt{a_2} + \sqrt{a_1}) = d$

$\Rightarrow$ $$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} = \frac{\sqrt{a_2}-\sqrt{a_1}}{d}$$

Similarly, $$\frac{1}{\sqrt{a_2}+\sqrt{a_3}} = \frac{\sqrt{a_3}-\sqrt{a_2}}{d}$$

... ...

... ...

... ...

$$\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d}$$

On adding these terms, we get

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{1}{d}[\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots+\sqrt{a_n}-\sqrt{a_{n-1}}]$$ [using above relations]

$$= \frac{1}{d}[\sqrt{a_n}-\sqrt{a_1}] \quad \ldots(i)$$

Again, $a_n = a_1 + (n-1)d$ [$\because T_n = a + (n-1)d$]

$\Rightarrow$ $a_n - a_1 = (n-1)d$

$\Rightarrow$ $(\sqrt{a_n})^2 - (\sqrt{a_1})^2 = (n-1)d$

$\Rightarrow$ $(\sqrt{a_n} - \sqrt{a_1})(\sqrt{a_n} + \sqrt{a_1}) = (n-1)d \Rightarrow \sqrt{a_n} - \sqrt{a_1} = \dfrac{(n-1)d}{\sqrt{a_n}+\sqrt{a_1}}$

On putting this value in Eq. (i), we get

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{(n-1)d}{d(\sqrt{a_n}+\sqrt{a_1})} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$ **Hence proved.**

Q. 11 Find the sum of the series

$(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) + \ldots$ to (i) n terms. (ii) 10 terms.

Sol. Given series, $(3^3 - 2^3) + (5^3 - 4^3) + (7^3 - 6^3) +$ ...(i)

$= (3^3 + 5^3 + 7^3 + \ldots) - (2^3 + 4^3 + 6^3 + \ldots)$

Let T_n be the nth term of the series (i),

then T_n = (nth term of $3^3, 5^3, 7^3, \ldots$) – (nth term of $2^3, 4^3, 6^3, \ldots$) $= (2n+1)^3 - (2n)^3$

$= (2n+1-2n)[(2n+1)^2 + (2n+1)2n + (2n)^2]$ [$\because a^3 - b^3 = (a-b)(a^2+ab+b^2)$]

$= [4n^2 + 1 + 4n + 4n^2 + 2n + 4n^2] = [12n^2 + 6n] + 1$

(i) Let S_n denote the sum of n term of series (i).

Then, $$S_n = \Sigma T_n = \Sigma(12n^2 + 6n)$$
$$= 12\Sigma n^2 + 6\Sigma n + \Sigma n$$
$$= 12 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$$
$$= 2n(n+1)(2n+1) + 3n(n+1) + n$$
$$= 2n(n+1)(2n+1) + 3n(n+1) + n$$
$$= (2n^2 + 2n)(2n+1) + 3n^2 + 3n + n$$
$$= 4n^3 + 2n^2 + 4n^2 + 2n + 3n^2 + 3n + n$$
$$= 4n^3 + 9n^2 + 6n$$

(ii) Sum of 10 terms, $$S_{10} = 4 \times (10)^3 + 9 \times (10)^2 + 6 \times 10$$
$$= 4 \times 1000 + 9 \times 100 + 60$$
$$= 4000 + 900 + 60 = 4960$$

Q. 12 Find the rth term of an AP sum of whose first n terms is $2n + 3n^2$.

Sol. Given that, sum of n terms of an AP,
$$S_n = 2n + 3n^2$$
$$T_n = S_n - S_{n-1}$$
$$= (2n + 3n^2) - [2(n-1) + 3(n-1)^2]$$
$$= (2n + 3n^2) - [2n - 2 + 3(n^2 + 1 - 2n)]$$
$$= (2n + 3n^2) - (2n - 2 + 3n^2 + 3 - 6n)$$
$$= 2n + 3n^2 - 2n + 2 - 3n^2 - 3 + 6n$$
$$= 6n - 1$$
$\therefore$ rth term $T_r = 6r - 1$

Long Answer Type Questions

Q. 13 If A is the arithmetic mean and G_1, G_2 be two geometric mean between any two numbers, then prove that $2A = \dfrac{G_1^2}{G_2} + \dfrac{G_2^2}{G_1}$.

Sol. Let the numbers be a and b.

Then, $$A = \frac{a+b}{2}$$
$\Rightarrow$ $$2A = a + b \quad \ldots \text{(i)}$$
and G_1, G_2 be geometric mean between a and b, then a, G_1, G_2, b are in GP.

Let r be the common ratio.

Then, $$b = ar^{4-1} \quad [\because a_n = ar^{n-1}]$$
$\Rightarrow$ $$b = ar^3 \Rightarrow \frac{b}{a} = r^3$$
$\therefore$ $$r = \left(\frac{b}{a}\right)^{1/3}$$

Now, $$G_1 = ar = a\left(\frac{b}{a}\right)^{1/3} \qquad \left[\because r = \left(\frac{b}{a}\right)^{1/3}\right]$$

and $$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/3}$$

$$\text{RHS} = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{\left[a\left(\frac{b}{a}\right)^{1/3}\right]^2}{a\left(\frac{b}{a}\right)^{2/3}} + \frac{\left[a\left(\frac{b}{a}\right)^{2/3}\right]^2}{a\left(\frac{b}{a}\right)^{1/3}}$$

$$= \frac{a^2\left(\frac{b}{a}\right)^{2/3}}{a\left(\frac{b}{a}\right)^{2/3}} + \frac{a^2\left(\frac{b}{a}\right)^{4/3}}{a\left(\frac{b}{a}\right)^{1/3}}$$

$$= a + a\left(\frac{b}{a}\right) = a + b = 2A \qquad \text{[using Eq. (i)]}$$

$$= \text{LHS}$$

Q. 14 If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP whose common difference is d, show that $$\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}.$$

Sol. Since, $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP.

$$\Rightarrow \qquad \theta_2 - \theta_1 = \theta_3 - \theta_2 = \cdots = \theta_n - \theta_{n-1} = d \qquad \ldots\text{(i)}$$

Now, we have to prove

$$\sec q_1 \sec q_2 + \sec q_2 \sec q_3 + \cdots + \sec q_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}$$

or it can be written as

$$\sin d\,[\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \cdots + \sec\theta_{n-1}\sec\theta_n] = \tan\theta_n - \tan\theta_1$$

Now, taking only first term of LHS

$$\sin d \sec\theta_1 \sec\theta_2 = \frac{\sin d}{\cos\theta_1 \cos\theta_2} = \frac{\sin(\theta_2 - \theta_1)}{\cos\theta_1\cos\theta_2} \qquad \text{[from Eq. (i)]}$$

$$= \frac{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2}$$

$$[\because \sin(A - B) = \sin A\cdot\cos B - \cos A\cdot\sin B]$$

$$= \frac{\sin\theta_2\cos\theta_1}{\cos\theta_1\cos\theta_2} - \frac{\cos\theta_2\sin\theta_1}{\cos\theta_1\cos\theta_2} = \tan\theta_2 - \tan\theta_1$$

Similarly, we can solve other terms which will be $\tan\theta_3 - \tan\theta_2, \tan\theta_4 - \tan\theta_3, \cdots$

$$\therefore \qquad \text{LHS} = \tan\theta_2 - \tan\theta_1 + \tan\theta_3 - \tan\theta_2 + \cdots + \tan\theta_n - \tan\theta_{n-1}$$

$$= -\tan\theta_1 + \tan\theta_n = \tan\theta_n - \tan\theta_1$$

$$= \text{RHS}$$

Hence proved.

Q. 15 If the sum of p terms of an AP is q and the sum of q terms is p, then show that the sum of $p+q$ terms is $-(p+q)$. Also, find the sum of first $p-q$ terms (where, $p>q$).

Sol. Let first term and common difference of the AP be a and d, respectively.

Then, $$S_p = q$$

$\Rightarrow$ $$\frac{p}{2}[2a+(p-1)d] = q$$

$$2a+(p-1)d = \frac{2q}{p} \quad \text{...(i)}$$

and $$S_q = p$$

$\Rightarrow$ $$\frac{q}{2}[2a+(q-1)d] = p$$

$\Rightarrow$ $$2a+(q-1)d = \frac{2p}{q} \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2a+(p-1)d-2a-(q-1)d = \frac{2q}{p} - \frac{2p}{q}$$

$\Rightarrow$ $$[(p-1)-(q-1)]\,d = \frac{2q^2-2p^2}{pq}$$

$\Rightarrow$ $$[p-1-q+1]\,d = \frac{2(q^2-p^2)}{pq}$$

$\Rightarrow$ $$(p-q)d = \frac{2(q^2-p^2)}{pq}$$

$\therefore$ $$d = \frac{-2(p+q)}{pq} \quad \text{...(iii)}$$

On substituting the value of d in Eq. (i), we get

$$2a+(p-1)\left(\frac{-2(p+q)}{pq}\right) = \frac{2q}{p}$$

$\Rightarrow$ $$2a = \frac{2q}{p} + \frac{2(p+q)(p-1)}{pq}$$

$\Rightarrow$ $$a = \left[\frac{q}{p} + \frac{(p+q)(p-1)}{pq}\right] \quad \text{...(iv)}$$

Now, $$S_{p+q} = \frac{p+q}{2}[2a+(p+q-1)d]$$

$$= \frac{p+q}{2}\left[\frac{2q}{p} + \frac{2(p+q)(p-1)}{pq} - \frac{(p+q-1)2(p+q)}{pq}\right]$$

$$= (p+q)\left[\frac{q}{p} + \frac{(p+q)(p-1)-(p+q-1)(p+q)}{pq}\right]$$

$$= (p+q)\left[\frac{q}{p} + \frac{(p+q)(p-1-p-q+1)}{pq}\right]$$

$$= p+q\left[\frac{q}{p} - \frac{p+q}{p}\right] = (p+q)\left[\frac{q-p-q}{p}\right]$$

$$S_{p+q} = -(p+q)$$

$$S_{p-q} = \frac{p-q}{2}[2a+(p-q-1)d]$$

$$= \frac{p-q}{2}\left[\frac{2q}{p} + \frac{2(p+q)(p-1)}{pq} - \frac{(p-q-1)2\,(p+q)}{pq}\right]$$

$$= (p-q)\left[\frac{q}{p} + \frac{p+q(p-1-p+q+1)}{pq}\right]$$

$$= (p-q)\left[\frac{q}{p} + \frac{(p+q)q}{pq}\right]$$

$$= (p-q)\left[\frac{q}{p} + \frac{p+q}{p}\right] = (p-q)\frac{(p+2q)}{p}$$

Q. 16 If pth, qth and rth terms of an AP and GP are both and c respectively, then show that $a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1.$

Sol. Let A, d are the first term and common difference of AP and x, R are the first term and common ratio of GP, respectively.

According to the given condition,

$$A + (p-1)d = a \quad \text{...(i)}$$

$$A + (q-1)d = b \quad \text{...(ii)}$$

$$A + (r-1)d = c \quad \text{...(iii)}$$

and

$$a = xR^{p-1} \quad \text{...(iv)}$$

$$b = xR^{q-1} \quad \text{...(v)}$$

$$c = xR^{r-1} \quad \text{...(vi)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$d(p-1-q+1) = a-b$$

$$\Rightarrow \quad a-b = d(p-q) \quad \text{...(vii)}$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$d(q-1-r+1) = b-c$$

$$\Rightarrow \quad b-c = d\,(q-r) \quad \text{...(viii)}$$

On subtracting Eq. (i) from Eq. (iii), we get

$$d(r-1-p+1) = c-a$$

$$\Rightarrow \quad c-a = d\,(r-p) \quad \text{...(ix)}$$

Now, we have to prove $a^{b-c}\, b^{c-a}\, c^{a-b} = 1$

$$\text{Taking LHS} = a^{b-c}\, b^{c-a}\, c^{a-b}$$

Using Eqs. (iv), (v), (vi) and (vii), (viii), (ix),

$$\text{LHS} = (xR^{p-1})^{d\,(q-r)}\,(xR^{q-1})^{d\,(r-p)}\,(xR^{r-1})^{d\,(p-q)}$$

$$= x^{d(q-r)+d(r-p)+d(p-q)}\,R^{(p-1)d(q-r)+(q-1)d\,(r-p)+(r-1)d\,(p-q)}$$

$$= x^{d(q-r+r-p+p-q)}$$

$$R^{d\,(pq-pr-q+r+qr-pq-r+p+rp-rq-p+q)} = x^0\,R^0 = 1$$

$$= \text{RHS}$$

Hence proved.

Objective Type Questions

Q. 17 If the sum of n terms of an AP is given by $S_n = 3n + 2n^2$, then the common difference of the AP is

(a) 3 (b) 2 (c) 6 (d) 4

Sol. *(d)* Given, $S_n = 3n + 2n^2$

First term of the AP,

$\therefore \quad T_1 = 3 \times 1 + 2\,(1)^2 = 3 + 2 = 5$

and $\quad T_2 = S_2 - S_1$

$= [3 \times 2 + 2 \times (2)^2] - [3 \times 1 + 2 \times (1)^2]$

$= 14 - 5 = 9$

$\therefore$ Common difference $(d) = T_2 - T_1 = 9 - 5 = 4$

Q. 18 If the third term of GP is 4, then the product of its first 5 terms is

(a) 4^3 (b) 4^4 (c) 4^5 (d) None of these

Sol. *(c)* It is given that, $T_3 = 4$

Let a and r the first term and common ratio, respectively.

Then, $\quad ar^2 = 4 \quad$ ...(i)

Product of first 5 terms $= a \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4$

$= a^5 r^{10} = (ar^2)^5 = (4)^5 \quad$ [using Eq. (i)]

Q. 19 If 9 times the 9th term of an AP is equal to 13 times the 13th term, then the 22nd term of the AP is

(a) 0 (b) 22 (c) 198 (d) 220

Sol. *(a)* Let the first term be a and common difference be d.

According to the question, $\quad 9 \cdot T_9 = 13 \cdot T_{13}$

$\Rightarrow \quad 9\,(a + 8d) = 13\,(a + 12d)$

$\Rightarrow \quad 9a + 72d = 13a + 156d$

$\Rightarrow \quad (9a - 13a) = 156d - 72d$

$\Rightarrow \quad -4a = 84d$

$\Rightarrow \quad a = -21d$

$\Rightarrow \quad a + 21d = 0 \quad$ ...(i)

$\therefore \quad$ 22nd term *i.e.*, $T_{22} = [a + 21d]$

$T_{22} = 0 \quad$ [using Eq. (i)]

Q. 20 If x, $2y$ and $3z$ are in AP where the distinct numbers x, y and z are in GP, then the common ratio of the GP is

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

Sol. *(b)* Given, $x, 2y$ and $3z$ are in AP.

Then, $\quad 2y = \dfrac{x + 3z}{2}$

$\Rightarrow \qquad y = \frac{x + 3z}{4}$

$\Rightarrow \qquad 4y = x + 3z \qquad$ ...(i)

and x, y, z are in GP.

Then, $\qquad \frac{y}{x} = \frac{z}{y} = \lambda$

$\Rightarrow \qquad y = x\lambda$ and $z = \lambda y = \lambda^2 x$

On substituting these values in Eq. (i), we get

$$4(x\lambda) = x + 3(\lambda^2 x)$$

$\Rightarrow \qquad 4\lambda x = x + 3\lambda^2 x$

$\Rightarrow \qquad 4\lambda = 1 + 3\lambda^2$

$\Rightarrow \qquad 3\lambda^2 - 4\lambda + 1 = 0$

$\Rightarrow \qquad (3\lambda - 1)(\lambda - 1) = 0$

$\therefore \qquad \lambda = \frac{1}{3}, \lambda = 1$

Q. 21 If in an AP, $S_n = q\,n^2$ and $S_m = qm^2$, where S_r denotes the sum of r terms of the AP, then S_q equals to

(a) $\frac{q^3}{2}$ (b) mnq (c) q^3 (d) $(m + n)\,q^2$

Sol. *(c)* Given, $S_n = qn^2$ and $S_m = qm^2$

$\therefore \quad S_1 = q, S_2 = 4q, S_3 = 9q$ and $S_4 = 16q$

Now, $\qquad T_1 = q$

$\therefore \qquad T_2 = S_2 - S_1 = 4q - q = 3q$

$\qquad T_3 = S_3 - S_2 = 9q - 4q = 5q$

$\qquad T_4 = S_4 - S_3 = 16q - 9q = 7q$

So, the series is $q, 3q, 5q, 7q, \ldots$

Here, $\qquad a = q$ and $d = 3q - q = 2q$

$\therefore \qquad S_q = \frac{q}{2}[2 \times q + (q - 1)2q]$

$$= \frac{q}{2} \times [2q + 2q^2 - 2q] = \frac{q}{2} \times 2q^2 = q^3$$

Q. 22 Let S_n denote the sum of the first n terms of an AP, if $S_{2n} = 3S_n$, then $S_{3n} : S_n$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10

Sol. *(b)* Let first term be a and common difference be d.

Then, $\qquad S_n = \frac{n}{2}[2a + (n - 1)d] \qquad$ ...(i)

$\therefore \qquad S_{2n} = \frac{2n}{2}[2a + (2n - 1)d]$

$\qquad S_{2n} = n[2a + (2n - 1)d] \qquad$ ...(ii)

$\qquad S_{3n} = \frac{3n}{2}[2a + (3n - 1)d] \qquad$ ...(iii)

According to the question, $S_{2n} = 3S_n$

$\Rightarrow \quad n[2a + (2n-1)d] = 3\frac{n}{2}[2a + (n-1)d]$

$\Rightarrow \quad 4a + (4n-2)d = 6a + (3n-3)d$

$\Rightarrow \quad -2a + (4n - 2 - 3n + 3)d = 0$

$\Rightarrow \quad -2a + (n+1)d = 0$

$\Rightarrow \quad d = \frac{2a}{n+1}$ ...(iv)

Now, $\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a + (3n-1)d]}{\frac{n}{2}[2a + (n-1)d]} = \frac{6a + (9n-3)\frac{2a}{n+1}}{2a + (n-1)\frac{2a}{n+1}}$

$= \frac{6an + 6a + 18an - 6a}{2an + 2a + 2an - 2a}$

$= \frac{24\,an}{4\,an} = \frac{S_{3n}}{S_n} = 6$

Q. 23 The minimum value of $4^x + 4^{1-x}$, $x \in R$ is

(a) 2 (b) 4 (c) 1 (d) 0

Sol. *(b)* We know that, AM $\geq$ GM

$\Rightarrow \quad \frac{4^x + 4^{1-x}}{2} \geq \sqrt{4^x \cdot 4^{1-x}}$

$\Rightarrow \quad 4^x + 4^{1-x} \geq 2\sqrt{4}$

$\Rightarrow \quad 4^x + 4^{1-x} \geq 2 \cdot 2$

$\Rightarrow \quad 4^x + 4^{1-x} \geq 4$

Q. 24 Let S_n denote the sum of the cubes of the first n natural numbers and s_n denote the sum of the first n natural numbers, then $\sum_{r=1}^{n} \frac{S_r}{S_4}$ equals to

(a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)}{2}$

(c) $\frac{n^2 + 3n + 2}{2}$ (d) None of these

Sol. *(a)* $\sum_{r=1}^{n} \frac{S_r}{S_r} = \frac{S_1}{s_1} + \frac{S_2}{s_2} + \frac{S_3}{s_3} + \ldots + \frac{S_n}{s_n}$

Let T_n be the nth term of the above series.

$\therefore \quad T_n = \frac{S_n}{s_n} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{\frac{n(n+1)}{2}}$

$= \frac{n(n+1)}{2} = \frac{1}{2}[n^2 + n]$

$\therefore$ Sum of the above series $= \Sigma T_n = \frac{1}{2}[\Sigma n^2 + \Sigma n]$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{1}{2}\cdot\frac{n(n+1)}{2}\left[\frac{(2n+1)}{3} + 1\right]$$

$$= \frac{1}{4}n(n+1)\left[\frac{2n+1+3}{3}\right] = \frac{1}{4\times 3}n(n+1)(2n+4)$$

$$= \frac{1}{12}n(n+1)(2n+4) = \frac{1}{6}n(n+1)(n+2)$$

Q. 25 If t_n denotes the nth term of the series $2 + 3 + 6 + 11 + 18 + \ldots$, then t_{50} is

(a) $49^2 - 1$ (b) 49^2 (c) $50^2 + 1$ (d) $49^2 + 2$

Sol. ***(d)*** Let S_n be sum of the series $2 + 3 + 6 + 11 + 18 + \ldots + t_{50}$.

$\therefore \quad S_n = 2 + 3 + 6 + 11 + 18 + \ldots + t_{50}$ ...(i)

and $\quad S_n = 0 + 2 + 3 + 6 + 11 + 18 + \ldots + t_{49} + t_{50}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$0 = 2 + 1 + 3 + 5 + 7 + \cdots + t_{50}$$

$\Rightarrow \quad t_{50} = 2 + 1 + 3 + 5 + 7 + \cdots$ upto 49 terms

$\therefore \quad t_{50} = 2 + [1 + 3 + 5 + 7 + \cdots$ upto 49 terms$]$

$$= 2 + \frac{49}{2}[2\times 1 + 48 \times 2]$$

$$= 2 + \frac{49}{2}\times[2 + 96]$$

$$= 2 + [49 + 49 \times 48]$$

$$= 2 + 49 \times 49 = 2 + (49)^2$$

Q. 26 The lengths of three unequal edges of a rectangular solid block are in GP. If the volume of the block is 216 cm^3 and the total surface area is 252 cm^2, then the length of the longest edge is

(a) 12 cm (b) 6 cm (c) 18 cm (d) 3 cm

Sol. ***(a)*** Let the length, breadth and height of rectangular solid block is $\frac{a}{r}$, a and ar, respectively.

$\therefore \quad$ Volume $= \frac{a}{r}\times a \times ar = 216 \text{ cm}^3$

$\Rightarrow \quad a^3 = 216 \Rightarrow a^3 = 6^3$

$\therefore \quad a = 6$

$$\text{Surface area} = 2\left(\frac{a^2}{r} + a^2 r + a^2\right) = 252$$

$\Rightarrow \quad 2a^2\left(\frac{1}{r} + r + 1\right) = 252$

$\Rightarrow \quad 2 \times 36\left(\frac{1 + r^2 + r}{r}\right) = 252$

$\Rightarrow \quad \frac{1 + r^2 + r}{r} = \frac{252}{2 \times 36}$

$$\Rightarrow \quad 1 + r^2 + r = \frac{126}{36} r \Rightarrow 1 + r^2 + r = \frac{21}{6} r$$

$$\Rightarrow \quad 6 + 6r^2 + 6r = 21r \Rightarrow 6r^2 - 15r + 6 = 0$$

$$\Rightarrow \quad 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0$$

$$\therefore \quad r = \frac{1}{2}, 2$$

For $r = \frac{1}{2}$:

$$\text{Length} = \frac{a}{r} = \frac{6 \times 2}{1} = 12$$

$$\text{Breadth} = a = 6$$

$$\text{Height} = ar = 6 \times \frac{1}{2} = 3$$

For $r = 2$:

$$\text{Length} = \frac{a}{r} = \frac{6}{2} = 3$$

$$\text{Breadth} = a = 6$$

$$\text{Height} = ar = 6 \times 2 = 12$$

Fillers

Q. 27 If a, b and c are in GP, then the value of $\frac{a - b}{b - c}$ is equal to ....

Sol. Given that, a, b and c are in GP.

Then, $$\frac{b}{a} = \frac{c}{b} = r \quad \text{[constant]}$$

$$\Rightarrow \quad b = ar \Rightarrow c = br$$

$$\therefore \quad \frac{a - b}{b - c} = \frac{a - ar}{ar - br} = \frac{a(1 - r)}{r(a - b)} = \frac{a(1 - r)}{r(a - ar)}$$

$$= \frac{a(1 - r)}{ar(1 - r)} = \frac{1}{r}$$

$$\therefore \quad \frac{a - b}{b - c} = \frac{1}{r} = \frac{a}{b} \text{ or } \frac{b}{c}$$

Q. 28 The sum of terms equidistant from the beginning and end in an AP is equal to ....

Sol. Let AP be $a, a + d, a + 2d \cdots a + (n - 1)d$

$$\therefore \quad a_1 + a_n = a + a + (n - 1)d$$

$$= 2a + (n - 1)d \quad \text{...(i)}$$

Now, $$a_2 + a_{n-1} = (a + d) + [a + (n - 2)d]$$

$$= 2a + (n - 1)d$$

$$a_2 + a_{n-1} = a_1 + a_n \quad \text{[using Eq. (i)]}$$

$$a_3 + a_{n-2} = (a + 2d) + [a + (n - 3)d]$$

$$= 2a + (n - 1)d$$

$$= a_1 + a_n \quad \text{[using Eq. (i)]}$$

Follow this pattern, we see that the sum of terms equidistant from the beginning and end in an AP is equal to [first term + last term].

Q. 29 The third term of a GP is 4, the product of the first five terms is .....

Sol. It is given that, $T_3 = 4$

Let a and r the first term and common ration, respectively.

Then, $ar^2 = 4$ ...(i)

Product of first 5 terms $= ar \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4$

$= a^5 r^{10} = (ar^2)^5 = (4)^5$ [using Eq. (i)]

True/False

Q. 30 Two sequences cannot be in both AP and GP together.

Sol. ***False***

Consider an AP $a, a + d, a + 2d, \dots$

Now, $$\frac{a_2}{a_1} = \frac{a + d}{a} \neq \frac{a + 2d}{a + d}$$

Thus, AP is not a GP.

Q. 31 Every progression is a sequence but the converse, *i.e.*, every sequence is also a progression need not necessarily be true.

Sol. ***True***

Consider the progression $a, a + d, a + 2d, \dots$

and sequence of prime number 2, 3, 5, 7, 11,...

Clearly, progression is a sequence but sequence is not progression because it does not follow a specific pattern.

Q. 32 Any term of an AP (except first) is equal to half the sum of terms which are equidistant from it.

Sol. ***True***

Consider an AP $a, a + d, a + 2d, \dots$

Now, $$a_2 + a_4 = a + d + a + 3d$$

$$= 2a + 4d = 2a_3$$

$\Rightarrow$ $$a_3 = \frac{a_2 + a_4}{2}$$

Again, $$\frac{a_3 + a_5}{2} = \frac{a + 2d + a + 4d}{2} = \frac{2a + 6d}{2}$$

$$= a + 3d = a_4$$

Hence, the statement is true.

Q. 33 The sum or difference of two GP, is again a GP.

Sol. ***False***

Let two GP are $a, ar_1, ar_1^2, ar_2^3, \dots$ and $b, br_2, br_2^2, br_2^3, \dots$

Now, sum of two GP $a + b, (ar_1 + br_2), (ar_1^2 + br_2^2), \dots$

Now, $$\frac{T_2}{T_1} = \frac{ar_1 + br_2}{a + b} \text{ and } \frac{T_3}{T_2} = \frac{ar_1^2 + br_2^2}{ar_1 + br_2}$$

$\therefore$ $$\frac{T_2}{T_1} \neq \frac{T_3}{T_2}$$

Again, difference of two GP is $a - b, ar_1 - br_2, ar_1^2 - br_2^2, \ldots$

Now, $\frac{T_2}{T_1} = \frac{ar_1 - br_2}{a - b}$ and $\frac{T_3}{T_2} = \frac{ar_1^2 - br_2^2}{ar_1 - br_2}$

$\therefore \quad \frac{T_2}{T_1} \neq \frac{T_3}{T_2}$

So, the sum or difference of two GP is not a GP.
Hence, the statement is false.

Q. 34 If the sum of n terms of a sequence is quadratic expression, then it always represents an AP.

Sol. ***False***

Let $S_n = an^2 + bn + c$

$S_1 = a + b + c$

$a_1 = a + b + c$

$S_2 = 4a + 2b + c$

$\therefore \quad a_2 = S_2 - S_1$

$= 4a + 4b + c - (a + b + c) = 3a + b$

$S_3 = 9a + 3b + c$

$\therefore \quad a_3 = S_3 - S_2 = 5a + b$

Now, $a_2 - a_1 = (3a + b) - (a + b + c) = 2a - c$

$a_3 - a_2 = (5a + b) - (3a + b) = 2a$

Now, $a_2 - a_1 \neq a_3 - a_2$

Hence, the statement is false.

Matching The Columns

Q. 35 Match the following.

	Column I		Column II
(i)	$4, 1, \frac{1}{4}, \frac{1}{16}$	(a)	AP
(ii)	2, 3, 5, 7	(b)	Sequence
(iii)	$13, 8, 3, -2, -7$	(c)	GP

Sol. (i) $4, 1, \frac{1}{4}, \frac{1}{16}$

$\Rightarrow \quad \frac{T_2}{T_1} = \frac{1}{4} \Rightarrow \frac{T_3}{T_2} = \frac{1}{4} \Rightarrow \frac{T_4}{T_3} = \frac{1/16}{1/4} = \frac{1}{4}$

Hence, it is a GP.

(ii) 2, 3, 5, 7

$\because \quad T_2 - T_1 = 3 - 2 = 1$

$T_3 - T_2 = 5 - 3 = 2$

$\because \quad T_2 - T_1 \neq T_3 - T_2$

Hence, it is not an AP.

Again, $\frac{T_2}{T_1} = 3/2 \Rightarrow \frac{T_3}{T_2} = 5/3$

$\because$ $\frac{T_2}{T_1} \neq \frac{T_3}{T_2}$

It is not a GP.

Hence, it is a sequence.

(iii) 13, 8, 3, – 2, – 7

$$T_2 - T_1 = 8 - 13 = -5$$
$$T_3 - T_2 = 3 - 8 = -5$$

$\because$ $T_2 - T_1 = T_3 - T_2$

Hence, it is an AP.

Q. 36 Match the following.

	Column I		Column II
(i)	$1^2 + 2^2 + 3^2 + \cdots + n^2$	(a)	$\left[\frac{n(n+1)}{2}\right]^2$
(ii)	$1^3 + 2^3 + 3^3 + \cdots + n^3$	(b)	$n(n+1)$
(iii)	$2 + 4 + 6 + \cdots + 2n$	(c)	$\frac{n(n+1)(2n+1)}{6}$
(iv)	$1 + 2 + 3 + \cdots + n$	(d)	$\frac{n(n+1)}{2}$

Sol. (i) $1^2 + 2^2 + 3^2 + \cdots + n^2$

Consider the identity, $(k+1)^3 - k^3 = 3k^2 + 3k + 1$

On putting $k = 1, 2, 3, \ldots, (n-1), n$ successively, we get

$$2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$$
$$3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$$
$$4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots$$
$$n^3 - (n-1)^3 = 3 \cdot (n-1)^2 + 3 \cdot (n-1) + 1$$
$$(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$$

Adding columnwise, we get

$$n^3 + 3n^2 + 3n = 3\left(\sum_{r=1}^{n} r^2\right) + 3\,\frac{n(n+1)}{2} + n \qquad \left[\because \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$$

$$\Rightarrow \quad 3\left(\sum_{r=1}^{n} r^2\right) = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \quad \left(\sum_{r=1}^{n} r^2\right) = \frac{2n^3 + 3n^2 + n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(ii) $1^3 + 2^3 + 3^3 + \cdots + n^3$

Consider the identity $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$

On putting $k = 1, 2, 3, \cdots (n-1), n$ successively, we get

$$2^4 - 1^4 = 4 \cdot 1^3 + 6 \cdot 1^2 + 4 \cdot 1 + 1$$
$$3^4 - 2^4 = 4 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 + 1$$
$$4^4 - 3^4 = 4 \cdot 3^3 + 6 \cdot 3^2 + 4 \cdot 3 + 1$$
$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$
$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4 \cdot n^3 + 6 \cdot n^2 + 4 \cdot n + 1$$

Adding columnwise, we get

$$(n+1)^4 - 1^4 = 4 \cdot (1^3 + 2^3 + \cdots + n^3) + 6\,(1^2 + 2^2 + 3^3 + \cdots + n^2)$$
$$+ 4\,(1 + 2 + 3 + \cdots + n) + (1 + 1 + \cdots + 1)\, n \text{ terms}$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{r=1}^{n} r^3\right) + 6\left(\sum_{r=1}^{n} r^2\right) + 4\left(\sum_{r=1}^{n} r\right) + n$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{r=1}^{n} r^3\right) + 6\left[\frac{n\,(n+1)\,(2n+1)}{6}\right] + 4\left[\frac{n\,(n+1)}{2}\right] + n$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^3 = \frac{n^2\,(n+1)^2}{4}$$

$$\Rightarrow \quad \sum_{r=1}^{n} r^3 = \left[\frac{n\,(n+1)}{2}\right]^2 = \left(\sum_{r=1}^{n} r\right)^2$$

Hence, $$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n\,(n+1)}{2}\right]^2 = \left(\sum_{r=1}^{n} r\right)^2$$

(iii) $$2 + 4 + 6 + \cdots + 2n = 2\,[1 + 2 + 3 + \cdots + n]$$
$$= 2 \times \frac{n\,(n+1)}{2} = n\,(n+1)$$

(iv) Let $S_n = 1 + 2 + 3 + \cdots + n$

Clearly, it is an arithmetic series with first term, $a = 1$,

common difference, $d = 1$

and last term $= n$

$$S_n = \frac{n}{2}(1 + n) = \frac{n\,(n+1)}{2}$$

Hence, $1 + 2 + 3 + \cdots + n = \dfrac{n\,(n+1)}{2}$.