Chapter - V

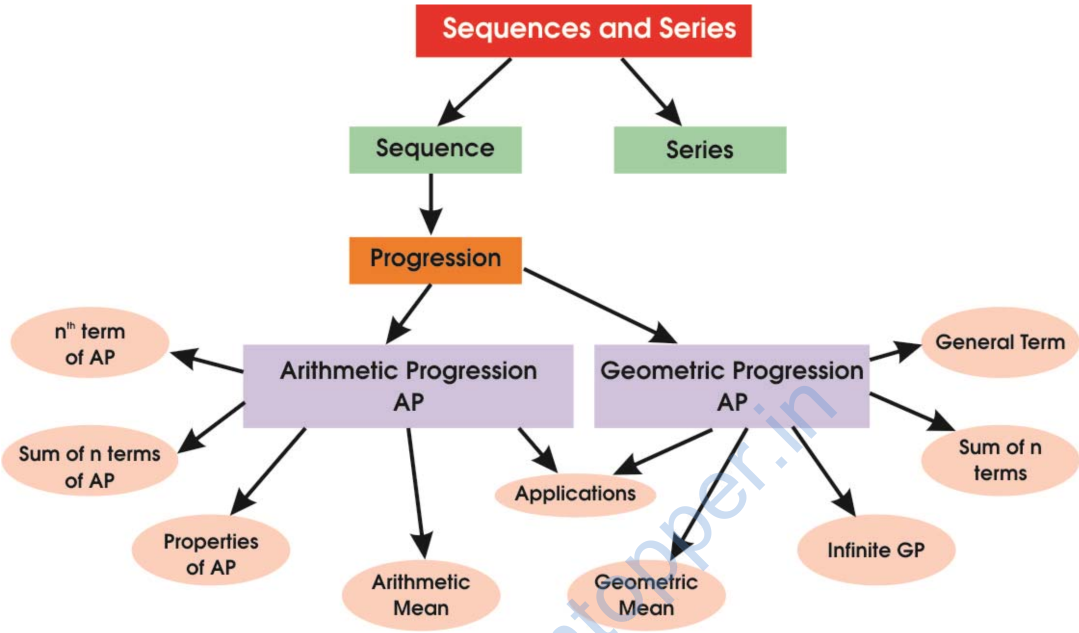
Sequences and Series

Learning Objectives:

After completion of this unit you will be able to:

- Differentiate among sequence, series and progressions
- Identity different type of progressions i.e. AP, GP
- Explain the concepts of AP and GP
- Derive the various formulas used in the chapter
- Solve problems based on different concepts
- Apply the knowledge gained to solve daily life problems

Concept Map :



5.1 Introduction

As in ordinary English, in mathematics too, sequence means a particular order in which one thing follow another.

This in turn means as in that ordered collection, we would be able to identify first, second, third members and so on. For example, the house numbers when you walk down the street, the population growth whether it be in human beings or bacteria, the amount of money deposited in bank, all form a sequence; hence conforming that sequences do have important applications in many spheres of life.

Sequence, following a specific pattern is called progression. In standard tenth, you have studied about arithmetic progression or A.P. In this chapter in addition to A.P., we would be discussing about arithmetic mean, geometric progression, geometric mean and relationship between arithmetic mean and geometric mean.

5.2 Sequences

Let us consider the following example:

A TV manufacturer has produced 1200 TVs in his first year of production. Assuming the production increases uniformly by a fixed number every year, say 200 per year. Find the number of TV sets produced in 5th year.

Now here, starting from the first year, the number of TV sets produced in 1st, 2nd, 3rd, 4th & 5th year would be 1200, 1400, 1600, 1800 and 2000 respectively. These numbers form a sequence.

Moreover, this type of sequence which has countable number of terms is called a finite sequences, otherwise it is an infinite sequence, where the number of elements is uncountable.

The various numbers occurring in the sequence are called terms and are denoted by $a_1, a_2, ... a_n$, where the subscripts denote the position of the term in the squence. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n.

This n^{th} term is called the general term of the sequence. So, in the above example,

$$a_1 = 1200, a_2 = 1400, ... a_5 = 2000$$

A sequence may not only be obtained by adding a fixed constant to the preceding term as seen in the above example, but it may also be obtained by multiplying a fixed non-zero real number to the preceding term.

For example, a child has 2 parents, further those parents have two parents, and so continuing in this manner, the child has 4 grandparents, 8 great grandparents, and so on. Now, in this case the sequence formed may be noted as 2, 4, 8, ... and so on.

This type of sequence is obtained by multiplying the preceding term by non-zero constant 2. This type of sequence is called a geometric progression. Here, $a_1 = 2, a_2 = 4$ and so on and the multiplying factor 2 is called the common ratio.

Series

Let a_1, a_2, ... a_n be a given sequence. Then the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ is called the series associated with a given sequence.

5.3 Arithmetic Progression

5.3.1 Definition 1: A sequence a_1, a_2, a_3, ..., a_n, ... is called an arithmetic progression (A.P.), if the difference of any term and its immediate preceding term is always the same, i.e.

$a_{n+1} - a_n$ = constant, say, 'd', for all $n \varepsilon N$,

or $a_{n+1} = a_n + d$. $\forall\, n \varepsilon N$

This constant difference 'd' is called the common difference. The term 'a_1' in the above sequence is called the first term.

Let us consider all the terms of the sequence a_1, a_2, ... a_n, ... in terms of only the first term and the common difference.

Let the first term 'a_1' be denoted by 'a' and common difference by 'd'. Therefore,

$a_2 = a_1 + d = a + d = a + (2 - 1)d$

$a_3 = a_2 + d = (a + d) + d = a + 2d = a + (3 - 1)d$

$a_4 = a_3 + d = (a + 2d) + d = a + 3d = a + (4 - 1)d$

$a_n = a + (n - 1)d$

Hence, the n^{th} term (general term) of the AP is:

$a_n = a + (n - 1)d$

Illustration 1: Find the first three terms of the arithmetic progressions whose n^{th} term is given. Also find the common difference and the 20th term in each case.

(i) $a_n = 2n + 5$

Now $a = a_1 = 2(1) + 5 = 7$

$a_2 = 2(2) + 5 = 9$

$a_3 = 2(3) + 5 = 11$

Therefore, $d = a_2 - a_1 = 9 - 7 = 2$

$\therefore$ a_{20} can either be found from 'a_n'=2n+5 putting n=20 or given by substituting the value of 'a' & 'd' in an.

$\therefore$ either; $a_{20} = 2(20) + 5 = 45$

or $a_{20} = a + (n - 1)\, d = 7 + (20 - 1) \times 2$

$= 7 + 2 \times 19 = 45$

(ii) $a_n = 3 - 4n$

$a_1 = 3 - 4\,(1) = -1$

$a_2 = 3 - 4\,(2) = -5$

$a_3 = 3 - 4\,(3) = -9$

Therefore, $d = a_2 - a_1 = -5 - (-) = -4$

You, may also calculate d as $d = a_3 - a_2 = -9 - (-5) = -4$

Hence, $a_{20} = a + (n - 1)d = -1 + 19\,(-4) = -77$

(iii) $a_n = \frac{n-3}{4}$

$a_1 = \frac{1-3}{4} = -\frac{1}{2}$

$a_2 = \frac{2-3}{4} = -\frac{1}{4}$

$a_3 = \frac{3-3}{4} = 0$

Therefore, $d = a_2 - a_1 = -\frac{1}{4} - \left(-\frac{1}{2}\right) = \frac{-1}{4} + \frac{1}{2} = \frac{1}{4}$

or $\quad d = a_3 - a_2 = 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$

Here $a_{20} = \frac{20-3}{4} = \frac{17}{4}$

Hence, from above Illustration we may note that the terms of an AP or the common difference, both can be any real number.

Let's Check on Some Properties of an A.P.

5.3.2 Properties of an A.P.

Property (i): If a constant is added to each term of an A.P., the resulting sequence is also an A.P. with same common difference as the previous one.

Example: Let us consider an AP: 1, 3, 5, 7, ...

In this sequence, d = 2

Now add a constant, say, 5, to each term,

We get a new sequence, 6, 8, 10, 12, ... which is also an A.P. with common difference d = 2.

Note that though the terms in the two sequences change, but the common difference remains the same.

Property (ii): If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P., again, with same common difference as the previous sequence.

Example: Let us consider an A.P.: 8, 5, 2, –1, –4, ...

In this sequence, d = –3

Now subtract a constant, say, 5, from each term.

We get a new sequence: 3, 0, –3, –6, –9, ..., which is again A.P., an with the same common difference d = –3.

Property (iii): Similarly if each term of an AP is multiplied by a constant, then the resulting sequence is also an A.P.

Example: Let us consider the sequence: 5, 9, 13, 17, ..., an A.P.

Here, d = 4

Now, multiply each term by, say, –4, we get a new sequence.

–20, –36, –52, –68, ... which is again an A.P., with d = –16

Please note in this case the common difference is –4d.

So, we may conclude that the common difference in this case is the product of the common difference and the number with which each term is multiplied.

So, if in an A.P., with common difference, 'd', each term is multiplied with a non-zero constant, say ' b', then the common difference of new AP would be 'bd'.

Property (iv): If each term of an A.P. is divided by a non-zero constant, then, the resulting sequence is also an A.P. In this case, the common difference of the new A.P. will be obtained by dividing the common difference of the original sequence by the non-zero constant with which each term was divided.

Property (v): A sequence is an A.P. if its n^{th} term is a unique linear expression in n i.e., a_n = An + B. In such cases, the coefficient of 'n' i.e. A is always the common difference.

Property (vi): In a finite A.P., a_1, a_2, a_3, ... a_n the sum of the terms equidistant from the beginning and end is always the same and is equal to the sum of its first and last term.

i.e. if $a_1, a_2, a_3, \ldots\ldots, a_{n-2}, a_{n-1}, a_n$ is an A.P. then

$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots\ldots$

Property (vii): If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

i.e. if $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ are in AP, then

a_1, a_3, a_5, a_7, chosen at same interval are also in A.P.

Property (viii): Three numbers a, b, c are in A.P. if 2b = a + b

This is because, $b - a = c - b \Rightarrow 2b = a + c$

Let us understand, what are the applications of arithmetic progression in daily life.

(a) **The Saving Schemes:** When you save some amount of money in the bank, you earn an increment to it. For example, if you deposit Rs. 1000 and every year you earn an increment of Rs. 100 this would make an A.P.. You can also calculate the total amount received by you in a particular year that you require.

(b) **A Taxi Fare:** A taxi ride can cost you fixed change. For example, for the first one kilometer, it can cost you Rs. 25 and the next subsequent kilometer that you cover, it would cost you additional Rs. 10. For each kilometer. Therefore, Rs. 25, Rs. 35, Rs. 45, ... would make an A.P.

Illustration 2: In a sequence, the n^{th} term is $a_n = 2n^2 + 5$. Show that it is not an A.P.

Solution: Given: $a_n = 2n^2 + 5$

$a_1 = 7, a_2 = 13, a_3 = 23, a_4 = 37\ldots\ldots$

Now, $a_2 - a_1 = 6$

$a_3 - a_2 = 10$

Since the common difference between any two consecutive terms is not the same, therefore it is not an A.P.

Illustration 3: Find the number of identical terms in the two sequences:

1, 5, 9, 13, 17, ..., 197 and 1, 4, 7, 10, 13, ... 196

Solution: We will first find the number of terms in the two sequences.

Let the number of terms in first sequence be 'n'.

Therefore, 197 = a + (n - 1)d, as 197 is the last n^{th} term of the sequence.

$\therefore \quad 197 = 1 + (n - 1) \times 4 = 1 + 4(n-1)$

$\Rightarrow \frac{196}{4} = n - 1 \Rightarrow n = 50$

Let the number of terms in the second sequence be 'm'.

This implies, 196 = a + (m-1)d

$\therefore \quad 196 = 1 + (m - 1) \times 3 = 1 + 3(m-1)$

$\Rightarrow \frac{195}{3} = m - 1$

$\Rightarrow m = 66$

Now, According to the question, let p^{th} term of first sequence be equal to the q^{th} term of the second sequence.

$\therefore \quad a_1 + (p - 1)\, d_1 = A_1 + (q - 1)\, D_1$

$1 + 4(p - 1) = 1 + 3(q - 1)$

$\therefore \quad \frac{p-1}{3} = \frac{q-1}{4} = k$ (say)

$\therefore \quad p = 3k + 1,\ q = 4k + 1$

Now since first A.P. contains 50 terms, therefore,

$p \leq 50 \Rightarrow 3k + 1 \leq 50 \Rightarrow K \leq \frac{49}{3}$

$\Rightarrow k \leq 16\frac{1}{3}$

Since second A.P. contains 66 terms, therefore,

$$q \le 66 \Rightarrow 4k + 1 \le 66$$

$$\Rightarrow k \le \frac{65}{4} = 16\frac{1}{4}$$

But since k is a natural number, therefore, $k \le 16$

$\therefore$ k = 1, 2, 3,..... 16

Hence, there are 16 identical terms in the given sequence.

Illustration 4: Find the 10^{th} term from the end of the sequence 7, 10, 13, ..., 130

Solution: The given sequence is an A.P. with common difference d = 3 and first term a = 7

Now, since 130 is the last term of the sequence, therefore,

$$a_n = 130$$

$$\Rightarrow \quad a + (n - 1)\, d = 130$$

$$\Rightarrow \quad 7 + (n - 1) \times 3 = 130$$

$$\Rightarrow \quad 123 = 3\,(n - 1)$$

$$\Rightarrow \quad n = 42$$

Now, 10^{th} term from the end of the sequence 7, 10, 13, ... 13, would be the 10^{th} term from the beginning of the sequence 130, ..., 13, 10, 7, obtained by reversing the order.

In this case, a = 130, d = –3, n = 42

$$\therefore \quad a_{10} = a + (n - 1)\, d$$

$$= 130 + (10 - 1)(-3)$$

$$= 130 - 27 = 103$$

Illustration 5: If p times the p^{th} term of an A.P. is q times then q^{th} term, the show that its $(p + q)^{th}$ term is zero.

Solution: Given p a_p = q a_q

To prove: $a_{p+q} = 0$

Now, let 'a' be the first term and 'd' be the common difference of the sequence.

$\therefore \quad a_p = a + (p-1)d$

$\quad a_q = a + (q-1)d$

According to the question,

$\quad p(a_p) = q(a_q)$

$\Rightarrow \quad p\{a + (p-1)d\} = q\{a + (q-1)d\}$

$\Rightarrow \quad ap + p(p-1)d = aq + q(q-1)d$

$\Rightarrow \quad a(p-q) + d(p^2 - p - q^2 + q) = 0$

$\Rightarrow \quad a(p-q) + d((p-q(p+q) - (p-q)) = 0$

$\Rightarrow \quad a(p-q) + d((p-q)(p+q-1)) = 0$

$\Rightarrow \quad (p-q)(a + d(p+q-1)) = 0$

$\Rightarrow \quad a + (p+q-1)d = 0 \qquad (\text{as } p - q \neq 0 \text{ as } p \neq q)$

$\Rightarrow \quad a_{p+q} = 0$

Hence Proved.

Illustration 6: If the numbers a, b, c, d, e form an A.P., then the value of a – 4b + 6c – 4d + e is:

(1) 1 (b) 2

(c) 0 (d) non of these

Solution: Let D be the common difference of the A.P. Then,

$\quad b = a+D,\ C = a+2D,\ d = 3D,\ e = a+4D$

So $a - 4b + 6c - 4d + e$

$= a - 4(a + D) + 6(a + 2D) - 4(a + 3D) + (a + 4D)$

$= a - 4a - 4D + 6a + 12D - 4a - 12D + a + 4D$

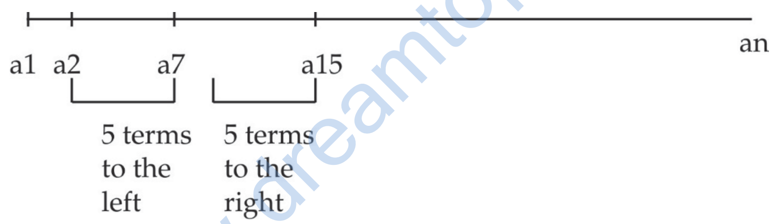
$= 0$

Illustration 7: If a_n be the n^{th} term of an A.P. and if $a_7 = 15$, then the value of the common difference that would make the product $a_2\, a_7\, a_{12}$ greatest is:

(1) 9 (b) 9/4

(c) 0 (d) 18

Solution: Let d be the common difference of the A.P. Then,



$a_2\, a_7\, a_{12} = (15 - 5d)\, 15\, (15 + 5d) = 375\,(9 - d^2)$ (1)

Therefore, if $a_7 = 15$, $a_2 = 15 - 5d$, $a_{15} = 15 + 5d$

Hence, RHS is greatest when $d = 0$ (as $d^2 \geq 0$)

Therefore, $a_2\, a_7\, a_{12}$ is greatest for $d = 0$

5.3.3 Selection of Terms of an A.P.

Sometimes, we require only certain number of terms of an A.P. The following ways of selecting terms are often convenient to find the solution of the problem.

Table 5.3.1

Number of Terms	Terms in an A.P.	Common Difference
3	$a - d, a, a + d$	d
4	$a - 3d, a - d, a + d, a + 3d$	$2d$
5	$a - 2d, a - d, a, a + d, a + 2d$	d
6	$a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$	$2d$

Please note that in case of odd number of terms, the middle term is 'a' and common difference is d while in case of an even number of terms the middle terms are $a - d$ and $a + d$ and the common difference is $2d$.

5.3.4 Sum of First 'n' Terms of an A.P.

Consider the terms of the A.P. $a_1, a_2, a_3, \ldots, a_n$, with common difference d.

Now, $a_1 = a$ (first term)

$a_2 = a + d$

$a_3 = a + 2d$

$a_n = a + (n - 1)d$

Adding these terms together we get,

$a_1 + a_2 + a_3 \ldots + a_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$

$= na + (1 + 2 + \ldots + (n - 1))d$

$= na + \frac{n(n-1)}{2}d$

$= \frac{n}{2}\left[2a+(n-1)d\right]$

$\therefore$ Sum of first 'n' terms of an A.P., denoted by Sn, is given by

$$Sn = \frac{n}{2}\left[2a+(n-1)d\right]$$

This can further be re-written as:

$$Sn = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{n}{2}\left[a+a+(n-1)d\right]$$

$= \frac{n}{2}\left[a+\ell\right]$, where '$\ell$' denotes the last term a_n

$$\therefore Sn = \frac{n}{2}\left[a+\ell\right]$$

Note: If Sn denotes the sum of 'n' terms, then $S_2 - S_1 = a_2$ as $S_1 = a_1$ (first term) and S_2 is the sum of first two terms a_1 and a_2.

Hence, $S_1 = a_1$; $S_2 = a_1 + a_2$

$$\Rightarrow S_2 - S_1 = a_2$$

If we generalise, $S_n - S_{n-1} = a_n$

Illustration 8: Find the sum of first 100 even natural number.

Solution: Consider the sequence 2, 4, 6, 8,..... of even natural numbers.

Now the first term is a = 2, first term and the common difference is d = 2

$\therefore \quad a_{100} = a + (n - 1)d$

$= 2 + 99 \times 2$

$= 200$

Now, $S_{100} = \frac{100}{2}(2 \times 2 + 99 \times 2)$

$= 50 \times (4 + 198)$

$= 50 \times 202$

$= 10100$

After $a_{100} = 200$

As $\quad S_n = \frac{n}{2}[a+\ell]$

$\therefore S_{100} = \frac{100}{2}$ (2 + 200)

$= 50 \times 202 = 10100$

Illustration 9: How many terms of the A.P. 17, 15, 13, are need to give the sum 72. Explain the double answer.

Solution: Let the number of terms to give the sum 72 be 'n'.

$\therefore \quad S_n = \frac{n}{2}(2 \times 17 + (n-1) \times (-2))$

Now, according to the question

$S_n = 72$

$\therefore \quad 72 = \frac{n}{2}(34 - 2n + 2)$

$= \frac{n}{2}(36 - 2n) = 72$

$= n(18 - n) = 72$

$\therefore \quad n(n - 18) + 72 = 0$

$= n^2 - 18n + 72 = 0$

$\Rightarrow \quad n = 12, 6$

The double answer is due to the fact that some terms in the A.P. are negative and these terms on addition with positive numbers gives zero.

Illustration 10: The sum of three numbers in A.P. is 24 and their product is 440. Find the numbers.

Solution: According to the table, 1.3.1 let the three numbers in A.P. be a – d, a, a + d.

According to the question, a – d + a + a + d = 24.

$3a = 24 \Rightarrow a = 8$

Also, (a – d) a (a + d) = 440

$a(a^2 - d^2) = 440$

$8(a^2 - d^2) = 440$

$a^2 - d^2 = \frac{440}{8} = 55$

$64 - d^2 = 55$

$d^2 = 9 \Rightarrow d = \pm 3$

When a = 8, d = 3, then the numbers are 5, 8, 11

When a = 8, d = –3, then the numbers are 11, 8, 5

Therefore, the required numbers are 5, 8, 11

Illustration 11: Solve for x: 1 + 4 + 7 + 10 + ... + x = 590

Solution: The given series is an arithmetic series, with a = 1 and d = 3.

Now, 'x' in the question represents the last term of the series and let us consider the number of terms to be 'n'. Therefore, $x = a_n$

According to the question, $\frac{n}{2}\left[2a+(n-1)d\right] = 590$

$\Rightarrow \frac{n}{2}\left[2\times1+(n-1)\times3\right] = 590$

$\Rightarrow n(2 + 3n - 3) = 590 \times 2$

$\Rightarrow n(3n - 1) = 1180$

$\Rightarrow 3n^2 - n - 1180 = 0$

$\Rightarrow$ (n – 20) (3n + 59) = 0

$\Rightarrow$ n = 20, n = $-\frac{59}{3}$

n = $-\frac{59}{3}$ is not possible as 'n' is a natural number.

$\therefore$ n = 20

$\therefore$ x = a_n = a + (n – 1)d = 1+ (20 – 1) × 3

= 1 + 57 = 58

Illustration 12: If the first, second and last terms of an A.P. are a, b & c respectively, then show that the sum of terms in A.P. is $\frac{(a+c)(b+c-2a)}{2(b-a)}$.

Solution: Let the number of terms in the given A.P. be in 'n' and common difference be 'd'

Therefore, according to the question,

a_1 = a

a_2 = b = a + d $\Rightarrow$ d = b – a

a_n = c = a + (n – 1) d

$\Rightarrow$ c = a+ (n – 1) (b – a)

$\Rightarrow$ c – a = (n – 1) (b – a)

$\Rightarrow$ $\frac{c-a}{b-a}$ = n – 1

$\Rightarrow$ $\frac{c-a}{b-a}$ + 1 = n

$\Rightarrow$ n = $\frac{b+c-2a}{b-a}$

Now, the sum of all the terms in the given A.P. is S_n = $\frac{n}{2}(a+\ell)$

$$\Rightarrow \quad S_n = \frac{n}{2}[a+c]$$

$$= \frac{(b+c-2a)}{2(b-a)}[a+c]$$

Illustration 13: The sum of n terms of two APs are in the ratio (3n + 8) : (7n + 15). Find the ratio of their 12th terms.

Solution: Let a_1, a_2 and d_1, d_2 be the first terms and common difference of the first and second A.P. respectively. According to the question,

$$\frac{\text{sum of n terms of first A.P.}}{\text{sum of n terms of second A.P.}} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}{\frac{n}{2}\left[2a_2+(n-1)d_2\right]} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2} = \frac{3n+8}{7n+15} \qquad (1)$$

Now, we need to find the ratio of 12^{th} terms

$$\therefore \quad \frac{12^{th}\text{ terms of first A.P.}}{12^{th}\text{ terms of second A.P.}} = \frac{a_1+11d_1}{a_2+11d_2}$$

$$= \frac{2a_1+22d_1}{2a_2+22d_2} = \frac{2a_1+(23-1)d_1}{2a_2+(23-1)d_2} \qquad (2)$$

From (1) and (2)

$$\frac{2a_1+(23-1)d_1}{2a_2+(23-1)d_2} = \frac{3\times(23)+8}{7\times(23)+15} \qquad \text{(Put n = 23)}$$

$$\Rightarrow \quad \frac{12^{th}\text{ terms of first A.P.}}{12^{th}\text{ terms of second A.P.}} = \frac{7}{16}$$

5.3.5 Arithmetic Mean

If between any two number say 'a' and 'b', we insert a number, say 'A' such that a, A, b is an A.P., then this number A is called the arithmetic mean of the number 'a' and 'b' i.e.,

$$A - a = b - A$$

$$\Rightarrow \qquad A = \frac{a+b}{2}$$

From this we may interpret that the AM between two numbers 'a' and 'b' is their average $\frac{a+b}{2}$

For example, consider two number 5 and 15. We may observe that 10 = $\frac{15+5}{2}$ = $\frac{a+b}{2}$ is the AM between 5 and 15.

More generally, given two numbers a and b, we need not restrict to only one arithmetic mean between any two given number. We can insert as many AMs between any two numbers as we like such that the resulting sequence is an A.P.

Let $A_1, A_2, A_3, \ldots A_n$ be 'n' numbers between 'a' and 'b' such that a, $A_1, A_2, A_3, \ldots A_n$, b forms an AP. These $A_1, A_2, \ldots, A_n$ are called the AMs of the given sequence. Note that the first AM is the second term of the sequence, second AM is the third term of the sequence and so on.

Illustration 14: Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Solution: Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that 3, $A_1, A_2, A_3, A_4, A_5, A_6$, 24 are in A.P.

Here a = 3, n = 8 and $a_8 = 24$

Now, to find A_1, which is the second term of the sequence, we need to find the common difference first.

$$a_8 = 24$$

$\Rightarrow \quad a + 7d = 24$

$\Rightarrow \quad 3 + 7d = 24$

$\Rightarrow \quad d = 3$

$\therefore \quad A_1 = a + d = 3 + 3 = 6$

$A_2 = a + 2d = 3 + 6 = 9$

$A_3 = a + 3d = 3 + 9 = 12$

$A_4 = a + 4d = 3 + 12 = 15$

$A_5 = a + 5d = 3 + 15 = 18$

and $\quad A_6 = a + 6d = 3 + 6 \times 3 = 21$

Hence the six numbers between 3 and 23 are 6, 9, 12, 15, 18 and 21.

We may also observe that in general, an arithmetic mean An = a + nd; where a is the first term and d is the common difference.

Illustration 15: If $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between 'a' and 'b', then find the value of 'n'.

Solution: Since $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ in the A.M. between 'a' and 'b',

then, $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}} = \dfrac{a+b}{2}$

$\Rightarrow 2(a^n + b^n) = (a^{n-1} + b^{n-1})(a + b)$

$\Rightarrow 2a^n + 2b^n = a^n + ab^{n-1} + a^{n-1} b + b^n$

$a^n + b^n = ab^{n-1} + a^{n-1} b$

$\Rightarrow \quad a^n - a^{n-1} b = ab^{n-1} - b^n$

$\Rightarrow \quad a^{n-1} (a - b) = b^{n-1} (a - b)$

$\Rightarrow \quad a^{n-1} = b^{n-1} \quad (\text{as } a - b \neq 0)$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n-1} = 1$

$\Rightarrow \quad n - 1 = 0$

$\Rightarrow \quad n = 1$

Check Your Progress 1

Multiple Choice Questions:

Choose the correct answer from the given four options.

1. If for a sequence $S_n = 2n - 3$, then the common difference is;

   (a) -1 (b) 2

   (c) -2 (d) 3

2. The number of integers from 100 to 500 that are divisible by 5 are:

   (a) 80 (b) 81

   (c) 75 (d) none of these

3. The number of two digit numbers divisible by 6 are:

   (a) 24 (b) 14

   (c) 15 (d) 20

4. If the fourth term of an A.P. is 4, then the sum of its 7 terms is:

   (a) 28 (b) 26

   (c) 32 (d) none of these

5. Which of the following terms are not the term of the A.P. $-3, -7, -11, -15,$ ..... $-403,$ ....., -799.

   (a) -500 (b) -399

   (c) -503 (d) -51

Short Answer Type Questions:

6. If each term in a given A.P. is doubled, then is the new sequence obtained an A.P. If yes, then find its common difference.

7. Find the middle term in the A.P. 20, 16, 12, ..., –176.

8. In an A.P., if $a_4 : a_7 = 2 : 3$, then find $a_6 : a_8$

9. Find the sum of integers from 100 to 500 that are divisible by 2 and 3.

10. Find the sum of 20 terms of the A.P. whose n^{th} terms is $2n + 1$.

Long Answer Type Questions:

11. Inert 'n' arithmetic means between 1 and 31 such that the ratio of the 7^{th} mean and the $(n-1)^{th}$ mean is 5 : 9. Find the value of n and the resulting A.P.

12. Find the sum of the following series.

    (a) $4 + 7 + 10 + ...$ to 100 terms

    (b) $1 + \frac{4}{3} + \frac{5}{3} + 2 + ...$ to 19 terms

    (c) $0.5 + 0.51 + 0.52 .....$ to 1000 terms

13. If the sum of term is an A.P. is 72, find the number of terms given that the first term of the sequence is 17 and common difference is –2.

14. If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then proven that the sum of the first (p + q) terms is zero.

15. Find the first negative term in the given A.P..

    $19, \frac{91}{5}, \frac{87}{5}, ...$

16. If p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively, then prove that.

    $(a - b)r + (b - c)p + (c - a)q$

17. Find the middle terms in the A.P. 5, 8, 11, 14,..... whose last term is 95.

18. In an A.P., if p^{th} terms is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$ then prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$

19. The sum of 'n' term of two A.P.s are in the ratio 5n + 4 : 9n + 6. Find the ratio of the their 18^{th} terms.

20. On a certain day in a hospital, during covid crisis, the patients in the OPD were 1000. Due to efforts of the doctors and health care warriors and precautions taken by general public numbers declined by 50 per day. As per the decline in the number of patients, do you think that there would be a day with no patients in the OPD? If yes, which day would it be from the day when there were 1000 patients.

5.4 Geometric Progression (GP)

5.4.1 Definition

A sequence $a_1, a_2, a_3, ...a_n, ...$ is called a geometric progression, if each term is non-zero and $\frac{a_{k+1}}{a_k}$ = r (constant) for $k \geq 1$.

By letting $a_1 = a$, we obtain the geometric progression a, ar, ar^2, ar^3, ..., where a is called the first term and r is called the common ratio of the G.P.

Thus we define, GP is sequence in which each term except the first is obtained by multiplying the previous term by a non-zero constant. The first term is denoted by a and the constant number by which we multiply any term to obtain the next term is called common ratio and is denoted by r.

Some example of GP are:

(i) 3, 6, 12, 24, ... Here, a = 3, r = $\frac{a_2}{a_1} = \frac{6}{3}$ = 2

(ii) $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}$, ... Here, a = $\frac{1}{2}$, r = $\frac{a_2}{a_1} = -\frac{1}{3}$

5.4.2 General Term of a G.P.

Let a be the first term and the r be the common ratio of a G.P.

Hence the G.P. is a, ar, ar^2, ar^3, ar^4, ...

also

$a_1 = a = ar^{1-1}$

$a_2 = ar = ar^{2-1}$

$a_3 = ar^2 = ar^{3-1}$

$a_4 = ar^3 = ar^{4-1}$

Do you see a pattern. What will be the 17^{th} term.

$a_{17} = ar^{17-1}$

Therefore continuing the pattern we get

$a_n = ar^{n-1}$

Thus, a, GP can be written as a, ar, ar^2, ar^3, ...ar^{n-1} if it is finite or a, ar, ar^2, ...ar^{n-1} ..... if it is infinite. The corresponding geometric series are $a + ar + ar^2 + ... + ar^{n-1}$ and $a + ar + ar^2 + ... + ar^{n-1} + ...$ respectively.

Illustration 16 : Find the 10^{th} and n^{th} terms of the G.P.

2, 4, 8, 16, ...

Solution: Here a = 2, $r = \frac{a_2}{a_1} = \frac{4}{2} = 2$

Thus, $a_{10} = ar^9 = 2(2)^9 = 2^{10} = 1024$

and $a_n = ar^{n-1} = 2\,(2^{n-1}) = 2^n$

5.4.3 Selection of Terms of a G.P.

Sometimes, we require only certain number of terms of an A.P. The following ways of selecting terms are often convenient to find the solution of the problem:

Table 1.3.3

Number of Terms	Terms in a G.P.	Common Ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r
6	$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$	r^2

Please note that in case of odd number of terms, the middle terms is 'a' and common ratio is 'r' while in case of an even number of term two middle terms are $\frac{a}{r}, ar$ and the common ratio is r^2

Illustration 17: The third term of a G.P. is 12. Find the product of its first five terms.

Solution: Let the first term and common ratio of the GP be a and r respectively.

Given that, $t_3 = ar^2 = 12$ (1)

Now, $t_1 t_2 t_3 t_4 t_5 = a (ar) (ar^2) (ar^3) (ar^4)$

$= a^5 r^{\frac{4\times5}{2}}$

$= a^5r^{10} = (ar^2)^5$

$= (12)^5$ Using (1)

$= 248832$

Illustration 18: If the m^{th} and n^{th} terms of a G.P are n and m respectively, show that its $(m+n)^{th}$ term is $\left(\frac{n^m}{m^n}\right)^{\frac{1}{m-n}}$

Solution: Let a be the first term and r the common ratio of the G.P

$$t_m = ar^{m-1} = n \qquad (1)$$

$$t_n = ar^{n-1} = m \qquad (2)$$

Dividing (1) by (2) we get

$$\frac{ar^{m-1}}{ar^{n-1}} = \frac{n}{m} \Rightarrow (r)^{m-n} = \frac{n}{m}$$

$$\Rightarrow r = \left(\frac{n}{m}\right)^{\frac{1}{m-n}} \qquad (3)$$

Now, $t_{m+n} = ar^{m+n-1}$

$$= ar^{m-1} \times r^n$$

$$= n \times \left[\left(\frac{n}{m}\right)^{\frac{1}{m-n}}\right]^n \qquad \text{Using (iii)}$$

$$= \frac{n^{1+\frac{n}{m-n}}}{m^{\frac{n}{m-n}}} = \frac{n^{\frac{m}{m-n}}}{m^{\frac{n}{m-n}}} = \left(\frac{n^m}{m^n}\right)^{\frac{1}{m-n}}$$

Hence proved

5.4.4 Sum of n Terms of a G.P

Let the first term of a G.P be a and the common ratio be r. Let S_n denote the sum of first n terms of the G.P. Then

$$S_n = a + ar + ar^2 + ... + ar^{n-1} \qquad (1)$$

If $r \neq 1$, multiplying equation (1) by r, we get

$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n$ (2)

Subtracting (2) from (1), we get

$(1-r)\, S_n = a - ar^n$

$\Rightarrow S_n \dfrac{a(1-r^n)}{1-r}$ or $\dfrac{a(r^n-1)}{r-1}$

Note: If r = 1, we obtain

$S_n = a + a + a + ...+ a$ (n terms)

i.e., $S_n = na$

Illustration 19: Find the sum of indicated number of terms in the following G.P.

(i) 3, 6, 9, ...10 terms, n terms

(ii) 1, –3, 9, –27, ...8 terms, p terms

Solution:

(i) Here a = 3, r = $\dfrac{6}{3} = 2$

$$S_n = \frac{a(r^n-1)}{r-1} = 3\,(2^n - 1)$$

$$S_{10} = \frac{3[2^{10}-1]}{2-1} = 3 \times 1023 = 3069$$

(ii) Here a = 1, r = –3

$$S_n = \frac{a(r^n-1)}{r-1} = \frac{1((-3)^n-1)}{-3-1}$$

$$S_n = \frac{(-3)^n-1}{-4}$$

$$\text{So, } S_p = \frac{1-(-3)^p}{4}$$

and $S_8 = \frac{1-(-3)^8}{4} = -1640$

Illustration 20: How many terms of the G.P. 32, 16, 8, ... are needed to give the sum $63\frac{3}{4}$?

Solution: Let n terms of the G.P. be needed to give the sum $63\frac{3}{4}$ or $\frac{255}{4}$

Now, a = 32, r = $\frac{16}{32} = \frac{1}{2}$ and $S_n = \frac{255}{4}$

$\because S_n = \frac{a(1-r^n)}{1-r}$

$\therefore \frac{255}{4} = \frac{32\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}$

$\Rightarrow \frac{255}{4} = 64\left(1-\left(\frac{1}{2}\right)^n\right)$

$\Rightarrow \left(\frac{1}{2}\right)^n = 1-\frac{255}{256}$

$\Rightarrow \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^8 \Rightarrow n = 8$

Illustration 21: The sum of three consecutive terms of a G.P. is 26 and their product is 216. Find the common ratio and the terms.

Solution: Let three consecutive terms of the G.P. be

$\frac{a}{r}$, a and ar

Then, $\left(\frac{a}{r}\right)(a)(ar) = 216 \Rightarrow a^3 = 216 = 6^3$

i.e., $a = 6$

Since, $\frac{a}{r} + a + ar = 26$

$\Rightarrow 6\left[\frac{1}{r}+1+r\right] = 26$

$\Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow (3r-1)(r-3) = 0$

$\Rightarrow r = \frac{1}{3}, 3$

If $r = \frac{1}{3}$, the numbers are 18, 6, 2

If $r = 3$, the numbers are 2, 6, 18.

Illustration 22: Find the sum of the sequence

4, 44, 444, ... to n terms

Solution: Let S_n be the sum of n terms of the sequence.

Observe carefully it is not a G.P., but can be converted into one as shown below:

$S_n = 4 + 44 + 444 + ...\text{n terms}$

$S_n = 4\,(1 + 11 + 111 + ...\text{n terms})$

$S_n = \frac{4}{9}\,(9 + 99 + 999 + ...\text{to n terms})$

$S_n = \frac{4}{9}\,((10-1) + (10^2-1) + (10^3-1) + ...\text{n terms}))$

$S_n = \frac{4}{9}\,((10 + 10^2 + 10^3 + ...\text{n terms}) - (1+1+1+.....\text{n terms}))$

$S_n = \frac{4}{9}\left[\frac{10\left(10^n-1\right)}{10-1} - n\right]$

$S_n = \frac{4}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$

Illustration 23: An insect population is growing in such a way that each generation is 2.5 times as large as the previous one. If there are 10,000 insects in the first generation, how many are there in the 5^{th} generation.

Solution: Here, a = 10000 and r = 2.5

Number of insects in the 5^{th} generation is given by

$t^5 = ar^4$

$= 10000 \times (2.5)^4$

$= 390625$

Illustration 24: The population of a country is 114 million at present. If the population is growing at the rate of 10% per year. What will be the population of the country in 5 years?

Solution: The population of different consecutive years forms a G.P. A growth of 10% per year, when added to the previous year makes value of r = 110% = 1.1

The value of a_1 = 114 million and n = 5.

Therefore the population of the country in 5 years is given by:

$a_5 = a_1 r^4$

$= 114 \times (1.1)^4$

$= 114 \times 1.4641$

$= 166.9074 = 166.91$ million (app.)

Illustration 25: A manufacturer reckons that the value of a machine, which costs him Rs. 56200, will depreciate each year by 20%. Find the estimated value at the end of 3 years.

Solution: The value of the machine in various years will form a G.P. with first term a = Rs. 56200 and common ratio, r = 80% = 0.8 (a depreciation of 20% when subtracted from the previous year).

The estimated value at the end of 3 years is given by its value in the 4th year.

So, $a_4 = ar^3 = 56200 \times (0.8)^3$

$= 56200 \times 0.512$

$= 28{,}774.4$

Hence the value of the machine at the end of 3 years is Rs.28774.4

Illustration 26: A man saves ₹ 500 in the first month and in successive months he saves twice as much as in the previous month. This process continued for 6 months. From the seventh month and onwards he is able to save Rs. 500 less than previous month. Find his total savings for the year.

Solution: For first 6 months the savings forms a G.P. with first term 500 and common ratio r = 2.

∴ Total savings in the first six months.

$$S_6 = \frac{a\left(r^n-1\right)}{r-1} = \frac{500\left(2^6-1\right)}{2-1}$$

= 500 × 63 = Rs. 31,500

During the next 6 months the savings forms an A.P. with first term ₹ 500 less than the savings of the 6th month i.e., $a_6 = ar^5 = 500\ (2)^5$

So the first term of the AP is a = 16000

Hence for next 6 months his total savings will be

$$S_{\text{next }6} = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{6}{2}\left[2\times16000+5x-(500)\right]$$

$= \frac{6}{2}[32000 - 25000] = 21000$

So the total savings for the year

= Rs. 31,500 + Rs. 21,000

= Rs. 52,500

5.4.5 Geometric Mean

The geometric mean of two positive real numbers *a* and *b* is the number $G = \sqrt{ab}$

Therefore the geometric mean of 4 and 9 is $\sqrt{4 \times 9} = 6$. We observe that 4,6,9 are consecutive terms of a GP. This leads to a generalisation of the concept of geometric mean of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we want between them to make the resulting sequence a G.P.

Illustration 27: Insert 3 numbers between 1and 256 so that the resulting sequence is a G.P.

Solution : Let G_1, G_2, G_3, be three numbers between 1 and 256 such that 1, G_1, G_2, G_3, 256 forms a G.P. Now 256 is the (3+2) i.e., 5th term of the GP having first term 1

Let r be the common ration

So $t_5 = 256 = ar^4 = 256 = 1r^4 = 256$

$\Rightarrow$ r=4 or -4

If r=4, the G.P. is 1,4,16,64,256 and

If r =-4, the G.P. is 1,-4,16,-64,256

So the three numbers are 4,16,,64 or -4,16,-64.

5.5 Relationship between AM and GM

Let *a* and *b* be two given positive real numbers.

Also let their AM and GM be A and G respectively.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Thus, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \qquad (1)$$

From (1), we obtain $A - G \geq 0$, i.e., $A \geq G$

Note:

1. A = G holds if a = b

2. For three positive real numbers a, b and c

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}} \qquad (\because \text{AM} \geq \text{GM})$$

Likewise the result can be generalised to n non-negative real numbers.

Illustration 28: If AM and GM of two positive numbers x and y are 13 and 12 respectively, find the numbers.

Solution: Given

$$\text{AM} = \frac{x+y}{2} = 13 \quad \Rightarrow \quad x + y = 26 \qquad (1)$$

$$\text{GM} = \sqrt{xy} = 12 \quad \Rightarrow \quad xy = 144 \qquad (2)$$

Substituting value of x+y and xy from (1) and (2) in the identity,

$$(x + y)^2 - (x - y)^2 = 4xy$$

We get, $26^2 - (x - y)^2 = 4 \times 144$

$\Rightarrow \quad (x - y)^2 = 100$

$\Rightarrow \quad x - y = \pm 10 \qquad (3)$

Solving (1) and (3), we obtain

$x = 18, y = 8$ or $x = 8, y = 18$

Thus the numbers x and y are 18, 8 or 8, 18 respectively.

Illustration 29: For any two positive real numbers a and b, prove that $\frac{a}{b} + \frac{b}{a} \geq 2$

Solution: Let $\frac{a}{b} = x$ and $\frac{b}{a} = y$

Now, $\frac{x+y}{2} \geq \sqrt{xy} \quad (\because \text{AM} \geq \text{GM})$

$$\therefore \quad \frac{\frac{a}{b}+\frac{b}{a}}{2} \geq \sqrt{\frac{a}{b} \times \frac{b}{a}}$$

$$\Rightarrow \quad \frac{a}{b} + \frac{b}{a} \geq 2$$

Illustration 30: If $x \in R$, find the minimum value of $3^x + 3^{(1-x)}$.

Solution: Since AM $\geq$ GM

Therefore, $\frac{3^x + 3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}}$

$$\Rightarrow \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3}$$

$$\Rightarrow \quad 3^x + 3^{(1-x)} \geq 2\sqrt{3}$$

Hence the minimum value of $3^x + 3^{1-x}$ is $2\sqrt{3}$

Illustration 31: If a, b, c and d are four distinct positive numbers in G.P then prove that $a + d \geq b + c$.

Solution:

$\because$ a, b, c, d are in GP

$\therefore$ a, b, c are in GP

$\therefore$ $a\,c = b^2$

Now for a and c,

$$\frac{a+c}{2} \geq \sqrt{ac} \qquad (\because \text{ AM > GM})$$

$$\Rightarrow \quad \frac{a+c}{2} \geq b \qquad (\because \; ac = b^2)$$

$\Rightarrow a + c > 2b$ (1)

Again a, b, c, d are in GP

$\therefore$ b, c, d are in GP

$\therefore$ $c\,d = c^2$

Now for numbers b and d,

$$\therefore \quad \frac{b+d}{2} \geq \sqrt{bd} \qquad (\because \text{ AM > GM})$$

$$\Rightarrow \frac{b+d}{2} \geq c \qquad (\because \; bd = c^2)$$

$\Rightarrow b + d \geq 2c$ (2)

Adding (1) and (2) we get

$$a + c + b + d \geq 2\,(b + c)$$

$\Rightarrow$ a + b + c + d $\geq$ 2b + 2c

$\Rightarrow$ a + d $\geq$ b + c

Illustration 32: For positive real numbers a, b and c if a + b + c = 18, find the maximum value of abc?

Solution: Since, AM $\geq$ GM

Therefore, $\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}}$

$\Rightarrow \frac{18}{3} \geq (abc)^{\frac{1}{3}}$

$\Rightarrow (abc)^{\frac{1}{3}} \leq 6$

$\Rightarrow$ abc $\leq 6^3$

$\Rightarrow$ abc $\leq$ 216

Hence the maximum value of abc is 216.

5.6 Infinite Geometric Progression

Infinite Geometric Progression is given as a, ar, ar^2, ar^3, ...

We know sum of GP a, ar, ar^2, ar^3, ... to n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} - \frac{a}{1-r}(r^n)$$

If $|r| < 1$, as $n \to \infty$, $r^n \to 0$

Therefore, $S_n \to \frac{a}{1-r}$ as $n \to \infty$

Symbolically if sum to infinity is denoted by S_∞ or S

Then, $S = S_\infty = \dfrac{a}{1-r}$

For example:

(i) $1+\dfrac{1}{3}+\dfrac{1}{3^2}+..... = \dfrac{1}{1-\dfrac{1}{3}} = \dfrac{3}{2}$

(ii) $1-\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+..... = \dfrac{1}{1-\left(-\dfrac{1}{2}\right)} = \dfrac{2}{3}$

Illustration 33: Find the sum to infinity of the G.P.

(i) 10, –8, 6.4, ...

(ii) $a, br, ar^2, br^3, ar^4, br^5, ...;\ |r| < l$

Solution:

(i) $\because\ |r| < l,$

$\therefore\ S_\infty = \dfrac{a}{1-r} = \dfrac{10}{1-\left(-\dfrac{4}{5}\right)} = \dfrac{10}{\dfrac{9}{5}} = \dfrac{50}{9}$

(ii) $S_\infty = (a + ar^2 + ar^4 + ...) + (br + br^3 + br^5 + ...)$

$= a(1 + r^2 + r^4 + ...) + br(1 + r^2 + r^4 + ...)$

$= a\left(\dfrac{1}{1-r^2}\right)+br\left(\dfrac{1}{1-r^2}\right)$

$= \dfrac{a+br}{1-r^2}$

Illustration 34: Represent the following as rational number

$0.3\overline{4}$

Solution: Let S = $0.3\overline{4}$

$$S = 0.3 + 0.04 + 0.004 + ...$$

$$S = \frac{3}{10} + \left(\frac{4}{100} + \frac{4}{1000} + ...\right)$$

$$S = \frac{3}{10} + \frac{4}{100}\left(1 + \frac{1}{10} + \frac{1}{10^2} + ...\right)$$

$$S = \frac{3}{10} + \frac{4}{100}\left(\frac{1}{1 - \frac{1}{10}}\right)$$

$$S = \frac{3}{10} + \frac{4}{100} \times \frac{10}{9} = \frac{3}{10} + \frac{2}{45} = \frac{31}{90}$$

Illustration 35: A substance initially weighing 64 grams is decaying at a rate such that after 8 hours there are only 32 grams remaining. In another 4 hours there are only 16 grams left, and after 2 more hours only 8 grams remain. How much time passes before none of the substance remains?

Solution: The weights of substance forms a GP given by 64 g, 32 g, 16 g, 8 g, ... with common ratio $\frac{1}{2}$. It is clear that $a_n \to 0$ as $n \to \infty$.

So the whole of substance will decay in S = 8 + 4 + 2 + ... hours which is again a G.P. with first term 8 and r = $\frac{1}{2}$.

Hence $S = \frac{8}{1 - \frac{1}{2}} = \frac{8}{\frac{1}{2}}$ = 16 hours.

Illustration 36: A pendulum swings through an arc of 25 cm. On each successive swing, the pendulum covers an arc equal to 90% of the previous swing. Find the length of the arc on the sixth swing and the total distance the pendulum travels before coming to rest.

Solution: Here a = 25 cm and r = $\frac{9}{10}$

The length of the arc on 6th swing = ar^5

= 25 × (0.9)5

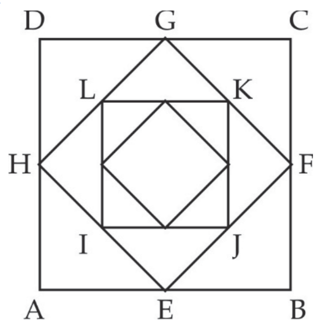
= 14.76 cm

Let the pendulum comes to rest after travelling a total distances

$$S = \frac{a}{1-r} = \frac{25}{1-0.9} = \frac{25}{\frac{1}{10}} = 250 \text{ cm}$$

Illustration 37: A square is drawn by joining the mid-points of the sides of a square. A third square is drawn inside the second square by joining the mid-points of the second square and the process is continued indefinitely. If the side of the original square is 8 cm, find the sum of the areas of all the square thus formed.

Solution: Given that AB = 8 cm,



$$HE = \sqrt{AE^2 + AH^2} = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

Also $LI = \sqrt{HI^2 + HL^2} = \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}$

= 4

and so on

Sum of the areas of all the squares is given below:

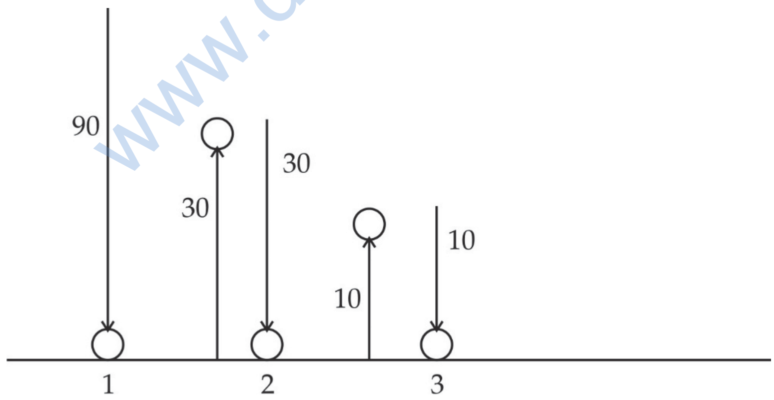
$$S_\infty = 8^2 + \left(4\sqrt{2}\right)^2 + (4)^2 + \ldots..$$

$$= 64 + 32 + 16 + \ldots..$$

$$= \frac{64}{1-\frac{1}{2}} = 128 \text{ cm}^2$$

Illustration 38: A ball is dropped from a height of 90 feet and always rebounds one-third the distance from which it falls. Find the total vertical distance the ball travelled when it hits the ground for the 3rd time. Also find the total distance travelled by the ball before it comes to rest.

Solution: First the ball falls 90 feet. Then the ball rebounds $\frac{1}{3}$ rd the distance i.e., 30 feet and falls 30 feet to hit the ground for the second time. On next rebound the ball rises 10 feet and falls back 10 feet to hit the ground for 3rd time as shown in the following figure.



Total distance travelled by the ball when it hits the ground for the 3rd time

$$= 90 + 2\,(30) + 2\,(10) = 170 \text{ feet}$$

Total distance travelled before coming to rest:

$$90 + 2\left(30 + 10 + \frac{10}{3} + ...\right) = 90 + 2\left(\frac{30}{1-\frac{1}{3}}\right) = 180 \text{ feet}$$

Check your Progress - 2

1. Find the indicated terms in each of the Geometric progressions given below:

   (i) 4, 12, 36, ...; 5^{th} term

   (ii) $3, -1, \frac{1}{3}, -\frac{1}{9}$, ...; 4^{th} term, nth term

2. Which term of the following sequences.

   (i) 5, 10, 20, 40, ... is 5120

   (ii) $2, 2\sqrt{2}, 4$, ... is 128

   (iii) $2, 1, \frac{1}{2}, \frac{1}{4}$, ... is $\frac{1}{128}$

3. Find the sum to indicated number of terms in each of the geometric progressions

   (i) $\sqrt{3}, 3, 3\sqrt{3}$, ... 6 terms

   (ii) 0.15 + 0.015 + 0.0015 + ... 20 terms.

4. Evaluate $\sum_{R=1}^{10}(3+2^k)$

5. The sum of the first two terms of a G.P. is 36 and the product of first term and third term is 9 times the second term. Find the sum of first 8 terms.

6. Find the sum to n terms of the sequence.

   7, 77, 777, 7777, ...

7. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

8. Find four numbers forming a G.P. in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

9. Insert 6 geometric means between 27 and $\frac{1}{81}$.

10. If the AM of two unequal positive real numbers a and b (a > b), be twice as much as their G.M., show that

$$a : b = (2 + \sqrt{3}) : (2 - \sqrt{3})$$

11. If a, b, c, d are in G.P., show that

(i) $a^2 + b^2, b^2 + c^2, c^2 + d^2$ are n G.P.

(ii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

12. Let S be the sum, P the product and R the sum of reciprocals of n terms of a G.P. Prove that $P^2 R^n = S^n$.

13. What will Rs. 5000 amount to in 10 years after it is deposited in a bank which pays annual interest of 8% compounded annually?

14. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

15. A certain type of bacteria doubles its population every 20 minutes. Assuming no bacteria die, how many bacteria will be there after 3 hours if there are 1 million bacteria at present.

16. One side of an equilateral triangle is 24 cm. The mid points of its sides are joined to form another triangle whose mid points are joined to form yet another triangle and so on. This process continues indefinitely. Find the sum of the perimeters of all the triangles.

17. After striking a floor a certain ball rebounds $\frac{4}{5}^{th}$ of the height from which it has fallen. If the ball is dropped from a height of 240 cm, find the total distance the ball travels before coming to rest.

18. An object decelerates such that it travels 60 m during the first second, 20 m during the second and $6\frac{2}{3}$ m during the third second. Determine the total distance the object travels before coming to rest.

19. Suppose a person mails a letter to five of his friends. He asks each one of them to mail it further to five additional friends with instruction that they move the chain further. Assuming the chain is not broken and no person receives the mail more than once, determine the amount spent on postage when the 8^{th} set of letters is mailed, if cost of postage of each letter is 50 paisa.

20. Due to reduced taxes an individual has an extra Rs. 30,000 in spendable income. If we assume that an individual spends 70% of this on consumer goods and the producers of these goods in turn spends 70% on consumer goods and this process continues indefinitely. What is the total amount spent on consumer goods.

21. A machine depreciates in value by one-fifth each year. If the machine is now worth Rs. 51000, how much will it be worth 3 years from now.

22. The sum of an infinite G.P. is 3 and the sum of the squares of its terms is also 3, than its first term and common ratio are:

(i) $1, \frac{1}{2}$

(ii) $\frac{1}{2}, \frac{3}{2}$

(iii) $\frac{3}{2}, \frac{1}{2}$

(iv) $1, \frac{1}{4}$

23. An antiques present worth is Rs. 9000. If its value appreciates at the rate of 10% per year. Its worth 3 years from now is:

(i) Rs. 6561

(ii) Rs. 10,890

(iii) Rs. 11,979

(iv) Rs. 12,000

24. Venessa invests Rs. 5000 in a bond that pays 6% interest compounded semi-annually. The value of the bond in rupees after 5 years is:

(i) $5000\ (1.06)^5$

(ii) $5000\ (1.03)^5$

(iii) $5000\ (1.06)^{10}$

(iv) $5000\ (1.03)^{10}$

Illustration - 39

Economy Stimulation

The government through a subsidy program infuses 10 lakh crore to boost the sagging economy. If we assume each state or UT spends 80% of what is received on infrastructure, health and industry and in turn each institution or agency spends 80% of what is received and so on. How much total increase in spending results from this government action?

Solution: According to the multiplier principle in economics, the effect of stimulus of 10 lakh crore on the economy is multiplied many times.

To find the amount spent if the process continues as given is obtained by applying infinite G.P. whose first term a_1 is the first amount spend i.e.,

$$a_1 = \frac{80}{100} \times 10 \text{ lakh crore} = 8 \text{ lakh crore}$$

Also, $r = 0.8$

Hence $S_\infty = 8 + 8\ (0.8) + 8\ (0.8)^2 + \ldots..$

$$S_\infty = \frac{8}{1-0.8} = \frac{8}{0.2} = 40 \text{ lakh crore}$$

So an economic stimulus of 10 lakh crore would result in spending of about 40 lakh crore.

Illustration - 40

Facing the Pandemic (The Virus Spread)

The population of a certain city is 51,20,000. Let us assume 5000 persons in the city are infected by corona-virus. The virus is so contagious that the number of infected persons doubles every 12 days. If the rate of doubling remains constant then in how many days 50% of the population gets infected.

Solution: Here, the first term is the initial number of persons infected by the virus.

i.e., $a_1 = 5000$

Since the number of patients doubles every 12 days

So $r = 2$

If the required number of infected persons be denoted by a_n then

$a_1 r^{n-1} = a_n$

$\Rightarrow 5000\ (2)^{n-1} = \frac{1}{2} \times 5120000$

$\Rightarrow 2^n = 1024 = 2^{10}$

$\Rightarrow n = 10$

Number of days required to infect half the entire population is

$= 12\ (n-1) = 12 \times 9 = 108$ days

Summary

- A sequence means arrangement of numbers in a definite order according to some rule. A sequence having countable number of terms is called finite sequence and a sequence is called infinite if it is not a finite sequence
- Let $a_1, a_2, a_3, \ldots$ be a given sequence, Then $a_1+a_2+a_3+\ldots$ is called a series

- An arithmetic progression (AP) is a sequence in which each term except the first is obtained by adding a constant number to the previous term. The first term is denoted by a, common difference by d and the last term by ℓ.

- The general term of the AP is given by an=a+(n-1) d

- The sum of n terms of the AP is given by

$$Sn = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}(a+R)$$

- Arithmetic Mean of two positive real numbers a and b is the number $A = \frac{a+b}{2}$

- A sequence is said to be geometric progression (GP) if the ratio of any term to its preceding term is same throughout. The first term and common ratio of the GP is denoted by a and r respectively.

∴ The general term of teh GP is given by $a_n = ar^{n-1}$

∴ The sum sn of the first n terms of the GP is given by $S_n = \frac{a(r^n-1)}{r-1} or \frac{a(1-r^n)}{1-r}$

- Relation between arithmetic mean A and Geometric mean G is given by

  A $\geq$ G

- Sum of infinite GP a, ar, ar^2,... where $|r|<1$ is denoted by S_∞ sm and is given by $S_\infty = \frac{a}{1-r}$

- Application problems based on GP.

Solution to check your progress

Check your Progress - 1

1 (d) 2(b) 3(c) 4(a) 5(a) 6 Yes; common difference is twice the common difference of original sequence

7. a_{25} and a_{26} are the middle terms.

$a_{25} = -76, \ a_{26} = -080$

8. $a_6 : a_8 = 4 : 5$

9. 20100

10. 440

11. 14, 1, 3, 5, 7, 9, 11, ..., 31

12 5495

13 n=6, n=12

15 25 term = $\frac{-1}{5}$

17 50, the 16^{th} term

19 179 : 321

20 on 21^{st} day there would be no Covid Patient report in the hospital

Check your Progress-2

Q1 (i) 324 (ii) $-\frac{1}{9}, \frac{(-1)^{n-1}}{3^{n-2}}$

Q2 (i) 11 (ii) 13 (iii) 9

Q3 (i) $39+13\sqrt{3}$ ii $\frac{1}{6}\left[1-(0.1)^{20}\right]$

Q4 $28+2^{11}$

Q5 $\frac{3280}{81}$

Q6 $\frac{70}{81}(10^n-1)-\frac{7n}{9}$

Q7 $r=\frac{5}{2} or \frac{2}{5}; term\ are\ \frac{2}{5},1,\frac{5}{2}\ or\ \frac{5}{2},1,\frac{2}{5}$

Q8 3, -6, 12, -24

Q9 9, 3, 1, $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

Q13 Rs 5000 $(1.08)^{10}$

Q15 512 million

Q16 144 cm

Q17 21.6m

Q18 90^{m}

Q19 Rs 24,4140.60

Q20 Rs 70,000

Q21 Rs 26,112

Q22 (iii)

Q23 (iii)

Q24 (iv)

Practice Questions:

Multiple Choice Questions:

1) Find the sum of the G.P.: 8/10, 8/100, 8/1000, 8/10000, ... to n terms.

   a) $\frac{-8}{9}\left(\frac{1}{10^n}-1\right)$

   b) $\frac{8}{81}\left(\frac{1}{10^n}-1\right)$

   c) $\frac{9}{8}\left(\frac{1}{10^n}-1\right)$

   d) $\frac{8}{90}\left(\frac{1}{10^n}-1\right)$

2) If the first term of a GP be 5 and common ratio is (-5) then which term is 3125.

   a) 6^{th}

b) 8^{th}

c) 5^{th}

d) 4^{th}

3) Which number should be added to the numbers 3, 8, 13 to make the resulting numbers should be GP.

a) 4

b) 2

c) 5

d) -2

4) If the third term of a G.P. is 6 then the product of its first 5 term is

a) 5^6

b) 6^5

c) 5^2

d) 6^2

5) If a, b and c are in A.P. as well as in G.P., then which of the following is true.

a) $a=b \neq c$

b) $a \neq b \neq c$

c) $a=b=c$

d) $a \neq b = c$

Problems:

1) You friend has invested in a 'Grow Your Money Scheme' that promises to return Rs 11,000 after a year if you invest 1,000 at the rate of 10% compounded annually. You are bit skeptic about the claim of this scheme. Your Friend shows the calculation done by the agent of the scheme as follows. Would you be willing to invest in this scheme? Explain.

Here is how I have done

Given: P = Rs. 1000

r = 10%

n = 1 year

$A = P\,(1 + r)^{t}$

$= 1000\,(1 + 10)^{1}$

$= 1000 * 11$

2) Write the first four terms of a geometric series for which S_8 = 39,360 and r =3. (Adopted from augusta.k.12)

3) Using Geometric series write 0.175175175175.... in fraction.

4) In a mock test of Sequence & Series, Rohan & Shweta have solved a question in the following manner. Find the 10th term of the Geometric series 9,3,1, ...

Shweta:

$r = \frac{3}{9}$ or

$a_{10} = ar^{9} = 9(\frac{1}{3})^{9}$

$a_{10} = \frac{1}{2187}$

Roshan:

$r = \frac{9}{3}$ or $r = 3$

$a_{10} = ar^{9} = 9(3)^{9}$

$a_{10} = 177147$

a) Who is correct explain your reasoning?

b) Can you guess the correct answer without solving? If yes what argument would you use?

5) On the first day, a music video of Arjit Singh was posted online, got 120 views in Delhi. The number of viewership gets increased by 5% per day. How many total views did the video get over the course of the first 29 days? Express your answer in exponential form. (From Delta Math)

6) In the book of Harry Potter and Deathly Hallows, Harry traced his family history and got to know that he is related with Peverell brothers. Let's assume that he had traced his family back for 15 generations starting with his parents, how many ancestors he had in total. Try this exercise at home. To make it more interesting you could draw the family tree with the photos of your relative. See if you have more relatives than Harry Potter. (Adopted from augusta.k12.va.us)

7) You and your sibling decide to ask for a raise in your pocket money from your Dad. Your Dad gives both of you a choice. You two could have 1000 Rs at once or can get Rs 2 on day one, Rs.4 on day two so on receiving twice as many rupees each day as the previous day, for 12 days. While you opted for getting Rs 1000 at once your brother opted for the later. Which one of you made a better decision and why?

8) If $a^x = b^y = c^z$ such that a, b and c are in GP and x, y and z are unequal positive integer then show that $\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$.

9) A person sends a fake news on WhatsApp to 4 of his friends on Monday. Each of those friends forward the fake news on to the four of their friends on Tuesday. Each person who receives the fake news on Tuesday send it four more people by Wednesday and this process goes on for a week. After a week a leading news agency verifies the news and finds it to be fake. Find how many people have received the fake news on WhatsApp

till then? Usually the fake news is shared and forwarded on a bigger scale. Try this question.

10) Three positive numbers form an increasing G.P. If the middle term of the series is doubled then the new numbers are in A.P. Find the common ratio of the G.P.

11) Priyanka invested Rs. 1300 in an account that pays 4% interest compounded annually. Assuming no deposits or withdrawals are made, find how much money she would have in the account 6 years after her initial investment. (Adapted from DeltaMath)

12) Dividend: A sum of money paid regularly by a company to its shareholders out of its profits.

A financial analyst is analyzing the prospects of a certain company. The company pays an annual dividend on its stock. A dividend of Rs. 500 has just been paid and the analyst estimates that the dividends will grow by 20% per year for the next 3 years, followed by annual growth of 10% per year for 2 years. (Adapted from Finance and Growth)

(a) Complete the following table:

Year	1	2	3	4	5
Dividend					

(b) Then calculate the total dividend that will be paid for the next five years.

13) In general, it has been assumed that compound interest is compounded once a year. In reality, interest may be compounded several times a

year, e.g. daily, weekly, quarterly, semi-annually or even continuously. The value of an investment at the end of m compounding periods is:

$$P_t = P_0(1 + r/m)^{m * t}$$

where m is the number of compounding periods per year and t is the number of years.

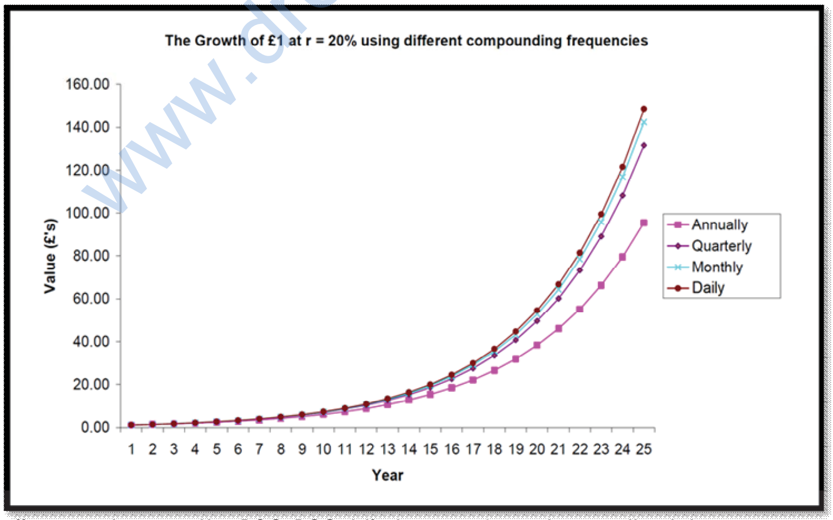
Using this information, solve the following problem:

(a) Rs.1,000 is invested for three years at 6% per annum compounded semi-annually. Calculate the total return after three years.

(b) What would the answer be if the interest was compounded annually?

(c) Using your answers of (a-b), what can you infer about the frequency of compounding and the size of the total return? (From Finance and Growth)

14) From this graph, what conclusion can be drawn regarding the frequency of compounding? (Adapted from Finance and Growth)



15) A small country emits 130,100 kilotons of carbon dioxide per year. In a recent global agreement, the country agreed to cut its carbon emissions by 3.1% per year for the next 3 years. In the first year by as per a special agreement, the country will keep its emissions at 130,000 kilotons and the

emissions will decrease 3.1% in each of the next two years. How many kilotons of carbon dioxide would the country emit over the course of the 3-year period? (Geometric Series- Deltamath.com)

16) https://www.khanacademy.org/math/geometry-home/geometry-volume-surface-area/koch-snowflake/v/koch-snowflake-fractal

17) A stock begins to pay dividends with the first dividend, one year from now, expected to be Rs. 100. Each year the dividend is 10% larger than the previous year's dividend. In what year the dividend paid is larger than Rs. 1000?

(Use concept of logarithm to solve the question)