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Relations and Functions

Short Answer Type Questions

Q. 1 Let $A = \{a, b, c\}$ and the relation R be defined on A as follows

$$R = \{(a, a), (b, c), (a, b)\}$$

Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Sol. Given relation, $R = \{(a, a), (b, c), (a, b)\}$.

To make R is reflexive we must add (b, b) and (c, c) to R. Also, to make R is transitive we must add (a, c) to R.

So, minimum number of ordered pair is to be added are (b, b), (c, c), (a, c).

Q. 2 Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then, write D.

Sol. Given function is, $f(x) = \sqrt{25 - x^2}$

For real valued of $f(x)$ $\quad 25 - x^2 \geq 0$

$$x^2 \leq 25$$

$$-5 \leq x \leq +5$$

$$\therefore \quad D = [-5, 5]$$

Q. 3 If $f, g : R \rightarrow R$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall\, x \in R$, respectively. Then, find gof.

Thinking Process

If $f, g: R \rightarrow R$ be two functions, then $gof(x) = g\{f(x)\}\ \forall x \in R$.

Sol. Given that, $\quad f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in R$

$$\therefore \quad gof = g\{f(x)\}$$

$$= g(2x + 1) = (2x + 1)^2 - 2$$

$$= 4x^2 + 4x + 1 - 2$$

$$= 4x^2 + 4x - 1$$

Q. 4 Let $f : R \to R$ be the function defined by $f(x) = 2x - 3, \forall x \in R$. Write f^{-1}.

Sol. Given that, $f(x) = 2x - 3, \forall x \in R$

Now, let

$$y = 2x - 3$$
$$2x = y + 3$$
$$x = \frac{y+3}{2}$$

$\therefore$ $$f^{-1}(x) = \frac{x+3}{2}$$

Q. 5 If $A = \{a, b, c, d\}$ and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}.

Sol. Given that, $A = \{a, b, c, d\}$

and $f = \{(a, b), (b, d), (c, a), (d, c)\}$

$$f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$$

Q. 6 If $f : R \to R$ is defined by $f(x) = x^2 - 3x + 2$, write $f\{f(x)\}$.

Thinking Process

To solve this problem use the formula i.e., $(a + b + c)^2 = (a^2 + b^2 + c^2 + 2ab + 2bc + 2ca)$

Sol. Given that, $f(x) = x^2 - 3x + 2$

$\therefore$ $$f\{f(x)\} = f(x^2 - 3x + 2)$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$$
$$= x^4 + 10x^2 - 6x^3 - 3x$$
$$f\{f(x)\} = x^4 - 6x^3 + 10x^2 - 3x$$

Q. 7 Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β?

Sol. Given that, $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$.

Here, each element of domain has unique image. So, g is a function.

Now given that, $g(x) = \alpha x + \beta$

$$g(1) = \alpha + \beta$$
$$\alpha + \beta = 1 \quad \ldots(i)$$
$$g(2) = 2\alpha + \beta$$
$$2\alpha + \beta = 3 \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$2(1 - \beta) + \beta = 3$$
$\Rightarrow$ $$2 - 2\beta + \beta = 3$$
$\Rightarrow$ $$2 - \beta = 3$$
$$\beta = -1$$

If $\beta = -1$, then $\alpha = 2$

$$\alpha = 2, \ \beta = -1$$

Q. 8 Are the following set of ordered pairs functions? If so examine whether the mapping is injective or surjective.

(i) $\{(x, y) : x$ is a person, y is the mother of $x\}$.

(ii) $\{(a, b) : a$ is a person, b is an ancestor of $a\}$.

Sol. **(i)** Given set of ordered pair is $\{(x, y) : x$ is a person, y is the mother of $x\}$.

It represent a function. Here, the image of distinct elements of x under f are not distinct, so it is not a injective but it is a surjective.

(ii) Set of ordered pairs $= \{(a, b) : a$ is a person, b is an ancestor of $a\}$

Here, each element of domain does not have a unique image. So, it does not represent function.

Q. 9 If the mappings f and g are given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write fog.

Sol. Given that, $f = \{(1, 2), (3, 5), (4, 1)\}$

and $g = \{(2, 3), (5, 1), (1, 3)\}$

Now,
$$fog(2) = f\{g(2)\} = f(3) = 5$$
$$fog(5) = f\{g(5)\} = f(1) = 2$$
$$fog(1) = f\{g(1)\} = f(3) = 5$$
$$fog = \{(2, 5), (5, 2), (1, 5)\}$$

Q. 10 Let C be the set of complex numbers. Prove that the mapping $f : C \to R$ given by $f(z) = |z|, \forall z \in C$, is neither one-one nor onto.

Sol. The mapping $f : C \to R$

Given,
$$f(z) = |z|, \forall z \in C$$
$$f(1) = |1| = 1$$
$$f(-1) = |-1| = 1$$
$$f(1) = f(-1)$$

But $1 \neq -1$

So, $f(z)$ is not one-one. Also, $f(z)$ is not onto as there is no pre-image for any negative element of R under the mapping $f(z)$.

Q. 11 Let the function $f : R \to R$ be defined by $f(x) = \cos x, \forall x \in R$. Show that f is neither one-one nor onto.

Sol. Given function, $f(x) = \cos x, \forall x \in R$

Now,
$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$\Rightarrow$
$$f\left(\frac{-\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$\Rightarrow$
$$f\left(\frac{\pi}{2}\right) = f\left(\frac{-\pi}{2}\right)$$

But
$$\frac{\pi}{2} \neq \frac{-\pi}{2}$$

So, $f(x)$ is not one-one.

Now, $f(x) = \cos x, \forall x \in R$ is not onto as there is no pre-image for any real number. Which does not belonging to the intervals $[-1, 1]$, the range of $\cos x$.

Q. 12 Let $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$. Find whether the following subsets of $X \times Y$ are functions from X to Y or not.

(i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$ (ii) $g = \{(1, 4), (2, 4), (3, 4)\}$

(iii) $h = \{(1, 4), (2, 5), (3, 5)\}$ (iv) $k = \{(1, 4), (2, 5)\}$

Sol. Given that, $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$

$$X \times Y = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

(i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$

f is not a function because f has not unique image.

(ii) $g = \{(1, 4), (2, 4), (3, 4)\}$

Since, g is a function as each element of the domain has unique image.

(iii) $h = \{(1, 4), (2, 5), (3, 5)\}$

It is clear that h is a function.

(iv) $k = \{(1, 4), (2, 5)\}$

k is not a function as 3 has not any image under the mapping.

Q. 13 If functions $f : A \to B$ and $g : B \to A$ satisfy $gof = I_A$, then show that f is one-one and g is onto.

Sol. Given that,

$$f : A \to B \text{ and } g : B \to A \text{ satisfy } gof = I_A$$

$\because$ $gof = I_A$

$\Rightarrow$ $gof\{f(x_1)\} = gof\{f(x_2)\}$

$\Rightarrow$ $g(x_1) = g(x_2)$ $[\because gof = I_A]$

$\therefore$ $x_1 = x_2$

Hence, f is one-one and g is onto.

Q. 14 Let $f : R \to R$ be the function defined by $f(x) = \dfrac{1}{2 - \cos x}$, $\forall\ x \in R$. Then, find the range of f.

Thinking Process

Range of $f = \{y \in Y : y = f(x)$: for some in $x\}$ and use range of $\cos x$ is $[-1, 1]$

Sol. Given function, $f(x) = \dfrac{1}{2 - \cos x}$, $\forall\ x \in R$

Let $y = \dfrac{1}{2 - \cos x}$

$\Rightarrow$ $2y - y\cos x = 1$

$\Rightarrow$ $y\cos x = 2y - 1$

$\Rightarrow$ $\cos x = \dfrac{2y - 1}{y} = 2 - \dfrac{1}{y}$ $\Rightarrow$ $\cos x = 2 - \dfrac{1}{y}$

$\Rightarrow$ $-1 \le \cos x \le 1$ $\Rightarrow$ $-1 \le 2 - \dfrac{1}{y} \le 1$

$\Rightarrow$ $-3 \le -\dfrac{1}{y} \le -1$ $\Rightarrow$ $1 \le \dfrac{1}{y} \le 3$

$\Rightarrow$ $\dfrac{1}{3} \le \dfrac{1}{y} \le 1$

So, y range is $\left[\dfrac{1}{3}, 1\right]$.

Q. 15 Let n be a fixed positive integer. Define a relation R in Z as follows $\forall$ a, $b \in Z$, aRb if and only if $a - b$ is divisible by n. Show that R is an equivalence relation.

Sol. Given that, $\forall$ $a, b \in Z$, aRb if and only if $a - b$ is divisible by n.

Now,

I. **Reflexive**

$aRa \Rightarrow (a - a)$ is divisible by n, which is true for any integer a as 'O' is divisible by n.

Hence, R is reflexive.

II. **Symmetric**

aRb

$\Rightarrow$ $a - b$ is divisible by n.

$\Rightarrow$ $-b + a$ is divisible by n.

$\Rightarrow$ $-(b - a)$ is divisible by n.

$\Rightarrow$ $(b - a)$ is divisible by n.

$\Rightarrow$ bRa

Hence, R is symmetric.

III. **Transitive**

Let aRb and bRc

$\Rightarrow$ $(a - b)$ is divisible by n and $(b - c)$ is divisible by n

$\Rightarrow$ $(a - b) + (b - c)$ is divisibly by n

$\Rightarrow$ $(a - c)$ is divisible by n

$\Rightarrow$ aRc

Hence, R is transitive.

So, R is an equivalence relation.

Long Answer Type Questions

Q. 16 If $A = \{1, 2, 3, 4\}$, define relations on A which have properties of being

(i) reflexive, transitive but not symmetric.

(ii) symmetric but neither reflexive nor transitive.

(iii) reflexive, symmetric and transitive.

Sol. Given that, $A = \{1, 2, 3, 4\}$

(i) Let $R_1 = \{(1, 1), (1, 2), (2, 3), (2, 2), (1, 3), (3, 3)\}$

R_1 is reflexive, since, (1, 1) (2, 2) (3, 3) lie in R_1.

Now, $(1, 2) \in R_1, (2, 3) \in R_1 \Rightarrow (1, 3) \in R_1$

Hence, R_1 is also transitive but $(1, 2) \in R_1 \Rightarrow (2, 1) \notin R_1$.

So, it is not symmetric.

(ii) Let $R_2 = \{(1, 2), (2, 1)\}$

Now, $(1, 2) \in R_2, (2, 1) \in R_2$

So, it is symmetric.

(iii) Let $R_3 = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3)\}$

Hence, R_3 is reflexive, symmetric and transitive.

Q. 17 Let R be relation defined on the set of natural number N as follows, $R = \{(x, y) : x \in N, y \in N, 2x + y = 41\}$. Find the domain and range of the relation R. Also verify whether R is reflexive, symmetric and transitive.

Sol. Given that, $R = \{(x, y) : x \in N, y \in N, 2x + y = 41\}$.

$$\text{Domain} = \{1, 2, 3, \ldots, 20\}$$
$$\text{Range} = \{1, 3, 5, 7, \ldots, 39\}$$
$$R = \{(1, 39), (2, 37), (3, 35), \ldots, (19, 3), (20, 1)\}$$

R is not reflexive as $(2, 2) \notin R$

$$2 \times 2 + 2 \neq 41$$

So, R is not symmetric.

As $(1, 39) \in R$ but $(39, 1) \notin R$

So, R is not transitive.

As $(11, 19) \in R, (19, 3) \in R$

But $(11, 3) \notin R$

Hence, R is neither reflexive, nor symmetric and nor transitive.

Q. 18 Given, $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following

(i) an injective mapping from A to B.

(ii) a mapping from A to B which is not injective.

(iii) a mapping from B to A.

Sol. Given that, $A = \{2, 3, 4\}, B = \{2, 5, 6, 7\}$

(i) Let $f : A \to B$ denote a mapping

$$f = \{(x, y) : y = x + 3\}$$

i.e., $f = \{(2, 5), (3, -6), (4, 7)\}$, which is an injective mapping.

(ii) Let $g : A \to B$ denote a mapping such that $g = \{(2, 2), (3, 5), (4, 5)\}$, which is not an injective mapping.

(iii) Let $h : B \to A$ denote a mapping such that $h = \{(2, 2), (5, 3), (6, 4), (7, 4)\}$, which is a mapping from B to A.

Q. 19 Give an example of a map

(i) which is one-one but not onto.

(ii) which is not one-one but onto.

(iii) which is neither one-one nor onto.

Sol. (i) Let $f : N \to N$, be a mapping defined by $f(x) = 2x$ which is one-one.

For $f(x_1) = f(x_2)$

$\Rightarrow$ $2x_1 = 2x_2$

$$x_1 = x_2$$

Further f is not onto, as for $1 \in N$, there does not exist any x in N such that $f(x) = 2x + 1$.

(ii) Let $f : N \to N$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for every $x > 2$ is onto but not one-one. f is not one-one as $f(1) = f(2) = 1$. But f is onto.

(iii) The mapping $f : R \to R$ defined as $f(x) = x^2$, is neither one-one nor onto.

Q. 20 Let $A = R - \{3\}$, $B = R - \{1\}$. If $f : A \to B$ be defined by $f(x) = \frac{x-2}{x-3}$, $\forall\ x \in A$. Then, show that f is bijective.

Thinking Process

A function $f : x \to y$ is said to be bijective, if f is both one-one and onto.

Sol. Given that, $A = R - \{3\}$, $B = R - \{1\}$.

$f : A \to B$ is defined by $f(x) = \frac{x-2}{x-3}$, $\forall\ x \in A$

For injectivity

Let $f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$

$\Rightarrow -3x_1 - 2x_2 = -3x_2 - 2x_1$

$\Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

So, $f(x)$ is an injective function.

For surjectivity

Let $y = \frac{x-2}{x-3} \Rightarrow x - 2 = xy - 3y$

$\Rightarrow x(1-y) = 2 - 3y \Rightarrow x = \frac{2-3y}{1-y}$

$\Rightarrow x = \frac{3y-2}{y-1} \in A, \forall\ y \in B$ [codomain]

So, $f(x)$ is surjective function.

Hence, $f(x)$ is a bijective function.

Q. 21 Let $A = [-1, 1]$, then, discuss whether the following functions defined on A are one-one onto or bijective.

(i) $f(x) = \frac{x}{2}$ (ii) $g(x) = |x|$

(iii) $h(x) = x|x|$ (iv) $k(x) = x^2$

Sol. Given that, $A = [-1, 1]$

(i) $f(x) = \frac{x}{2}$

Let $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{2} = \frac{x_2}{2} \Rightarrow x_1 = x_2$

So, $f(x)$ is one-one.

Now, let $y = \frac{x}{2}$

$\Rightarrow x = 2y \notin A, \forall\ y \in A$

As for $y = 1 \in A, x = 2 \notin A$

So, $f(x)$ is not onto.

Also, $f(x)$ is not bijective as it is not onto.

(ii) $g(x) = |x|$

Let $g(x_1) = g(x_2)$

$\Rightarrow$ $|x_1| = |x_2| \Rightarrow x_1 = \pm x_2$

So, $g(x)$ is not one-one.

Now, $y = |x| \Rightarrow x = \pm y \notin A, \forall y \in A$

So, $g(x)$ is not onto, also, $g(x)$ is not bijective.

(iii) $h(x) = x|x|$

Let $h(x_1) = h(x_2)$

$\Rightarrow$ $x_1|x_1| = x_2|x_2| \Rightarrow x_1 = x_2$

So, $h(x)$ is one-one.

Now, let $y = x|x|$

$\Rightarrow$ $y = x^2 \in A, \forall x \in A$

So, $h(x)$ is onto also, $h(x)$ is a bijective.

(iv) $k(x) = x^2$

Let $k(x_1) = k(x_2)$

$\Rightarrow$ $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$

Thus, $k(x)$ is not one-one.

Now, let $y = x^2$

$\Rightarrow$ $x = \sqrt{y} \notin A, \forall y \in A$

As for $y = -1$, $x = \sqrt{-1} \notin A$

Hence, $k(x)$ is neither one-one nor onto.

Q. 22 Each of the following defines a relation of N

(i) x is greater than y, $x, y \in N$.

(ii) $x + y = 10$, $x, y \in N$.

(iii) xy is square of an integer $x, y \in N$.

(iv) $x + 4y = 10$, $x, y \in N$

Determine which of the above relations are reflexive, symmetric and transitive.

Sol. **(i)** x is greater than y, $x, y \in N$

$(x, x) \in R$

For xRx $\quad x > x$ is not true for any $x \in N$.

Therefore, R is not reflexive.

Let $(x, y) \in R \Rightarrow xRy$

$x > y$

but $y > x$ is not true for any $x, y \in N$

Thus, R is not symmetric.

Let xRy and yRz

$x > y$ and $y > z \Rightarrow x > z$

$\Rightarrow$ xRz

So, R is transitive.

(ii) $x + y = 10, x, y \in N$

$$R = \{(x, y); x + y = 10, x, y \in N\}$$

$$R = \{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\}\ (1, 1) \notin R$$

So, R is not reflexive.

$$(x, y) \in R \Rightarrow (y, x) \in R$$

Therefore, R is symmetric.

$$(1, 9) \in R, (9, 1) \in R \Rightarrow (1, 1) \notin R$$

Hence, R is not transitive.

(iii) Given xy, is square of an integer $x, y \in N$.

$\Rightarrow$ $$R = \{(x, y) : xy \text{ is a square of an integer } x, y \in N\}$$

$$(x, x) \in R, \forall x \in N$$

As x^2 is square of an integer for any $x \in N$.

Hence, R is reflexive.

If $$(x, y) \in R \Rightarrow (y, x) \in R$$

Therefore, R is symmetric.

If $$(x, y) \in R, (y, z) \in R$$

So, xy is square of an integer and yz is square of an integer.

Let $xy = m^2$ and $yz = n^2$ for some $m, n \in Z$

$$x = \frac{m^2}{y} \text{ and } z = \frac{x^2}{y}$$

$$xz = \frac{m^2 n^2}{y^2}, \text{ which is square of an integer.}$$

So, R is transitive.

(iv) $$x + 4y = 10, x, y \in N$$

$$R = \{(x, y) : x + 4y = 10, x, y \in N\}$$

$$R = \{(2, 2), (6, 1)\}$$

$$(1, 1), (3, 3), \ldots, \notin R$$

Thus, R is not reflexive.

$$(6, 1) \in R \text{ but } (1, 6) \notin R$$

Hence, R is not symmetric.

$$(x, y) \in R \Rightarrow x + 4y = 10 \text{ but } (y, z) \in R$$

$$y + 4z = 10 \Rightarrow (x, z) \in R$$

So, R is transitive.

Q. 23 Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\, R\, (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation and also obtain the equivalent class $[(2, 5)]$.

Sol. Given that, $A = \{1, 2, 3, \ldots, 9\}$ and $(a, b)\ R(c, d)$ if $a + d = b + c$ for $(a, b) \in A \times A$ and $(c, d) \in A \times A$.

Let $(a, b)\, R\, (a, b)$

$\Rightarrow$ $$a + b = b + a, \forall a, b \in A$$

which is true for any $a, b \in A$.

Hence, R is reflexive.

Let $(a, b)\, R\, (c, d)$ $$a + d = b + c$$

$$c + b = d + a \Rightarrow (c, d)\, R\, (a, b)$$

So, R is symmetric.

Let $(a, b) R (c, d)$ and $(c, d) R (e, f)$

$$a + d = b + c \text{ and } c + f = d + e$$
$$a + d = b + c \text{ and } d + e = c + f$$
$$(a + d) - (d + e) = (b + c) - (c + f)$$
$$(a - e) = b - f$$
$$a + f = b + e$$
$$(a, b) R (e, f)$$

So, R is transitive.

Hence, R is an equivalence relation.

Now, equivalence class containing [(2, 5)] is {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}.

Q. 24 Using the definition, prove that the function $f : A \to B$ is invertible if and only if f is both one-one and onto.

Sol. A function $f : X \to Y$ is defined to be invertible, if there exist a function $g = Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function is called the inverse of f and is denoted by f^{-1}.

A function $f = X \to Y$ is invertible iff f is a bijective function.

Q. 25 Functions $f, g : R \to R$ are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find

(i) fog (ii) gof (iii) fof (iv) gog

Sol. Given that, $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$

(i)
$$fog = f\{g(x)\} = f(2x - 3)$$
$$= (2x - 3)^2 + 3(2x - 3) + 1$$
$$= 4x^2 + 9 - 12x + 6x - 9 + 1 = 4x^2 - 6x + 1$$

(ii)
$$gof = g\{f(x)\} = g(x^2 + 3x + 1)$$
$$= 2(x^2 + 3x + 1) - 3$$
$$= 2x^2 + 6x + 2 - 3 = 2x^2 + 6x - 1$$

(iii)
$$fof = f\{f(x)\} = f(x^2 + 3x + 1)$$
$$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$
$$= x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 + 3x^2 + 9x + 3 + 1$$
$$= x^4 + 6x^3 + 14x^2 + 15x + 5$$

(iv)
$$gog = g\{g(x)\} = g(2x - 3)$$
$$= 2(2x - 3) - 3$$
$$= 4x - 6 - 3 = 4x - 9$$

Q. 26 Let $*$ be the binary operation defined on Q. Find which of the following binary operations are commutative

(i) $a * b = a - b, \forall\, a, b \in Q$ (ii) $a * b = a^2 + b^2, \forall\, a, b \in Q$

(iii) $a * b = a + ab, \forall\, a, b \in Q$ (iv) $a * b = (a - b)^2, \forall\, a, b \in Q$

Sol. Given that $*$ be the binary operation defined on Q.

(i) $a * b = a - b, \forall\, a, b \in Q$ and $b * a = b - a$

So, $a * b \neq b * a$ $[\because b - a \neq a - b]$

Hence, $*$ is not commutative.

(ii) $a * b = a^2 + b^2$

$b * a = b^2 + a^2$

So, $*$ is commutative. [since, '+' is on rational is commutative]

(iii) $a * b = a + ab$

$b * a = b + ab$

Clearly, $a + ab \neq b + ab$

So, $*$ is not commutative.

(iv) $a * b = (a - b)^2, \forall a, b \in Q$

$b * a = (b - a)^2$

$\because \quad (a - b)^2 = (b - a)^2$

Hence, $*$ is commutative.

Q. 27 If $*$ be binary operation defined on R by $a * b = 1 + ab, \forall a, b \in R$. Then, the operation $*$ is

(i) commutative but not associative.
(ii) associative but not commutative.
(iii) neither commutative nor associative.
(iv) both commutative and associative.

Sol. (i) Given that, $a * b = 1 + ab, \forall a, b \in R$

$a * b = ab + 1 = b * a$

So, $*$ is a commutative binary operation.

Also, $a * (b * c) = a * (1 + bc) = 1 + a(1 + bc)$

$a * (b * c) = 1 + a + abc$ ...(i)

$(a * b) * c = (1 + ab) * c$

$= 1 + (1 + ab)c = 1 + c + abc$ ...(ii)

From Eqs. (i) and (ii),

$a * (b * c) \neq (a * b) * c$

So, $*$ is not associative

Hence, $*$ is commutative but not associative.

Objective Type Questions

Q. 28 Let T be the set of all triangles in the Euclidean plane and let a relation R on T be defined as aRb, if a is congruent to b, $\forall a, b \in T$. Then, R is

(a) reflexive but not transitive
(b) transitive but not symmetric
(c) equivalence
(d) None of these

Sol. *(c)* Consider that aRb, if a is congruent to b, $\forall a, b \in T$.

Then, $aRa \Rightarrow a \cong a$,

which is true for all $a \in T$

So, R is reflexive, ...(i)

Let $aRb \Rightarrow a \cong b$

$\Rightarrow b \cong a \Rightarrow b \cong a$

$\Rightarrow bRa$

So, R is symmetric. ...(ii)

Let aRb and bRc

$\Rightarrow a \cong b$ and $b \cong c$

$\Rightarrow a \cong c \Rightarrow aRc$

So, R is transitive. ...(iii)

Hence, R is equivalence relation.

Q. 29 Consider the non-empty set consisting of children in a family and a relation R defined as aRb, if a is brother of b. Then, R is

(a) symmetric but not transitive

(b) transitive but not symmetric

(c) neither symmetric nor transitive

(d) both symmetric and transitive

Sol. *(b)* Given, $aRb \Rightarrow a$ is brother of b

$\therefore aRa \Rightarrow a$ is brother of a, which is not true.

So, R is not reflexive.

$aRb \Rightarrow a$ is brother of b.

This does not mean b is also a brother of a and b can be a sister of a.

Hence, R is not symmetric.

$aRb \Rightarrow a$ is brother of b

and $bRc \Rightarrow b$ is a brother of c.

So, a is brother of c.

Hence, R is transitive.

Q. 30 The maximum number of equivalence relations on the set $A = \{1, 2, 3\}$ are

(a) 1 (b) 2 (c) 3 (d) 5

Sol. *(d)* Given that, $A = \{1, 2, 3\}$

Now, number of equivalence relations as follows

$$R_1 = \{(1, 1), (2, 2), (3, 3)\}$$
$$R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$
$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$$
$$R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$$
$$R_5 = \{(1, 2, 3) \Leftrightarrow A \times A = A^2\}$$

$\therefore$ Maximum number of equivalence relation on the set $A = \{1, 2, 3\} = 5$

Q. 31 If a relation R on the set $\{1, 2, 3\}$ be defined by $R = \{(1, 2)\}$, then R is

(a) reflexive (b) transitive (c) symmetric (d) None of these

Sol. *(b)* R on the set $\{1, 2, 3\}$ be defined by $R = \{(1, 2)\}$

It is clear that R is transitive.

Q. 32 Let us define a relation R in R as aRb if $a \geq b$. Then, R is

(a) an equivalence relation
(b) reflexive, transitive but not symmetric
(c) symmetric, transitive but not reflexive
(d) neither transitive nor reflexive but symmetric

Sol. *(b)* Given that, aRb if $a \geq b$

$\Rightarrow \quad aRa \Rightarrow a \geq a$ which is true.

Let aRb, $a \geq b$, then $b \geq a$ which is not true R is not symmetric.

But aRb and $b\,R\,c$

$\Rightarrow \quad a \geq b$ and $b \geq c$

$\Rightarrow \quad a \geq c$

Hence, R is transitive.

Q. 33 If $A = \{1, 2, 3\}$ and consider the relation

$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

Then, R is

(a) reflexive but not symmetric
(b) reflexive but not transitive
(c) symmetric and transitive
(d) neither symmetric nor transitive

Sol. *(a)* Given that, $A = \{1, 2, 3\}$

and $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$

$\because \quad (1, 1), (2, 2), (3, 3) \in R$

Hence, R is reflexive.

$(1, 2) \in R$ but $(2, 1) \notin R$

Hence, R is not symmetric.

$(1, 2) \in R$ and $(2, 3) \in R$

$\Rightarrow \quad (1, 3) \in R$

Hence, R is transitive.

Q. 34 The identity element for the binary operation $*$ defined on $Q - \{0\}$ as

$a * b = \dfrac{ab}{2}, \forall\, a, b \in Q - \{0\}$ is

(a) 1
(b) 0
(c) 2
(d) None of these

Thinking Process

For given binary operation $* : A \times A \rightarrow A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a, \forall\, a \in A$.

Sol. *(c)* Given that, $a * b = \dfrac{ab}{2}, \forall\, a, b \in Q - \{0\}$.

Let e be the identity element for $*$.

$\therefore \quad a * e = \dfrac{ae}{2}$

$\Rightarrow \quad a = \dfrac{ae}{2} \Rightarrow e = 2$

Q. 35 If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

(a) 720 (b) 120 (c) 0 (d) None of these

Sol. *(c)* We know that, if A and B are two non-empty finite set containing m and n elements respectively, then the number of one-one and onto mapping from A to B is

$$n!, \text{ if } m = n$$
$$0, \text{ if } m \neq n$$

Given that, $m = 5$ and $n = 6$

$\therefore$ $m \neq n$

Number of mapping = 0

Q. 36 If $A = \{1, 2, 3, \ldots, n\}$ and $B = \{a, b\}$. Then, the number of surjections from A into B is

(a) ${}^nP_2$ (b) $2^n - 2$ (c) $2^n - 1$ (d) None of these

Sol. *(d)* Given that, $A = \{1, 2, 3, \ldots, n\}$ and $B = \{a, b\}$.

We know that, if A and B are two non-empty finite sets containing m and n elements respectively, then the number of surjection from A into B is

$${}^nC_m \times m!, \text{ if } n \geq m$$
$$0, \text{ if } n < m$$

Here, $m = 2$

$\therefore$ Number of surjection from A into B is ${}^nC_2 \times 2! = \dfrac{n!}{2!(n-2)!} \times 2!$

$$= \frac{n(n-1)(n-2)!}{2 \times 1(n-2)} \times 2! = n^2 - n$$

Q. 37 If $f : R \to R$ be defined by $f(x) = \dfrac{1}{x}, \forall x \in R$. Then, f is

(a) one-one (b) onto (c) bijective (d) f is not defined

Thinking Process

In the given function at $x = 0$, $f(x) = \infty$. So, the function is not define.

Sol. *(d)* Given that, $f(x) = \dfrac{1}{x}, \forall x \in R$

For $x = 0$,

$f(x)$ is not defined.

Hence, $f(x)$ is a not define function.

Q. 38 If $f : R \to R$ be defined by $f(x) = 3x^2 - 5$ and $g : R \to R$ by $g(x) = \dfrac{x}{x^2 + 1}$. Then, gof is

(a) $\dfrac{3x^2 - 5}{9x^4 - 30x^2 + 26}$ (b) $\dfrac{3x^2 - 5}{9x^4 - 6x^2 + 26}$

(c) $\dfrac{3x^2}{x^4 + 2x^2 - 4}$ (d) $\dfrac{3x^2}{9x^4 + 30x^2 - 2}$

Sol. *(a)* Given that, $f(x) = 3x^2 - 5$ and $g(x) = \dfrac{x}{x^2 + 1}$

$$gof = g\{f(x)\} = g(3x^2 - 5)$$
$$= \frac{3x^2 - 5}{(3x^2 - 5)^2 + 1} = \frac{3x^2 - 5}{9x^4 - 30x^2 + 25 + 1}$$
$$= \frac{3x^2 - 5}{9x^4 - 30x^2 + 26}$$

Q. 39 Which of the following functions from Z into Z are bijections?

(a) $f(x) = x^3$ (b) $f(x) = x + 2$ (c) $f(x) = 2x + 1$ (d) $f(x) = x^2 + 1$

Sol. *(b)* Here, $f(x) = x + 2 \Rightarrow f(x_1) = f(x_2)$

$$x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$$

Let $y = x + 2$

$$x = y - 2 \in Z, \forall\, y \in x$$

Hence, $f(x)$ is one-one and onto.

Q. 40 If $f : R \to R$ be the functions defined by $f(x) = x^3 + 5$, then $f^{-1}(x)$ is

(a) $(x + 5)^{\frac{1}{3}}$ (b) $(x - 5)^{\frac{1}{3}}$ (c) $(5 - x)^{\frac{1}{3}}$ (d) $5 - x$

Sol. *(b)* Given that, $f(x) = x^3 + 5$

Let $y = x^3 + 5 \Rightarrow x^3 = y - 5$

$$x = (y - 5)^{\frac{1}{3}} \Rightarrow f(x)^{-1} = (x - 5)^{\frac{1}{3}}$$

Q. 41 If $f : A \to B$ and $g : B \to C$ be the bijective functions, then $(gof)^{-1}$ is

(a) $f^{-1}og^{-1}$ (b) fog (c) $g^{-1}of^{-1}$ (d) gof

Sol. *(a)* Given that, $f : A \to B$ and $g : B \to C$ be the bijective functions.

$$(gof)^{-1} = f^{-1}og^{-1}$$

Q. 42 If $f : R - \left\{\frac{3}{5}\right\} \to R$ be defined by $f(x) = \dfrac{3x + 2}{5x - 3}$, then

(a) $f^{-1}(x) = f(x)$ (b) $f^{-1}(x) = -f(x)$ (c) $(fof)x = -x$ (d) $f^{-1}(x) = \dfrac{1}{19}f(x)$

Sol. *(a)* Given that, $f(x) = \dfrac{3x + 2}{5x - 3}$

Let $y = \dfrac{3x + 2}{5x - 3}$

$$3x + 2 = 5xy - 3y \Rightarrow x(3 - 5y) = -3y - 2$$
$$x = \frac{3y + 2}{5y - 3} \Rightarrow f^{-1}(x) = \frac{3x + 2}{5x - 3}$$

$\therefore$ $f^{-1}(x) = f(x)$

Q. 43 If $f:[0, 1] \to [0, 1]$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$

then $(fof)x$ is

(a) constant (b) $1+x$ (c) x (d) None of these

Sol. *(c)* Given that, $f:[0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

$\therefore$ $(fof)x = f(f(x)) = x$

Q. 44 If $f:[2, \infty) \to R$ be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(a) R (b) $[1, \infty)$ (c) $[4, \infty)$ (d) $[5, \infty)$

Thinking Process

Range of $f = \{y \in Y : y = f(x) \text{ for some in } X\}$

Sol. *(b)* Given that, $f(x) = x^2 - 4x + 5$

Let $y = x^2 - 4x + 5$

$\Rightarrow$ $y = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$

$\Rightarrow$ $(x-2)^2 = y - 1 \Rightarrow x - 2 = \sqrt{y-1}$

$\Rightarrow$ $x = 2 + \sqrt{y-1}$

$\therefore$ $y - 1 \geq 0, y \geq 1$

Range $= [1, \infty)$

Q. 45 If $f: N \to R$ be the function defined by $f(x) = \dfrac{2x-1}{2}$ and $g: Q \to R$ be another function defined by $g(x) = x + 2$. Then, $(gof)\dfrac{3}{2}$ is

(a) 1 (b) 1 (c) $\dfrac{7}{2}$ (d) None of these

Sol. *(d)* Given that, $f(x) = \dfrac{2x-1}{2}$ and $g(x) = x + 2$

$$(gof)\frac{3}{2} = g\left[f\left(\frac{3}{2}\right)\right] = g\left(\frac{2 \times \frac{3}{2} - 1}{2}\right)$$

$$= g(1) = 1 + 2 = 3$$

Q. 46 If $f : R \to R$ be defined by $f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$

Then, $f(-1) + f(2) + f(4)$ is

(a) 9 (b) 14 (c) 5 (d) None of these

Sol. *(a)* Given that,

$$f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$$

$$f(-1) + f(2) + f(4) = 3(-1) + (2)^2 + 2 \times 4$$
$$= -3 + 4 + 8 = 9$$

Q. 47 If $f : R \to R$ be given by $f(x) = \tan x$, then $f^{-1}(1)$ is

(a) $\frac{\pi}{4}$ (b) $\left\{n\pi + \frac{\pi}{4} : n \in Z\right\}$

(c) Does not exist (d) None of these

Sol. *(a)* Given that, $f(x) = \tan x$

Let $y = \tan x \Rightarrow x = \tan^{-1} y$

$\Rightarrow f^{-1}(x) = \tan^{-1} x \Rightarrow f^{-1}(1) = \tan^{-1} 1$

$\Rightarrow = \tan^{-1} \tan \frac{\pi}{4} = \frac{\pi}{4}$ $\left[\because \tan \frac{\pi}{4} = 1\right]$

Fillers

Q. 48 Let the relation R be defined in N by aRb, if $2a + 3b = 30$. Then, $R = \ldots\ldots$.

Sol. Given that,

$$2a + 3b = 30$$
$$3b = 30 - 2a$$
$$b = \frac{30 - 2a}{3}$$

For

$a = 3, b = 8$
$a = 6, b = 6$
$a = 9, b = 4$
$a = 12, b = 2$

$R = \{(3, 8), (6, 6), (9, 4), (12, 2)\}$

Q. 49 If the relation R be defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b) : |a^2 - b^2| < 8\}$. Then, R is given by ……… .

Sol. Given, $A = \{1, 2, 3, 4, 5\}$,

$R = \{(a, b) : |a^2 - b^2| < 8\}$

$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 3), (3, 4), (4, 4), (5, 5)\}$

Q. 50 If $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, then $gof = \ldots\ldots\ldots$ and $fog = \ldots\ldots\ldots$.

Sol. Given that, $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$

$$gof(1) = g\{f(1)\} = g(2) = 3$$
$$gof(3) = g\{f(3)\} = g(5) = 1$$
$$gof(4) = g\{f(4)\} = g(1) = 3$$
$$gof = \{(1, 3), (3, 1), (4, 3)\}$$

Now,
$$fog(2) = f\{g(2)\} = f(3) = 5$$
$$fog(5) = f\{g(5)\} = f(1) = 2$$
$$fog(1) = f\{g(1)\} = f(3) = 5$$
$$fog = \{(2, 5), (5, 2), (1, 5)\}$$

Q. 51 If $f : R \to R$ be defined by $f(x) = \dfrac{x}{\sqrt{1 + x^2}}$, then $(fofof)(x) = \ldots\ldots\ldots$.

Sol. Given that, $f(x) = \dfrac{x}{\sqrt{1 + x^2}}$

$$(fofof)(x) = f[f\{f(x)\}]$$
$$= f\left[f\left(\frac{x}{\sqrt{1 + x^2}}\right)\right] = f\left(\frac{\frac{x}{\sqrt{1 + x^2}}}{\sqrt{1 + \frac{x^2}{1 + x^2}}}\right)$$
$$= f\left[\frac{x\sqrt{1 + x^2}}{\sqrt{1 + x^2}(\sqrt{2x^2 + 1})}\right] = f\left(\frac{x}{\sqrt{1 + 2x^2}}\right)$$
$$= \frac{\frac{x}{\sqrt{1 + 2x^2}}}{\sqrt{1 + \frac{x^2}{1 + 2x^2}}} = \frac{x\sqrt{1 + 2x^2}}{\sqrt{1 + 2x^2}\sqrt{1 + 3x^2}}$$
$$= \frac{x}{\sqrt{1 + 3x^2}} = \frac{x}{\sqrt{3x^2 + 1}}$$

Q. 52 If $f(x) = [4 - (x - 7)^3]$, then $f^{-1}(x) = \ldots\ldots\ldots$.

Sol. Given that, $f(x) = \{4 - (x - 7)^3\}$

Let $y = [4 - (x - 7)^3]$

$$(x - 7)^3 = 4 - y$$
$$(x - 7) = (4 - y)^{1/3}$$
$$\Rightarrow \quad x = 7 + (4 - y)^{1/3}$$
$$f^{-1}(x) = 7 + (4 - x)^{1/3}$$

True/False

Q. 53 Let $R = \{(3, 1), (1, 3), (3, 3)\}$ be a relation defined on the set $A = \{1, 2, 3\}$. Then, R is symmetric, transitive but not reflexive.

Sol. ***False***

Given that, $R = \{(3, 1), (1, 3), (3, 3)\}$ be defined on the set $A = \{1, 2, 3\}$

$$(1, 1) \notin R$$

So, R is not reflexive. $(3, 1) \in R, (1, 3) \in R$

Hence, R is symmetric.

Since, $(3, 1) \in R, (1, 3) \in R$

But $(1, 1) \notin R$

Hence, R is not transitive.

Q. 54 If $f : R \to R$ be the function defined by $f(x) = \sin(3x + 2)\ \forall\ x \in R$. Then, f is invertible.

Sol. ***False***

Given that, $f(x) = \sin(3x + 2),\ \forall\ x \in R$ is not one-one function for all $x \in R$.

So, f is not invertible.

Q. 55 Every relation which is symmetric and transitive is also reflexive.

Sol. ***False***

Let R be a relation defined by

$$R = \{(1, 2), (2, 1), (1, 1), (2, 2)\} \text{ on the set } A = \{1, 2, 3\}$$

It is clear that $(3, 3) \notin R$. So, it is not reflexive.

Q. 56 An integer m is said to be related to another integer n, if m is a integral multiple of n. This relation in Z is reflexive, symmetric and transitive.

Sol. ***False***

The given relation is reflexive and transitive but not symmetric.

Q. 57 If $A = \{0, 1\}$ and N be the set of natural numbers. Then, the mapping $f : N \to A$ defined by $f(2n - 1) = 0,\ f(2n) = 1,\ \forall\ n \in N$, is onto.

Sol. ***True***

Given, $A = \{0, 1\}$

$$f(2n - 1) = 0,\ f(2n) = 1,\ \forall\ n \in N$$

So, the mapping $f : N \to A$ is onto.

Q. 58 The relation R on the set $A = \{1, 2, 3\}$ defined as $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$ is reflexive, symmetric and transitive.

Sol. ***False***

Given that, $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$

$$(2, 2) \notin R$$

So, R is not reflexive.

Q. 59 The composition of function is commutative.

Sol. ***False***

Let $f(x) = x^2$

and $g(x) = x + 1$

$fog(x) = f\{g(x)\} = f(x + 1)$

$= (x + 1)^2 = x^2 + 2x + 1$

$gof(x) = g\{f(x)\} = g(x^2) = x^2 + 1$

$\therefore$ $fog(x) \neq gof(x)$

Q. 60 The composition of function is associative.

Sol. ***True***

Let $f(x) = x, g(x) = x + 1$

and $h(x) = 2x - 1$

Then, $fo\{goh(x)\} = f[g\{h(x)\}]$

$= f\{g(2x - 1)\}$

$= f(2x - 1 + 1)$

$= f(2x) = 2x$

$\therefore$ $(fog)oh(x) = (fog)\{h(x)\}$

$= (fog)(2x - 1)$

$= f\{g(2x - 1)\}$

$= f(2x - 1 + 1)$

$= f(2x) = 2x$

Q. 61 Every function is invertible.

Sol. ***False***

Only bijective functions are invertible.

Q. 62 A binary operation on a set has always the identity element.

Sol. ***False***

'+' is a binary operation on the set N but it has no identity element.