2

Relations and Functions

Short Answer Type Questions

Q. 1 If $A = \{-1, 2, 3\}$ and $B = \{1, 3\}$, then determine

(i) $A \times B$ (ii) $B \times A$ (iii) $B \times B$ (iv) $A \times A$

Sol. $A = \{-1, 2, 3\}$ and $B = \{1, 3\}$

(i) $A \times B = \{(-1, 1), (-1, 3), (2, 1), (2, 3), (3, 1), (3, 3)\}$

(ii) $B \times A = \{(1, -1), (1, 2), (1, 3), (3, -1), (3, 2), (3, 3)\}$

(iii) $B \times B = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$

(iv) $A \times A = \{(-1, -1), (-1, 2), (-1, 3), (2, -1), (2, 2), (2, 3), (3, -1), (3, 2), (3, 3)\}$

Q. 2 If $P = \{x : x < 3, x \in N\}$, $Q = \{x : x \le 2, x \in W\}$, then find $(P \cup Q) \times (P \cap Q)$, where W is the set of whole numbers.

Sol. We have, $P = \{x : x < 3, x \in N\} = \{1, 2\}$

and $Q = \{x : x \le 2, x \in W\} = \{0, 1, 2\}$

$\therefore$ $P \cup Q = \{0, 1, 2\}$ and $P \cap Q = \{1, 2\}$

$(P \cup Q) \times (P \cap Q) = \{0, 1, 2\} \times \{1, 2\}$

$= \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)\}$

Q. 3 If $A = \{x : x \in W, x < 2\}$, $B = \{x : x \in N, 1 < x < 5\}$ and $C = \{3, 5\}$, then find

(i) $A \times (B \cap C)$ (ii) $A \times (B \cup C)$

Sol. We have, $A = \{x : x \in W, x < 2\} = \{0, 1\}$

and $B = \{x : x \in N, 1 < x < 5\}$

$= \{2, 3, 4\}$ and $C = \{3, 5\}$

(i) $\because$ $B \cap C = \{3\}$

$\therefore$ $A \times (B \cap C) = \{0, 1\} \times \{3\} = \{(0, 3), (1, 3)\}$

(ii) $\because$ $(B \cup C) = \{2, 3, 4, 5\}$

$\therefore$ $A \times (B \cup C) = \{0, 1\} \times \{2, 3, 4, 5\}$

$= \{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5)\}$

Q. 4 In each of the following cases, find a and b.

(i) $(2a + b, a - b) = (8, 3)$

(ii) $\left(\frac{a}{4}, a - 2b\right) = (0, 6 + b)$

Sol. **(i)** We have, $(2a + b, a - b) = (8, 3)$

$\Rightarrow \quad 2a + b = 8$ and $a - b = 3$

[since, two ordered pairs are equal, if their corresponding first and second elements are equal]

On substituting, $b = a - 3$ in $2a + b = 8$, we get

$$2a + a - 3 = 8 \Rightarrow 3a - 3 = 8$$

$$\Rightarrow \quad 3a = 11 \Rightarrow a = \frac{11}{3}$$

Again, substituting $a = \frac{11}{3}$ in $b = a - 3$, we get

$$b = \frac{11}{3} - 3 = \frac{11 - 9}{3} = \frac{2}{3}$$

$$\therefore \quad a = \frac{11}{3} \text{ and } b = \frac{2}{3}$$

(ii) We have, $\left(\frac{a}{4}, a - 2b\right) = (0, 6 + b)$

$$\Rightarrow \quad \frac{a}{4} = 0 \Rightarrow a = 0$$

and $\quad a - 2b = 6 + b$

$\Rightarrow \quad 0 - 2b = 6 + b$

$\Rightarrow \quad -3b = 6$

$\therefore \quad b = -2$

$\therefore \quad a = 0, b = -2$

Q. 5 $A = \{1, 2, 3, 4, 5\}$, $S = \{(x, y) : x \in A, y \in A\}$, then find the ordered which satisfy the conditions given below.

(i) $x + y = 5$ (ii) $x + y < 5$

(iii) $x + y > 8$

Sol. We have, $A = \{1, 2, 3, 4, 5\}$ and $S = \{(x, y) : x \in A, y \in A\}$

(i) The set of ordered pairs satisfying $x + y = 5$ is,

$$\{(1, 4), (2, 3), (3, 2), (4, 1)\}.$$

(ii) The set of ordered pairs satisfying $x + y < 5$ is $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)$.

(iii) The set of ordered pairs satisfying $x + y > 8$ is $\{(4, 5), (5, 4), (5, 5)\}$.

Q. 6 If $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$, then find the domain and range of R.

Thinking Process

First, write the relation in Roaster form, then find the domain and range of R.

Sol. We have, $R = \{(x, y) : x, y \in W, x^2 + y^2 = 25\}$

$= \{(0, 5), (3, 4), (4, 3), (5, 0)\}$

Domain of R = Set of first element of ordered pairs in R

$= \{0, 3, 4, 5\}$

Range of R = Set of second element of ordered pairs in R

$= \{5, 4, 3, 0, \}$

Q. 7 If $R_1 = \{(x, y) | y = 2x + 7$, where $x \in R$ and $-5 \le x \le 5\}$ is a relation. Then, find the domain and range of R_1.

Sol. We have, $R_1 = \{x, y) | y = 2x + 7$, where $x \in R$ and $-5 \le x \le 5\}$

Domain of $R_1 = \{-5 \le x \le 5, x \in R\}$

$= [-5, 5]$

$\because$ $y = 2x + 7$

When $x = -5$, then $y = 2(-5) + 7 = -3$

When $x = 5$, then $y = 2(5) + 7 = 17$

$\therefore$ Range of $R_1 = \{-3 \le y \le 17, y \in R\}$

$= [-3, 17]$

Q. 8 If $R_2 = \{x, y) | x$ and y are integers and $x^2 + y^2 = 64\}$ is a relation, then find the value of R_2.

Sol. We have, $R_2 = \{(x, y)\}$ x and y are integers and $x^2 + y^2 = 64\}$

Since, 64 is the sum of squares of 0 and $\pm$ 8.

When $x = 0$, then $y^2 = 64 \Rightarrow y = \pm 8$

$x = 8$, then $y^2 = 64 - 8^2 \Rightarrow 64 - 64 = 0$

$x = -8$, then $y^2 = 64 - (-8)^2 = 64 - 64 = 0$

$\therefore$ $R_2 = \{(0, 8), (0, -8), (8, 0), (-8, 0)\}$

Q. 9 If $R_3 = \{(x, |x|) | x$ is a real number$\}$ is a relation, then find domain and range of R_3.

Sol. We have, $R_3 = \{(x, |x|) | x$ is real number$\}$

Clearly, domain of $R_3 = R$

Since, image of any real number under R_3 is positive real number or zero.

$\therefore$ Range of $R_3 = R^+ \cup \{0\}$ or $(0, \infty)$

Q. 10 Is the given relation a function? Give reason for your answer.

(i) $h = \{(4, 6), (3, 9), (-11, 6), (3, 11)\}$

(ii) $f = \{(x, x) | x \text{ is a real number}\}$

(iii) $g = \left\{\left(x, \frac{1}{x}\right) \middle| x \text{ is a positive integer}\right\}$

(iv) $s = \{(x, x^2) | x \text{ is a positive integer}\}$

(v) $t = \{(x, 3) | x \text{ is a real number}\}$

Sol. **(i)** We have, $h = \{(4, 6), (3, 9), (-11, 6), (3, 11)\}$.

Since, 3 has two images 9 and 11. So, it is not a function.

(ii) We have, $f = \{(x, x) | x$ is a real number.

We observe that, every element in the domain has unique image. So, it is a function.

(iii) We have, $g = \left\{\left(x, \frac{1}{x}\right) \middle| x \text{ is a positive integer}\right\}$

For every x, it is a positive integer and $\frac{1}{x}$ is unique and distinct. Therefore, every element in the domain has unique image. So, it is a function.

(iv) We have, $s = \{(x, x^2) | x \text{ is a positive integer}\}$

Since, the square of any positive integer is unique. So, every element in the domain has unique image. Hence, it is a function.

(v) We have, $t = \{(x, 3) | x \text{ is a real number}\}$.

Since, every element in the domain has the image 3. So, it is a constant function.

Q. 11 If f and g are real functions defined by $f(x) = x^2 + 7$ and $g(x) = 3x + 5$. Then, find each of the following.

(i) $f(3) + g(-5)$

(ii) $f\left(\frac{1}{2}\right) \times g(14)$

(iii) $f(-2) + g(-1)$

(iv) $f(t) - f(-2)$

(v) $\frac{f(t) - f(5)}{t - 5}$, if $t \neq 5$

Sol. Given, f and g are real functions defined by $f(x) = x^2 + 7$ and $g(x) = 3x + 5$.

(i) $f(3) = (3)^2 + 7 = 9 + 7 = 16$ and $g(-5) = 3(-5) + 5 = -15 + 5 = -10$

$\therefore f(3) + g(-5) = 16 - 10 = 6$

(ii) $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + 7 = \frac{1}{4} + 7 = \frac{29}{4}$

and $g(14) = 3(14) + 5 = 42 + 5 = 47$

$$\therefore \quad f\left(\frac{1}{2}\right) \times g(14) = \frac{29}{4} \times 47 = \frac{1363}{4}$$

(iii) $f(-2) = (-2)^2 + 7 = 4 + 7 = 11$ and $g(-1) = 3(-1) + 5 = -3 + 5 = 2$

$\therefore \quad f(-2) + g(-1) = 11 + 2 = 13$

(iv) $f(t) = t^2 + 7$ and $f(-2) = (-2)^2 + 7 = 4 + 7 = 11$

$\therefore \quad f(t) - f(-2) = t^2 + 7 - 11 = t^2 - 4$

(v) $f(t) = t^2 + 7$ and $f(5) = 5^2 + 7 = 25 + 7 = 32$

$$\therefore \quad \frac{f(t) - f(5)}{t - 5}, \text{ if } t \neq 5$$

$$= \frac{t^2 + 7 - 32}{t - 5}$$

$$= \frac{t^2 - 25}{t - 5} = \frac{(t - 5)(t + 5)}{(t - 5)}$$

$$= t + 5 \qquad [\because t \neq 5]$$

Q. 12 Let f and g be real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$.

(i) For what real numbers x, $f(x) = g(x)$?

(ii) For what real numbers x, $f(x) < g(x)$?

Sol. We have, $f(x) = 2x + 1$ and $g(x) = 4x - 7$

(i) $\because \quad f(x) = g(x)$

$\Rightarrow \quad 2x + 1 = 4x - 7 \Rightarrow 2x = 8$

$\therefore \quad x = 4$

(ii) $\because \quad f(x) < g(x)$

$\Rightarrow \quad 2x + 1 < 4x - 7$

$\Rightarrow \quad 2x - 4x + 1 < 4x - 7 - 4x$

$\Rightarrow \quad -2x + 1 < -7$

$\Rightarrow \quad -2x < -7 - 1$

$\Rightarrow \quad -2x < -8$

$$\Rightarrow \quad \frac{-2x}{-2} > \frac{-8}{-2}$$

$\therefore \quad x > 4$

Q. 13 If f and g are two real valued functions defined as $f(x) = 2x + 1$ and $g(x) = x^2 + 1$, then find

(i) $f + g$ (ii) $f - g$ (iii) fg (iv) $\frac{f}{g}$

Sol. We have, $f(x) = 2x + 1$ and $g(x) = x^2 + 1$

(i) $(f + g)(x) = f(x) + g(x)$

$= 2x + 1 + x^2 + 1 = x^2 + 2x + 2$

(ii) $(f - g)(x) = f(x) - g(x) = (2x + 1) - (x^2 + 1)$

$= 2x + 1 - x^2 - 1 = 2x - x^2 = x(2 - x)$

(iii) $(fg)(x) = f(x) \cdot g(x) = (2x + 1)(x^2 + 1)$

$= 2x^3 + 2x + x^2 + 1 = 2x^3 + x^2 + 2x + 1$

(iv) $$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{2x + 1}{x^2 + 1}$$

Q. 14 Express the following functions as set of ordered pairs and determine their range.

$$f : x \to R,\ f(x) = x^3 + 1, \text{ where } x = \{-1, 0, 3, 9, 7\}$$

Sol. We have, $f : X \to R,\ f(x) = x^3 + 1.$

Where $X = \{-1, 0, 3, 9, 7\},$

When $x = -1$, then $f(-1) = (-1)^3 + 1 = -1 + 1 = 0$

$x = 0$, then $f(0) = (0)^3 + 1 = 0 + 1 = 1$

$x = 3$, then $f(3) = (3)^3 + 1 = 27 + 1 = 28$

$x = 9$, then $f(9) = (9)^3 + 1 = 729 + 1 = 730$

$x = 7$, then $f(7) = (7)^3 + 1 = 343 + 1 = 344$

$f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$

$\therefore$ Range of $f = \{0, 1, 28, 730, 344\}$

Q. 15 Find the values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.

Sol. $\because$ $f(x) = g(x)$

$\Rightarrow$ $3x^2 - 1 = 3 + x$

$\Rightarrow$ $3x^2 - x - 4 = 0$

$\Rightarrow$ $3x^2 - 4x + 3x - 4 = 0$

$\Rightarrow$ $x(3x - 4) + 1(3x - 4) = 0$

$\Rightarrow$ $(3x - 4)(x + 1) = 0$

$\therefore$ $x = -1, \frac{4}{3}$

Long Answer Type Questions

Q. 16 Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7),\}$ a function, justify. If this is described by the relation, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?

Thinking Process

First, find the two equation by substitutions different values of x and $g(x)$.

Sol. We have, $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$

Since, every element has unique image under g. So, g is a function.

Now, $g(x) = \alpha x + \beta$

When $x = 1$, then $g(1) = \alpha(1) + \beta$

$\Rightarrow$ $1 = \alpha + \beta$ ...(i)

When $x = 2$, then $g(2) = \alpha(2) + \beta$

$\Rightarrow$ $3 = 2\alpha + \beta$ ...(ii)

On solving Eqs. (i) and (ii), we get

$\alpha = 2, \beta = -1$

Q. 17 Find the domain of each of the following functions given by

(i) $f(x)=\dfrac{1}{\sqrt{1-\cos x}}$ (ii) $f(x)=\dfrac{1}{\sqrt{x+|x|}}$

(iii) $f(x)=x|x|$ (iv) $f(x)=\dfrac{x^3-x+3}{x^2-1}$

(v) $f(x)=\dfrac{3x}{28-x}$

Sol. **(i)** We have, $f(x)=\dfrac{1}{\sqrt{1-\cos x}}$

$\because \quad -1\le \cos x\le 1$

$\Rightarrow \quad -1\le -\cos x\le 1$

$\Rightarrow \quad 0\le 1-\cos x\le 2$

So, $f(x)$ is defined, if $1-\cos x\ne 0$

$\cos x\ne 1$

$x\ne 2n\pi - \forall n\in Z$

$\therefore \quad$ Domain of $f=R-\{2n\pi : n\in Z\}$

(ii) We have, $f(x)=\dfrac{1}{\sqrt{x+|x|}}$

$\because \quad x+|x|=x-x=0,\ x<0$

$=x+x=2x,\ x\ge 0$

Hence, $f(x)$ is defined, if $x>0$.

$\therefore \quad$ Domain of $f=R^+$

(iii) We have, $f(x)=x|x|$

Clearly, $f(x)$ is defined for any $x\in R$.

$\therefore \quad$ Domain of $f=R$

(iv) We have, $f(x)=\dfrac{x^3-x+3}{x^2-1}$

$f(x)$ is not defined, if $x^2-1=0$

$\Rightarrow \quad (x-1)(x+1)=0$

$\Rightarrow \quad x=-1,1$

$\therefore \quad$ Domain of $f=R-\{-1,1\}$

(v) We have, $f(x)=\dfrac{3x}{28-x}$

Clearly, $f(x)$ is defined, if $28-x\ne 0$

$\Rightarrow \quad x\ne 28$

$\therefore \quad$ Domain of $f=R-\{28\}$

Q. 18 Find the range of the following functions given by

(i) $f(x)=\dfrac{3}{2-x^2}$ (ii) $f(x)=1-|x-2|$

(iii) $f(x)=|x-3|$ (iv) $f(x)=1+3\cos 2x$

Thinking Process

First, find the value of x in terms of y, where $y = f(x)$. Then, find the values of y for which x attain real values.

Sol. **(i)** We have, $$f(x) = \frac{3}{2 - x^2}$$

Let $$y = f(x)$$

Then, $$y = \frac{3}{2 - x^2} \Rightarrow 2 - x^2 = \frac{3}{y}$$

$$\Rightarrow \quad x^2 = 2 - \frac{3}{y} \Rightarrow x = \sqrt{\frac{2y - 3}{y}}$$

x assums real values, if $2y - 3 \geq 0$ and $y > 0 \Rightarrow y \geq \frac{3}{2}$

$$\therefore \quad \text{Range of } f = \left[\frac{3}{2}, \infty\right)$$

(ii) We know that, $|x - 2| \geq 0 \Rightarrow -|x - 2| \leq 0$

$\Rightarrow$ $1 - |x - 2| \leq 1 \Rightarrow f(x) \leq 1$

$\therefore$ Range of $f = (-\infty, 1]$

(iii) We know that, $|x - 3| \geq 0 \Rightarrow f(x) \geq 0$

$\therefore$ Range of $f = [0, \infty)$

(iv) We know that, $-1 \leq \cos 2x \leq 1 \Rightarrow -3 \leq 3\cos 2x \leq 3$

$\Rightarrow$ $1 - 3 \leq 1 + 3\cos 2x \leq 1 + 3 \Rightarrow -2 \leq 1 + 3\cos 2x \leq 1 + 3$

$\Rightarrow$ $-2 \leq f(x) \leq 4$

$\therefore$ Range of $f = [-2, 4]$

Q. 19 Redefine the function

$$f(x) = |x - 2| + |2 + x|, -3 \leq x \leq 3$$

Thinking Process

First find the interval in which $|x - 2|$ and $|2 + x|$ is defined, then find the value of $f(x)$ in that interval.

Sol. Since, $|x - 2| = -(x - 2), x < 2$

$x - 2, x \geq 2$

and $|2 + x| = -(2 + x), x < -2$

$(2 + x), x \geq -2$

$$\therefore \quad f(x) = |x - 2| + |2 + x|, -3 \leq x \leq 3$$

$$= \begin{cases} -(x-2) - (2+x), & -3 \leq x < -2 \\ -(x-2) + 2 + x, & -2 \leq x < 2 \\ x - 2 + 2 + x, & 2 \leq x \leq 3 \end{cases}$$

$$= \begin{cases} -2x, & -3 \leq x < -2 \\ 4, & -2 \leq x < 2 \\ 2, & 2 \leq x \leq 3 \end{cases}$$

Q. 20 If $f(x) = \frac{x-1}{x+1}$, then show that

(i) $f\left(\frac{1}{x}\right) = -f(x)$ (ii) $f\left(-\frac{1}{x}\right) = \frac{-1}{f(x)}$

Sol. We have, $f(x) = \frac{x-1}{x+1}$

(i) $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = \frac{(1-x)/x}{(1+x)/x} = \frac{1-x}{1+x} = \frac{-(x-1)}{x+1} = -f(x)$

(ii) $f\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x}-1}{-\frac{1}{x}+1} = \frac{(-1-x)/x}{(-1+x)/x} \Rightarrow f\left(-\frac{1}{x}\right) = \frac{-(x+1)}{x-1}$

Now, $\frac{-1}{f(x)} = \frac{-1}{\frac{x-1}{x+1}} = \frac{-(x+1)}{x-1}$

$\therefore$ $f\left(-\frac{1}{x}\right) = -\frac{1}{f(x)}$

Q. 21 If $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined in the domain $R^+ \cup \{0\}$, then find the value of

(i) $(f+g)(x)$ (ii) $(f-g)(x)$

(iii) $(fg)(x)$ (iv) $\left(\frac{f}{g}\right)(x)$

Sol. We have, $f(x) = \sqrt{x}$ and $g(x) = x$ be two function defined in the domain $R^+ \cup \{0\}$.

(i) $(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$

(ii) $(f-g)(x) = f(x) - g(x) = \sqrt{x} - x$

(ii) $(fg)(x) = f(x) \cdot g(x) = \sqrt{x} \cdot x = x^{\frac{3}{2}}$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

Q. 22 Find the domain and range of the function $f(x) = \frac{1}{\sqrt{x-5}}$.

Sol. We have, $f(x) = \frac{1}{\sqrt{x-5}}$

$f(x)$ is defined, if $x - 5 > 0 \Rightarrow x > 5$

$\therefore$ Domain of $f = (5, \infty)$

Let $f(x) = y$

$\therefore$ $y = \frac{1}{\sqrt{x-5}} \Rightarrow \sqrt{x-5} = \frac{1}{y}$

$\Rightarrow$ $x - 5 = \frac{1}{y^2}$

$\therefore$ $x = \frac{1}{y^2} + 5$

$\because$ $x \in (5, \infty) \Rightarrow y \in R^+$

Hence, range of $f = R^+$

Q. 23 If $f(x) = y = \dfrac{ax - b}{cx - a}$, then prove that $f(y) = x$.

Sol. We have, $f(x) = y = \dfrac{ax - b}{cx - a}$

$$\therefore \quad f(y) = \frac{ay - b}{cy - a} = \frac{a\left(\frac{ax - b}{cx - a}\right) - b}{c\left(\frac{ax - b}{cx - a}\right) - a}$$

$$= \frac{a(ax - b) - b(cx - a)}{c(ax - b) - a(cx - a)} = \frac{a^2x - ab - bcx + ab}{acx - bc - acx + a^2}$$

$$= \frac{a^2x - bcx}{a^2 - bc} = \frac{x(a^2 - bc)}{(a^2 - bc)} = x$$

$$\therefore \quad f(y) = x$$

Hence proved.

Objective Type Questions

Q. 24 Let $n(A) = m$ and $n(B) = n$. Then, the total number of non-empty relations that can be defined from A to B is

(a) m^n (b) $n^m - 1$ (c) $mn - 1$ (d) $2^{mn} - 1$

Thinking Process

First find the number of element in $A \times B$ and then find the number of relation by using $2^{m(A \times B)} - 1$.

Sol. (d) We have, $n(A) = m$ and $n(B) = n$

$$n(A \times B) = n(A) \cdot n(B) = mn$$

Total number of relation from A to $B = 2^{mn} - 1 = 2^{n(A \times B)-1} - 1$

Q. 25 If $[x]^2 - 5[x] + 6 = 0$, where $[\cdot]$ denote the greatest integer function, then

(a) $x \in [3, 4]$ (b) $x \in (2, 3]$ (c) $x \in [2, 3]$ (d) $x \in [2, 4)$

Thinking Process

If a and b are two successive positive integer and $[x] = a, b$, then $x \in [a, b]$

Sol. (c) We have, $[x]^2 - 5[x] + 6 = 0$

$$\Rightarrow \quad [x]^2 - 3[x] - 2[x] + 6 = 0$$

$$\Rightarrow \quad [x]([x] - 3) - 2([x] - 3) = 0$$

$$\Rightarrow \quad ([x] - 3)([x] - 2) = 0$$

$$\Rightarrow \quad [x] = 2, 3$$

$$\therefore \quad x \in [2, 3]$$

Q. 26 Range of $f(x) = \dfrac{1}{1 - 2\cos x}$ is

(a) $\left[\dfrac{1}{3}, 1\right]$ (b) $\left[-1, \dfrac{1}{3}\right]$

(c) $(-\infty, -1] \cup \left[\dfrac{1}{3}, \infty\right)$ (d) $\left[-\dfrac{1}{3}, 1\right]$

Sol. ***(b)*** We know that, $-1 \le -\cos x \le 1$

$$\Rightarrow \quad -2 \le -2\cos x \le 2$$
$$\Rightarrow \quad 1 - 2 \le 1 - 2\cos x \le 1 + 2$$
$$\Rightarrow \quad -1 \le 1 - 2\cos x \le 3$$
$$\Rightarrow \quad -1 \le \frac{1}{1 - 2\cos x} \le \frac{1}{3}$$
$$\Rightarrow \quad -1 \le f(x) \le \frac{1}{3}$$
$$\therefore \quad \text{Range of } f = \left[-1, \frac{1}{3}\right]$$

Q. 27 Let $f(x) = \sqrt{1 + x^2}$, then

(a) $f(xy) = f(x) \cdot f(y)$ (b) $f(xy) \ge f(x) \cdot f(y)$

(c) $f(xy) \le f(x) \cdot f(y)$ (d) None of these

Sol. ***(c)*** We have, $f(x) = \sqrt{1 + x^2}$

$$f(xy) = \sqrt{1 + x^2y^2}$$
$$f(x) \cdot f(y) = \sqrt{1 + x^2} \cdot \sqrt{1 + y^2}$$
$$= \sqrt{(1 + x^2)(1 + y^2)}$$
$$= \sqrt{1 + x^2 + y^2 + x^2y^2}$$
$$\because \quad \sqrt{1 + x^2y^2} \le \sqrt{1 + x^2 + y^2 + x^2y^2}$$
$$\Rightarrow \quad f(xy) \le f(x) \cdot f(y)$$

Q. 28 Domain of $\sqrt{a^2 - x^2}$ $(a > 0)$ is

(a) $(-a, a)$ (b) $[-a, a]$

(c) $[0, a]$ (d) $(-a, 0]$

Sol. ***(b)*** Let $f(x) = \sqrt{a^2 - x^2}$

$f(x)$ is defined, if $a^2 - x^2 \ge 0$

$$\Rightarrow \quad x^2 - a^2 \le 0$$
$$\Rightarrow \quad (x - a)(x + a) \le 0$$
$$\Rightarrow \quad -a \le x \le a \quad [\because a > 0]$$
$$\therefore \quad \text{Domain of } f = [-a, a]$$

Q. 29 If $f(x) = ax + b$, where a and b are integers, $f(-1) = -5$ and $f(3) = 3$, then a and b are equal to

(a) $a = -3, b = -1$
(b) $a = 2, b = -3$
(c) $a = 0, b = 2$
(d) $a = 2, b = 3$

Sol. *(b)* We have,
$$f(x) = ax + b$$
$$f(-1) = a(-1) + b$$
$$-5 = -a + b \quad \text{...(i)}$$
and,
$$f(3) = a(3) + b$$
$$3 = 3a + b \quad \text{...(ii)}$$
On solving Eqs. (i) and (ii), we get
$$a = 2 \text{ and } b = -3$$

Q. 30 The domain of the function f defined by
$$f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}} \text{ is equal to}$$

(a) $(-\infty, -1) \cup (1, 4]$
(b) $(-\infty, -1] \cup (1, 4]$
(c) $(-\infty, -1) \cup [1, 4]$
(d) $(-\infty, -1) \cup [1, 4)$

Sol. *(a)* We have,
$$f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}}$$
$f(x)$ is defined, if
$$4 - x \ge 0 \text{ or } x^2 - 1 > 0$$
$$x - 4 \le 0 \text{ or } (x + 1)(x - 1) > 0$$
$$x \le 4 \text{ or } x < -1 \text{ and } x > 1$$
$\therefore$ Domain of $f = (-\infty, -1) \cup (1, 4]$

Q. 31 The domain and range of the real function f defined by $f(x) = \dfrac{4 - x}{x - 4}$ is given by

(a) Domain $= R$, Range $= \{-1, 1\}$
(b) Domain $= R - \{1\}$, Range $= R$
(c) Domain $= R - \{4\}$, Range $= \{-1\}$
(d) Domain $= R - \{-4\}$, Range $= \{-1, 1\}$

Thinking Process

A function $\dfrac{f(x)}{g(x)}$ is defined, if $g(x) \ne 0$.

Sol. *(c)* We have,
$$f(x) = \frac{4 - x}{x - 4}$$
$f(x)$ is defined, if $x - 4 \ne 0$ *i.e.*, $x \ne 4$

$\therefore$ Domain of $f = R - \{4\}$

Let
$$f(x) = y$$
$$\therefore \quad y = \frac{4 - x}{x - 4} \Rightarrow xy - 4y = 4 - x$$
$$\Rightarrow \quad xy + x = 4 + 4y \Rightarrow x(y + 1) = 4(1 + y)$$
$$\therefore \quad x = \frac{4(1 + y)}{y + 1}$$
x assumes real values, if $y + 1 \ne 0$ *i.e.*, $y = -1$.

$\therefore$ Range of $f = R - \{-1\}$

Q. 32 The domain and range of real function f defined by

$$f(x) = \sqrt{x-1} \text{ is given by}$$

(a) Domain $= (1, \infty)$, Range $= (0, \infty)$ (b) Domain $= [1, \infty)$, Range $= (0, \infty)$

(c) Domain $= (1, \infty)$, Range $= [0, \infty)$ (d) Domain $= [1, \infty)$, Range $= [0, \infty)$

Thinking Process

A function is defined $f(x) = \sqrt{x}$ is defined $x \geq 0$.

Sol. ***(d)*** We have, $f(x) = \sqrt{x-1}$

$f(x)$ is defined, if $x - 1 \geq 0$.

$\Rightarrow \quad x \geq 1$

$\therefore \quad$ Domain of $f = [1, \infty)$

Let $\quad f(x) = y$

$\therefore \quad y = \sqrt{x-1}$

$\Rightarrow \quad y^2 = x - 1$

$\therefore \quad x = y^2 + 1$

x assumes real values for $y \in R$.

but $\quad y \geq 0$

$\therefore \quad$ Range of $f = [0, \infty)$

Q. 33 The domain of the function f given by $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$.

(a) $R - \{3, -2\}$ (b) $R - \{-3, 2\}$ (c) $R - [3, -2]$ (d) $R - (3, -2)$

Sol. ***(a)*** We have, $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - x - 6}$

$f(x)$ is defined, if $x^2 - x - 6 = 0$

$\Rightarrow \quad x^2 - 3x + 2x - 6 = 0$

$\Rightarrow \quad x(x-3) + 2(x-3) = 0$

$\Rightarrow \quad (x-3)(x+2) = 0$

$\therefore \quad x = -3, -2$

$\therefore \quad$ Domain of $f = R - \{3, -2\}$

Q. 34 The domain and range of the function f given by $f(x) = 2 - |x - 5|$ is

(a) Domain $= R^+$, Range $= (-\infty, 1]$

(b) Domain $= R$, Range $= [-\infty, 2]$

(c) Domain $= R$, Range $= (-\infty, 2)$

(d) Domain $= R^+$, Range $= (-\infty, 2]$

Sol. ***(b)*** We have, $f(x) = 2 - |x - 5|$

$f(x)$ is defined for all $x \in R$

$\therefore \quad$ Domain of $f = R$

We know that, $\quad |x-5| \geq 0 \Rightarrow -|x-5| \leq 0$

$\Rightarrow \quad 2 - |x-5| \leq 2$

$\therefore \quad f(x) \leq 2$

$\therefore \quad$ Range of $f = [-\infty, 2]$

Q. 35 The domain for which the functions defined by $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal to

(a) $\left[-1, \frac{4}{3}\right]$ (b) $\left[1, \frac{4}{3}\right]$

(c) $\left[-1, -\frac{4}{3}\right]$ (d) $\left[-2, -\frac{4}{3}\right]$

Sol. ***(a)*** We have, $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$

$$f(x) = g(x)$$

$$\Rightarrow \quad 3x^2 - 1 = 3 + x$$

$$\Rightarrow \quad 3x^2 - x - 4 = 0$$

$$\Rightarrow \quad 3x^2 - 4x + 3x - 4 = 0$$

$$\Rightarrow \quad x(3x - 4) + 1(3x - 4) = 0$$

$$\Rightarrow \quad (3x - 4)(x + 1) = 0$$

$$\therefore \quad x = -1, \frac{4}{3}$$

So, domain for which $f(x)$ and $g(x)$ are equal to $\left[-1, \frac{4}{3}\right]$.

Fillers

Q. 36 Let f and g be two real functions given by

$$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$$

and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$,

then the domain of $f \cdot g$ is given by...........

Thinking Process

First find the domain of f and domain of g. Then,

domain of $f \cdot g$ = domain of $f \cap$ domain of g.

Sol. We have, $f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$

and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$

$\therefore$ Domain of $f = \{0, 2, 3, 4, 5\}$,

and Domain of $g = \{1, 2, 3, 4, 5\}$

$\therefore$ Domain of $(f \cdot g)$ = Domain of $f \cap$ Domain of $g = \{2, 3, 4, 5\}$

Q. 37 Let $f = \{(2, 4), (5, 6), (8, -1), (10, -3)\}$
and $g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, 5)\}$
be two real functions. Then, match the following.

Column I		Column II	
(i)	$f - g$	(a)	$\left\{\left(2, \frac{4}{5}\right), \left(8, \frac{-1}{4}\right), \left(10, \frac{-3}{13}\right)\right\}$
(ii)	$f + g$	(b)	$\{(2, 20), (8, -4), (10, -39)\}$
(c)	$f \cdot g$	(c)	$\{(2, -1), (8, -5), (10, -16)\}$
(d)	$\frac{f}{g}$	(d)	$\{(2, 9), (8, 3), (10, -10)\}$

The domain of $f - g, f + g, f \cdot g, \frac{f}{g}$ is domain of $f \cap$ domain of g. Then, find their images.

Sol. We have,

$$f = \{(2, 4), (5, 6), (8, 1), (10, -3)\}$$

and $$g = \{(2, 5), (7, 1), (8, 4), (10, 13), (11, 5)\}$$

So, $f - g, f + g, f \,.\, g, \frac{f}{g}$ are defined in the domain (domain of $f \cap$ domain of g)

i.e., $\{2, 5, 8, 10\} \cap \{2, 7, 8, 10, 11\} \Rightarrow \{2, 8, 10\}$

(i) $(f - g)(2) = f(2) - g(2) = 4 - 5 = -1$

$$(f - g)(8) = f(8) - g(8) = -1 - 4 = -5$$
$$(f - g)(10) = f(10) - g(10) = -3 - 13 = -16$$
$$\therefore \quad f - g = \{(2, -1), (8, -5), (10, -16)\}$$

(ii) $(f + g)(2) = f(2) + g(2) = 4 + 5 = 9$

$$(f + g)(8) = f(8) + g(8) = -1 + 4 = 3$$
$$(f + g)(10) = f(10) + g(10) = -3 + 13 = 10$$
$$\therefore \quad f + g = \{(2, 9), (8, 3), (10, 10)\}$$

(iii) $(f \cdot g)(2) = f(2) \cdot g(2) = 4 \times 5 = 20$

$$(f \cdot g)(8) = f(8) \cdot g(8) = -1 \times 4 = -4$$
$$(f \cdot g)(10) = f(10) \cdot g(10) = -3 \times 13 = -39$$
$$\therefore \quad fg = \{(2, 20), (8, -4), (10, -39)\}$$

(iv) $\left(\frac{f}{g}\right)(2) = \frac{f(2)}{g(2)} = \frac{4}{5}$

$$\left(\frac{f}{g}\right)(8) = \frac{f(8)}{g(8)} = \frac{-1}{4}$$
$$\left(\frac{f}{g}\right)(10) = \frac{f(10)}{g(10)} = \frac{-3}{13}$$
$$\therefore \quad \frac{f}{g} = \left\{\left(2, \frac{4}{5}\right), \left(8, -\frac{1}{4}\right), \left(10, \frac{-3}{13}\right)\right\}$$

Hence, the correct matches are (i) → (c), (ii) → (d), (iii) → (b), (iv) → (a).

True/False

Q. 38 The ordered pair (5, 2) belongs to the relation
$$R = \{(x, y) : y = x - 5, x, y \in Z\}$$

Sol. ***False***

We have, $R = \{(x, y) : y = x - 5, x, y \in Z\}$

If $x = 5$, then $y = 5 - 5 = 0$

Hence, (5, 2) does not belong to R.

Q. 39 If $P = \{1, 2\}$, then $P \times P \times P = \{(1, 1, 1), (2, 2, 2), (1, 2, 2), (2, 1, 1)\}$

Sol. ***False***

We have, $P = \{1, 2\}$ and $n(P) = 2$

$\therefore$ $n(P \times P \times P) = n(P) \times n(P) \times n(P) = 2 \times 2 \times 2 = 8$

But given $P \times P \times P$ has 4 elements.

Q. 40 If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then $(A \times B) \cup (A \times C)$ $= \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

Thinking Process

First, we find $A \times B$ and $A \times C$, then we will find $(A \times B) \cup (A \times C)$.

Sol. ***True***

We have, $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$

$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$

$A \times C = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$(A \times B) \cup (A \times C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

Q. 41 If $(x - 2, y + 5) = \left(-2, \frac{1}{3}\right)$ are two equal ordered pairs, then $x = 4$, $y = \frac{-14}{3}$

Sol. ***False***

We have, $(x - 2, y + 5) = \left(-2, \frac{1}{3}\right)$

$\Rightarrow$ $x - 2 = -2, y + 5 = \frac{1}{3} \Rightarrow x = -2 + 2, y = \frac{1}{3} - 5$

$\therefore$ $x = 0, y = \frac{-14}{3}$

Q. 42 If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, then $A = \{a, b\}$ and $B = \{x, y\}$.

Sol. ***True***

We have, $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$

A = Set of first element of ordered pairs in $A \times B = \{a, b\}$

B = Set of second element of ordered pairs in $A \times B = \{x, y\}$