Chapter - IX

Probability

Learning Objectives:

Students will be able to:

- Find out sample space in a random experiment
- Recognize whether events are mutually exclusive, independent or dependent
- Apply theorem of total probability in different situations
- Apply Bayes' theorem in practical situations.

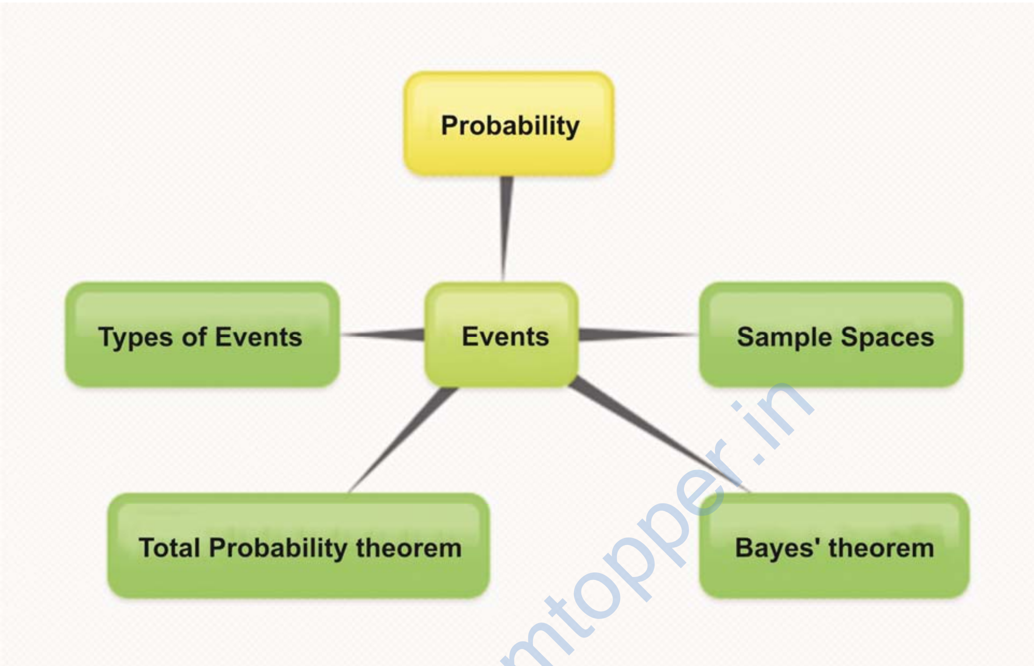
Before you start you should know

Sets, Union and Intersection of sets, Definition of probability and Addition and multiplication rule of probability.

Revise your ideas on probability here:

https://www.khanacademy.org/math/probability/probability-geometry/probability-basics/a/probability-the-basics

Concept Map



9.1 Introduction

Probabilities are part and parcel of life. Some examples from daily life where probability calculations are involved are the determination of premium of insurance, the introduction of new medicines in the market, opinion and exit polls and weather forecasts, In everyday life, many of us use probabilities in our language and say things like "I am one hundred percent sure".

Many important practical problems involving probability can be solved using ***Total Probability Theorem and Bayes' theorem***. These theorems can be used to solve the problems related to insurances, to find out the rate of risk of lending money to potential borrowers, determine the accuracy of medical test results etc. In this unit we are going to discuss more about the practical applications of these two theorems. To understand and apply these two theorems we should know some basic concepts like **random experiment, sample space, events, mutually exclusive events, independent and dependent events.**

9.2 Random experiment and sample space:

An experiment is called random if it has more than one possible outcome and it is not possible to predict the outcome in advance. For example before rolling a die, we do not know the result and hence is random in nature. Rolling a die, therefore is an example of a random experiment.

A **random experiment** is a process by which we observe something uncertain.

After the experiment, the outcome of the random experiment is known. An **outcome** is a result of a random experiment.

The set of all possible outcomes is called the **sample space**.

Sample space can be represented as a **Set** of all possible outcomes. Thus in the context of a random experiment, the sample space is our universal set.

Here are some examples of random experiments and their sample spaces represented in the form of sets.

Random experiment: Tossing a coin	**Random experiment: Rolling a die**
	
Sample space: S={head, tail}	Sample space: D={1,2,3,4,5,6}

Random experiment : Observing the number of android phones sold by a company in 2019.  Sample space: L={0,1,2,3,⋯}	**Random experiment:** Observing the number of goals scored in a soccer match.  Sample space: K={0,1,2,3,⋯}

When a random experiment consists of performance repeated several times, we call each one of them a **trial**. Thus, **a trial is a particular performance of a random experiment.** In tossing a coin, each trial will result in either heads or tails. **Note that the sample space is defined based on how you define the outcome of your random experiment.** For example, if we toss a coin three times and observe the sequence of heads/tails as an outcome, which is usually the understanding, the sample space may be defined as:

S={(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)}.

And if the outcome of the experiment of tossing a coin three times is defined as the number of heads, then S = {0,1,2,3}

Example 1 :

In each of the following experiments specify appropriate sample space

(i) A girl has a one rupee coin, a two rupee coin and a five rupee coin in her pocket. She takes out two coins out of her pocket, one after the other.

(ii) A person is noting down the number of accidents along a busy road during a year.

Solution:

(i) Let Q denote a 1 rupee coin, H denote a 2 rupee coin and R denote a 5 rupee coin. The first coin she takes out of her pocket may be any one of the three coins Q, H or R. Corresponding to Q, the second draw may be H or R. So the result of two draws may be QH or QR. Similarly, corresponding to H, the second draw may be Q or R. Therefore, the outcomes may be HQ or HR. Lastly, corresponding to R, the second draw may be H or Q. So, the outcomes may be RH or RQ.

Thus, the sample space is S={QH, QR, HQ, HR, RH, RQ}

(ii) The number of accidents along a busy road during the year of observation can be either 0 (for no accident) or 1 or 2, or some other positive integer.

Thus, a sample space associated with this experiment is S= {0,1,2,...}

Example-2 : A bag has 3 blue and 4 white balls in it. If a tossed coin shows head, we draw a ball from the bag. If it shows tail we throw a die to show a number. Describe the sample space of this experiment.

Solution: Let us denote blue balls by B1, B2, B3 and the white balls by W1, W2, W3, W4.

Then a sample space of the experiment is

S = { HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6}.

Here HB_1 means head on the coin and ball B_1 HW_1 means head on the coin and ball W_1. For any number i, HB_i means head on the coin and ball B_i. HW_i means head on the coin and ball W_i. Similarly, Tj means tail on the coin and the number j on the die. Here, i = 1, 2, 3; j = 1, 2, 3, 4, 5, 6

Check your Progress -1 :

1. Write down an experiment in practical life whose sample space is $S = \{0,1,2,...\}$

2. Suppose 3 bulbs are selected at random from a lot of bulbs. Each bulb is tested and classified as defective (D) or non – defective (N). Write the sample space of this Experiment.

9.3 Event and its probability:

When we say "Event" associated with a random experiment we mean a set of one (or more) outcomes. We have seen earlier that sample space of tossing a coin three times is defined as.

$$S=\{(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)\}.$$

Suppose, we want to find only the outcomes which have at least two heads; then the set of all such possibilities (outcomes) can be given as:

$$E=\{(H,H,H),(H,H,T),(H,T,H),(T,H,H)\}.$$

E is an event associated with this experiment. Thus, an event is a subset of the sample space, i.e., E is a subset of S. Even the empty set is a subset of S, hence an event, called the Impossible event. The S, itself is a subset of S, hence an event, called 'Sure' event.

There could be a lot of events associated with a Random Experiment. For any event E to occur, the outcome of the experiment must be an element of the set E. **Then how to calculate the probability of an event?**

The ratio of number of favourable outcomes to the total number of outcomes is defined as the probability of occurrence of an event.

Note: This definition applies when the outcomes of the experiment are equally likely to occur.

So, the probability that an event will occur is given as:

$$P(E) = \frac{\text{Number of Favourable Outcomes}}{\text{Total Number of Outcomes}}$$

Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability **of each outcome is** $\frac{1}{10}$

Now the probability of event A is

P(A) = P(2) + P(4) + P(6) + P(8)

(Following an axiom of Probability theory)

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

We have calculated the probability of an event as sum of probabilities of each outcome in that event. While calculating the probability of each outcome we have taken into account sample space for finding the total number of outcomes.

Similarly we can also find an event which is **not A**. Let us write down the event not A with reference to the example we have discussed earlier.

A′ = {1, 3, 5, 7, 9, 10}

P(A′) = P(1) + P(3) + P(5) + P(7) + P(9) + P(10) = $6 \times \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$

We can easily see that **P(A’) = 1- P(A)**

This means that , if the event A occurs event ‘*not A*’ cannot occur. They are called mutually exclusive events.

Note : Two mutually exclusive events need not be compliment of each other.

Two events are said to be mutually exclusive if the occurrence of one rules out the occurrence of the other. E and F are mutually exclusive if $E \cap F = \varnothing$

Some terms Similar to this term related to events are used in our future discussions. Let us understand these terms.

Independent Events and Dependent Events:

If the probability of occurrence of an event is completely unaffected by the occurrence (or non-occurrence) of another event, such events are termed as independent events. The events that are not independent are called **dependent events.**

Impossible and Sure Events:

If the probability of occurrence of an event is zero such an event is called an impossible event. If the probability of occurrence of an event is 1, it is called a sure event. In other words, the empty set ϕ represents an impossible event and the sample space S represents a sure event.

Exhaustive Events:

A set of events is called exhaustive if all the events together consume the entire sample space. Their union is the sample space.

Mutually Exclusive Events:

If the occurrence of one event excludes the simultaneous occurrence of another event, then such events are called mutually exclusive events. For example, if $S = \{1, 2, 3, 4, 5, 6\}$ and E_1, E_2 are two events such that E_1 consists of numbers less than 4 and E_2 consists of a number greater than 5.

So, $E_1 = \{1,2,3\}$ and $E_2 = \{6\}$.

Then, E_1 and E_2 are also called mutually exclusive events.

Dependent Events

Mr. Verma and Ms. Banerjee plan to invest some money in the shares of one of the companies in the following sectors:

1. Banking
2. Software development
3. Metal
4. Automobiles

They approach the same financial analyst for advice and his recommendation has definite influence on their decision.
The two events

E = Mr. Verma invests in shares of banking sector
F = Ms. Banerjee invests in metal stocks
are dependent events as they have the same financial advisor influencing their decision.

Independent Events

A missile radar detective system consists of two radar screens 'RSO2A' and 'RQ54L'. The probability that these will detect an incoming missile are 0.9 and 0.85 respectively.
The two events
E = radar screen 'RS02A' detects the missile
F = radar screen 'RQ54L' detects the missile
are independent events as the probability of detection by any one is not affected by the detection of missile by of the other.

Impossible and Sure Events

Impossible Event

The cut off list for centralized admission to B.Com program in various colleges is as under:

College	Minimum Percentage
Hindu College	95%
St. Stephens	97%
Hansraj	95%
SRCC	96%

If Sarthak who scored 92% in class 12 has applied for admission in B.Com program then the event

E = Sarthak gets admission in one of these colleges

is an impossible event as his marks are less than the minimum cut off required

Sure Event

A manufacturer of earphone claims that not more than 2 pieces of every 1000 earphones manufactured are defective. It also states that a consignment of earphones for export is rejected if it contains atleast 3 defective pieces per 1000.

The event:

A = The consignment for export by the manufacturer is accepted is a sure event.

Exhaustive Events

An inhabitant is selected at random from a city.

The events

S_1 = The person selected is covid +ve

S_1' = The person selected is not covid +ve

S_2 = The person is diabetic

The events S_1 and S_1' are exhaustive events as they cover the whole sample space.

Mutually Exclusive Events

A packet containing flower seeds contains 30 seeds of which 12 seeds are of red flower, 8 seeds of yellows flowers and 10 seeds of blue flowers.

The two events

E_1 = the seed produces red flower

E_2 = the seed produces yellow flower

are mutually exclusive events as the flower can be either red or yellow in colour.

Mutually Exclusive and Exhaustive Events

A financial institution sanctioned the following type of loans in 2019.

Type of loans	Number of Approvals
1. Business loan	18
2. Debit consolidation loan	15
3. Economic opportunity loan	17

If an application from approved loans is selected at random for review then the events:

E_1 = application for business loan

E_2 = application for debit consolidation loan

E_3 = application for economic opportunity loan

are mutually exclusive and exhaustive events as E_1, E_2 and E_3 have nothing in common and all the applications are of one of these types.

Check your progress-2:

Give real life examples of the following:

1. Independent Events and Dependent Events
2. Impossible and Sure Events
3. Exhaustive Events
4. Mutually Exclusive Events

Mutually exclusive and exhaustive Events

9.4 Conditional Probability

If we have two events associated with the same experiment, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur.

Consider the experiment of tossing three fair coins. The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point

Let us consider two events E & F. Let E be the event where 'at least two heads appear' and F be the event where 'first coin shows tail'.

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E?

In the above example above we can clearly see that we have placed a condition: When F occurs, then what is the probability of occurrence of E?

Do you think the probability of event E with the condition will be same or different?

Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'.

Then we can write the sample spaces as:

E = {HHH, HHT, HTH, THH} and F = {THH, THT, TTH, TTT}

$\therefore$ P(E) = P (HHH) + P (HHT) + P (HTH) + P (THH)

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$P(F) = P(\{THH\}) + P(\{THT\}) + P(\{TTH\}) + P(\{TTT\})$

$$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Now we have to consider only those outcomes which are favourable to the occurrence of the event F to answer the question "Suppose we are given that the first coin shows tail, what is the probability of occurrence of E?". That means we have to find out what **is common between E and F** or the condition given in F matches with that in the Event E. This can be found as given below.

$E \cap F = \{THH\}$

Now, the sample point of F which is favourable to event E is THH.

Thus, Probability of E considering F as the sample space = $\frac{1}{4}$

or Probability of E given that the event F has occurred = $\frac{1}{4}$

This probability of the event E is called the conditional probability of E given that F has already occurred, and is denoted by **P (E|F).**

Hence, $P(E|F) = \frac{1}{4}$

Note that the elements of F which favour the event E are the common elements of E and F, i.e. the sample points of $E \cap F$.

If E and F are the two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred is given by:

$$P(E|F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \text{ provided } P(F) \neq 0$$

Example: A family has two children. What is the probability that both the children are boys given that at least one of them is a boy?

Solution :

Let E and F denote the following events :

E : both the children are boys

F : at least one of the children is a boy

Then E = {(b,b)} and F = {(b,b), (g,b), (b,g)}

Now E$\cap$ F = {(b,b)}

Thus P(E | F) = $\frac{1}{3}$

We have used the formula :

P(E | F) = $\dfrac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$

You can also verify the following formula in the above case.

P(E | F) = $\dfrac{P(E \cap F)}{P(F)}$ provided P (F) $\neq$ 0

Check your progress-3

1. In a group of 100 sports car buyers, 40 bought alarm systems, 30 purchased bucket seats, and 20 purchased an alarm system and bucket seats. If a car buyer chosen at random, bought an alarm system, what is the probability they also bought bucket seats?

2. From the given data, find out the probability that a randomly selected person is male, given that he owns a pet?

	Have pets	Do not have pets	Total
Male	0.41	0.08	0.49
Female	0.45	0.06	0.51
Total	0.86	0.14	1

9.5 Theorem of Total Probability

Theorem of total probability "expresses the total probability of an outcome which can be realized via several distinct events".

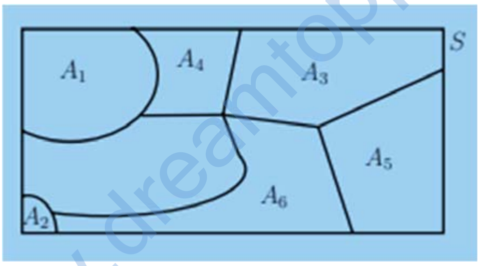
The collection of events $A_1, A_2, \ldots, A_n$ is said to be a **partition** in a sample space S if

(a) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(b) $A_i \cap A_j = \phi$ for all $i, j, i \neq j$

(c) $A_i \neq \phi$ for all i

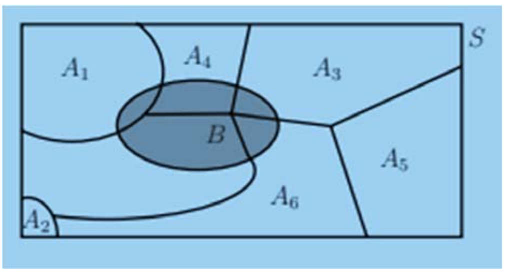
In essence, **a partition is a collection of non-empty, non-overlapping subsets of a sample space whose union is the sample space itself.**



If B is any event within S then we can express B as the union of subsets:

$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$

This is illustrated in the diagram that follows in which an event B in S is represented by the shaded region.



The bracketed events $(B \cap A_1), (B \cap A_2), \ldots (B \cap A_n)$ are mutually exclusive (if one occurs then none of the others can occur). Using the addition law of probability for mutually exclusive events, we can write probability of event B as:

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$$

Each of the probabilities on the right-hand side may be expressed in terms of conditional probabilities:

$$P(B \cap A_i) = P(B \mid A_i)P(A_i) \text{ for all } i \quad \left(\text{as } P(B|A) = \frac{P(B \cap A)}{P(A)}\right)$$

Using these in the expression for P(B), above, gives:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n)$$

$$\sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Let us understand this theorem with an example. A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Now we have to determine the probability that the construction job will be completed on time.

To solve this problem let us see if this comprise of two mutually exclusive events. We can see that there is either strike (B) or there is no strike (B′). The sum of these probabilities should be 1. If we have strike then we cannot have 'no strike' and vice versa. That means events B and B′ form partition of the sample space.

Hence we can write probability of strike as

$$P(B) = 0.65,$$

$$P(B') = P(\text{no strike}) = 1 - P(B) = 1 - 0.65 = 0.35$$

Now there is another event : Completion of work A. This can happen when there is strike and also when there is no strike for various reasons. This fits the requirement of application of theorem of total probability, we have

It is given that the probability of completion of work when there **is strike** (B) as

$P(A\,|\,B) = 0.32$

and probability of completion of work when there **is no strike** (B′) as

$P(A\,|\,B') = 0.80$

Now we can easily find out the probability of completion of work in any case as:

$P(A) = P(B)\,P(A\,|\,B) + P(B')\,P(A\,|\,B')$

$= 0.65 \times 0.32 + 0.35 \times 0.8$

$= 0.208 + 0.28 = 0.488$

Thus, the probability that the construction job will be completed in time is 0.488.

WATCH THIS VIDEO : https://www.youtube.com/watch?v=U3_783xznQI

Check your progress-4 :

Its given that 80% of people attend their family and doctor regularly; 35% of these people have no health problems cropping up during the following year. Out of the 20% of people who don't see their doctor regularly, only 5% have no health issues during the following year. What is the probability a person selected at random will have no health problems in the following year?

9.6 Bayes' theorem

Bayes' theorem, named after 18th-century British mathematician Thomas Bayes, is a mathematical formula for determining conditional probability. It provides a way to revise existing predictions or theories (update probabilities) given new or additional evidence. This theorem is the foundation of the field of Bayesian statistics.

Statement of the theorem:

If $A_1, A_2, \ldots A_n$ are n non empty events which constitute a partition of sample space S

i.e., $A_i \cap A_j = \phi$ for all i, j $i \neq j$ which means

$A_1, A_2, \ldots A_n$ are pairwise disjoint

and $A_1 \cup A_2 \cup \ldots\ldots \cup A_n = S$

Also B is any event of non-zero probability, then

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)} \text{ for any } i = 1, 2, 3, \ldots, n$$

Proof: By formula of conditional probability, we know

$$P(A_i|B) = \frac{P(B \cap A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B|A_i)}{P(B)} \qquad \text{(By multiplication rule)}$$

$$= \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)} \qquad \text{(By theorem of total probability)}$$

Remark: In Bayes' theorem the following terminology is applied.

The events $A_1, A_2, \ldots A_n$ are called hypotheses.

The probability $P(A_i)$ is called the priori probability of the hypotheses.

The conditional probability $P(A_i|B)$ is called a posteriori probability of the hypothesis A_i.

Let us try to understand this theorem with a practical example given below:

Illustration: Let us find out patients' probability of having liver disease if they are alcoholic. "Being an alcoholic" is the **test** for liver disease.

- **A** could mean the event "Patient has liver disease." Let us say 10% of patients entering a clinic have liver disease as per data available with the doctor. Probability of event A = P(A) = 0.10
- **B** could mean the confirmatory test that "Patient is an alcoholic." Let us assume that five percent of the clinic's patients are alcoholics. Probability of event B = P(B) = 0.05

You might also know that among those patients diagnosed with liver disease, 7% are alcoholics. This is **B | A:** the probability that a patient is alcoholic, given that they have liver disease, is 7%. Applying Bayes' theorem, we get

$$P(A|B) = (0.07 \times 0.1)/0.05 = 0.14$$

In other words, if the patient is an alcoholic, his/her chances of having liver disease is 0.14 (14%). This is a large increase from the 10% suggested by past data. But it's still unlikely that any particular patient has liver disease.

WATCH THESE VIDEOS NOW TO INTERNALIZE THE THEOREM.

Bayes' theorem : https://www.youtube.com/watch?v=HZGCoVF3YvM

Implications of Bayes' theorem :
https://www.youtube.com/watch?v=R13BD8qKeTg

Prior probability, in Bayesian statistical inference, is the probability of an event before new data is collected. This is the best way of assessing the probability of an outcome based on the current knowledge before an experiment is performed. **Posterior probability** is the revised probability of an event occurring after taking into consideration new information. Posterior probability is calculated by updating the prior probability by using Bayes' theorem. In statistical terms, the posterior probability is the probability of event A occurring given that event B has occurred.

Bayes' theorem thus gives the probability of an event based on new information that is, or may be related, to that event. The formula can also be used to see how the probability of an event occurring is affected by hypothetical new information, supposing the new information will turns out to be true.

Exercise on Bayes' theorem

1. The number of loans sanctioned by a particular branch of a bank under different heads and the percentage of defaults in each category is given above:

Types of Loan	Number of Loans Approved	Defaults (%)
Personal Loan	15	3%
Education Loan	5	1%
Housing Loan	10	2%
Car Loan	10	5%

If the loan application form picked at random for review is found to be of a person who has defaulted then find the probability that the application was for car loan.

2. A courier service company sends 30% of its orders by air, 50% by combination of bus and local transport and remaining 20% by train. Past record shows the courier is delivered late 2%, 7% and 5% of the time when orders are sent by air, bus local transport and train respectively. Find

   (i) the probability that the order will be delivered late

   (ii) the probability that the parcel delivered to a customer is sent by train if it is delivered late.

3. A young entrepreneur imports high tech machines for a startup venture. The imported machines are to be set up by an expert who is sent by the firm making these machines. From experience it is known that 80% of the times the expert is able to correctly set up the machines. If the setup is correctly done the machine produces 90% acceptable items and in case of an incorrect set up the machine produces only 50% acceptable item. If after a certain set up the machine produces an acceptable item followed by an unacceptable item find the probability that the machine is incorrectly set up.

4. An insurance company insures scooter drivers, car drivers and bus drivers in the ratio 4:5:3. The probability of a scooter driver, car driver and bus driver meeting with an accident is 0.7%, 0.4% and 1.2% respectively. If an insured

person meets with an accident find the probability that the person is a scooter driver.

5. Two cards from a pack of 52 cards are lost. From the remaining cards of the pack a card is drawn at random and is found to be spade. Find the probability that the lost cards are both spades.

6. A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested. If 0.2% of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive?

Check your progress - 5

1. Car Repair Diagnosis

   The manager of a car repair workshop knows from past experience that when a call is received from a person who is stuck up far away and has problem starting the car, the probabilities of various troubles are as follows:

Event	**Trouble**	**Probability**
A_1	Battery problem	0.4
A_2	No petrol	0.3
A_3	Flooded	0.1
A_4	Some other reason	0.2

   Assuming that no two faults occur simultaneously and E be the event that the car starts if instructions given by the manager are followed by the driver with probabilities:

   $P(E|A_1) = 0.3$, $P(E|A_2) = 0$, $P(E|A_3) = 0.8$, $P(E|A_4) = 0.5$

   (a) If a person follows the instructions given by the manager, what is the probability that the car starts?

   (b) If the car starts on following the instructions of the manager, find the probability that car had a battery problem.

2. In a factory which manufactures bulbs, units A, B and C manufacture respectively 25%, 35% and 40% of the bulbs. Of their outputs, 5, 4 and 2 percent are respectively defective bulbs. A bulbs is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the unit B?

Summary

1. A random experiment is a process by which we observe something uncertain
2. An outcome is a result of a random experiment.
3. The set of all possible outcomes is called the sample space. The sample space is defined based on how you define your random experiment.
4. A trial is a particular performance of a random experiment.
5. An event is a subset of the sample space.
6. The ratio of number of favourable outcomes to the total number of outcomes is defined as the probability of occurrence of an event.
7. We have Independent or dependent Events, Impossible and Sure Events, Exhaustive Events, Mutually Exclusive Events according to the probabilities with reference to the sample space.
8. If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred,

$$P(E \mid F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, \text{ provided } P(F) \neq 0$$

9. Theorem of total probability "expresses the total probability of an outcome which can be realized via several distinct events".

if

(a) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(b) $A_i \cap A_j = \phi$ for all i, j

(c) $A_i \neq \phi$ for all i

Then :

$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_n)P(A_n)$

$$= \sum_{i=1}^{n} P(B|A_1)P(A_i)$$

10. Bayes' Theorem If A_1, A_2,... An are n non empty events which constitute a partition of sample space S. Also B is any event of non zero probability, then

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)} \quad \text{for any } i = 1,2,3, \ldots, n$$

Practical and project work

1. Find out different examples of practical applications of Total Probability theorem and Bayes' theorem in finance, research and industry

2. Search for data and frame a problem to be solved by Bayes' theorem. Use excel sheet to calculate the result.

Further exploration

1. https://www.cimt.org.uk/projects/mepres/alevel/stats_ch1.pdf

Solutions to Check your Progress - 1

1. Observing the Number of people voted in a constituency.

2. S= {DDD, DDN, DND, NDD, DNN, NDN, NND, NNN}

Solutions to check your Progress-2

1. Independent events

   A : Person having black hair

   B : Person working in a MNC

   Dependent events : Traffic and Condition of road Accident

2. Impossible event : Sun rising in the west

   Sure event : Getting sum of number ≤ 12 when a pair of dice are rolled

3. Exhaustive events

   A: Getting a head in a single throw of a coin

   B: Getting a tail in a single throw of a coin

4. Mutually exclusive events

   A: A person running forward

   B: A person running backward avB=0

5. Mutually exclusive and exhaustive events

   A: getting an even number in a throw of a die

   B: getting an odd number in a single throw of a die.

Solution to check your Progress - 3

1. Let A denotes the event of car buyer buying alarm system and B denotes the event of car buyer buying bucket seat

P(A) = 40%, or 0.4.

P(A∩B) = $\frac{20}{100} = 0.2$

Substitude the values in the formula

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.4} = 0.5$$

The probability that a buyer bought bucket seats, given that they purchased an alarm system, is 50%.

2. Let M stands for event of person being male and P stands for a event that person is a pet owner, we have:

$P(M \cap P) = 0.41$.

$P(P) = 0.86$

Substituting in the formula:

$$P(M \mid P) = \frac{P(M \cap P)}{P(M)} = \frac{0.41}{0.86} = 0.477, \text{or } 47.7\%$$

Solution to check your Progress – 4

1. Let D denotes the event of seeing the doctor regularly and H denotes the event of facing no health issues the following year.

   Probability of those who see doctor regularly, $P(D) = 0.80$

   Probability of those who do not see the doctor regularly, $P(D') = 0.20$

   Probability of those who see the doctor given that have no health issues $P(H \mid D) = 0.35$

   Probability of those who do not see doctor and have no health issues $P(H \mid D') = 0.05$

   $P(H) = P(D)\,P(H \mid D) + P(D')\,P(H \mid D')$

   $= 0.80 \times 0.35 + 0.20 \times 0.05 = 0.28 + 0.01$

Answer to exercise on Baye's Theorem

1. 5/12
2. 51/1000, 10/51
3. 25/61
4. 1/3
5. .051
6. 198/697

Solution to Check your Progress - 5

1 (a) We need to compute P(E)

Using theorem of total probability, we get

P(E) $= P(A_1)\,P(E\mid A_1) + P(A_2)\,P(E\mid A_2) + P(A_3)\,P(E\mid A_3) + P(A_4)\,P(E\mid A_4)$

$= 0.4 \times 0.3 + 0.3 \times 0 + 0.1 \times 0.8 + 0.2 \times 0.5$

$= 0.12 + 0 + 0.08 + 0.10$

$= 0.3$ ... (1)

The probability that the car starts is 0.3

(b) We use Bayes' formula to compute the posteriori probability $P(A_1 \mid E)$:

$$P(A_1/E) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{4} P(A_j)P(B|A_i)}$$

$$= \frac{0.4 \times 0.3}{0.3} \qquad \text{(Using 1)}$$

$= 0.4$

Thus the probability that the car had battery problem if it started on following managers instruction is 0.4.

2. Let B_1, B_2, B_3 be the event that the bulb is manufactured by units A, B and C respectively and E be the event that bulb manufactured is defective

$P(B_1) = 0.25$, $P(B_2) = 0.35$ and $P(B_3) = 0.40$

$P(E \mid B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by unit A = 5% = 0.05

Also $P(E \mid B_2) = 0.04$, $P(E \mid B_3) = 0.02$.

$$P(B_2|E) = \frac{P(B_2)\times P(E|B_2)}{P(B_1)\times P(E|B_1)+P(B_2)\times P(E|B_2)+P(B_3)\times P(E|B_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05+0.35\times 0.04+0.40\times 0.02}$$

$$= \frac{0.014}{0.0345} = \frac{28}{69}$$