Unit 8

Linear Programming Problem

8.0 LEARNING OUTCOMES

At the end of this unit, the student will be able to:

- Understand the concept of Linear Programming Problem.
- Know the Mathematical Formulation of Linear Programming Problem.
- Conceptualize the feasible region and infeasible region.
- Distinguish between the feasible solution and optimal solution.
- Find the optimal solution of LPP by Graphical Method.
- Know the meaning of Optimization.

Before you start you should know:

- Graphing a given linear equation or a linear inequality
- Knowledge of linear inequalities
- Solving the simultaneous linear equations.
- Finding the coordinates of intersection point of linear equations/ inequalities

CONTENT

- Introduction and related terminologies (constraints, objective function, optimization)
- Mathematical formulation of LPP
- Application of LPP on different types of real life situations
- Graphical method of solution for problems in two variables
  - ₹ Corner- method
  - ₹ Iso-profit/iso-cost method
- Feasible and Infeasible regions (Bounded and Unbounded)
- Feasible and Infeasible solution, optimal feasible solution (up to three non-trivial constraints.

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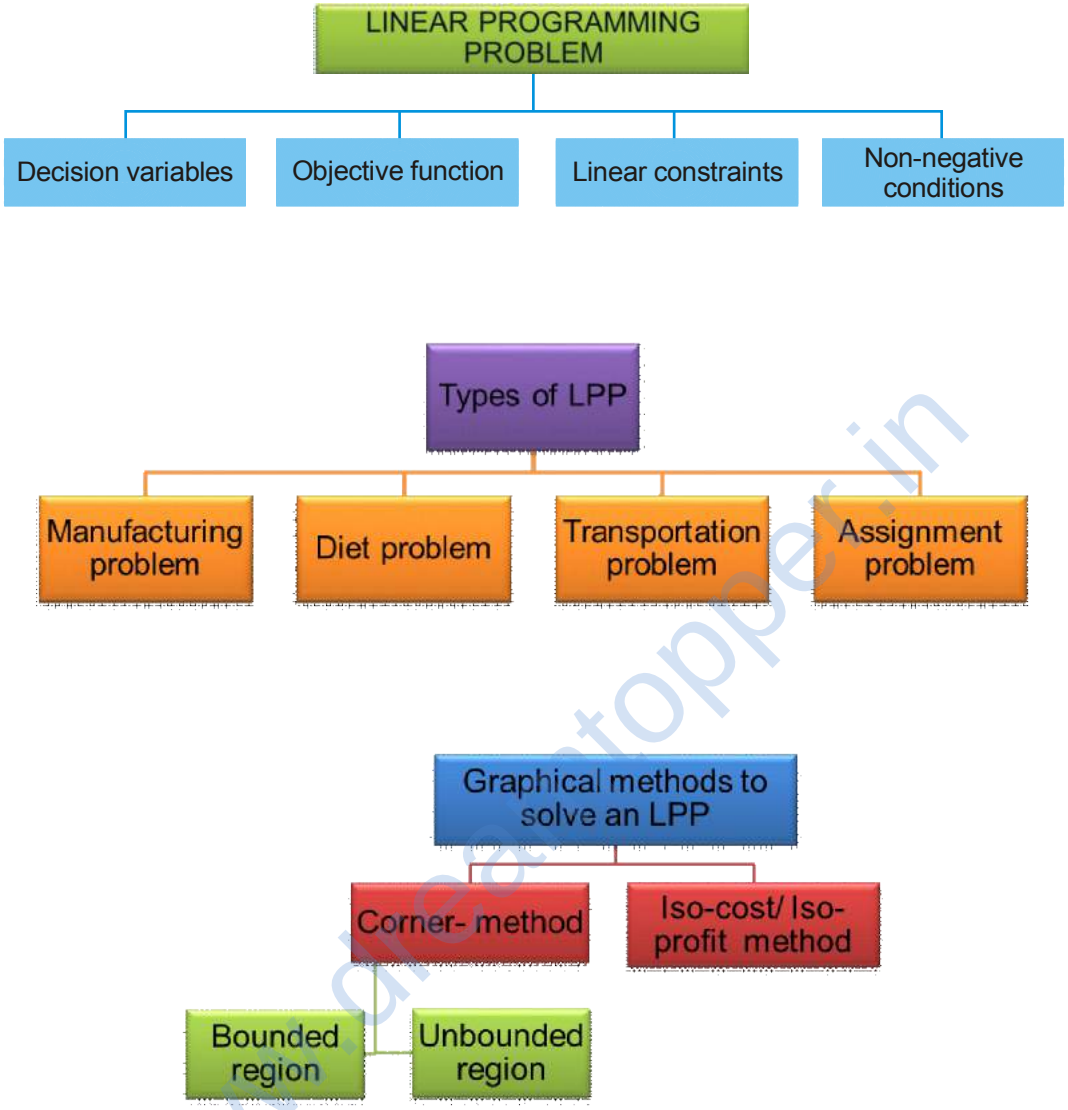
MIND MAP



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8.0 INTRODUCTION

Most of the organizations, big or small are concerned with a problem of planning and optimizing its available resources to yield the maximum production (or to maximize profit) or in some cases, to minimize the cost of production. Dealing with–such problems using mathematics are referred to as the problems of constrained optimization.

Linear Programming is a one of the techniques for determining an optimal solution of interdependent constraints and factors in view of the available resources. It refers to a particular plan of action from amongst several alternatives for maximizing profit or production or minimizing cost of production or transport etc. The word linear stands for indicating that all inequations or equation used in a particular problem are linear.

Thus, a linear programming problem deals with the optimization (Minimization or Maximization) of a linear function having number of variables; subject to a number of conditions on the variables in the form of linear inequations or equations in the variables involved.

In this chapter, we shall discuss mathematical formulation of LPP and also learn graphical method to solve it. We shall also try to understand and appreciate the wide applicability of LPP in industry, commerce, management and sciences. The graphical method is used to optimize and find possible solutions for an LPP in two-variables.

8.1 LINEAR PROGRAMMING PROBLEM:

A Linear programming problem (LPP) consists of three important components:

(i) Decision variables

(ii) The Objective function

(iii) The Linear Constraints

1. **Decision Variable**: - The decision variables refer to the limitations or the activities that are competing with one another for sharing the available resources. These variables are usually inter-related in terms of utilization of resources and need simultaneous solution. All the decision variables are considered to be continuous, controllable and non-negative and represented as variables x, y etc.

2. **The Objective function: -** As every linear programming problem is aimed to have an objective to be measured in quantitative terms such as profit (sales) maximization, cost (time) minimization and so on. The relationship among the variables representing objective must be linear.

A linear objective is a real valued function, represented as $Z = ax + by$, where a, b are arbitrary constants, where Z is to be maximized or minimized.

3. **The Constraints**: - There are always certain limitations (constraints) on the use of resources, such as labor, space, availability of raw material or restrictions on transportation variables etc. that limit the extent to which an objective can be achieved. Such constraints are expressed as linear inequalities or equalities in terms of decision variables.

The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions on the decision variables.

Basic Assumptions:

A Linear programming problem is based on the following four basic assumptions:

(i) Certainty: It is assumed that in LPP, all the parameters; such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit decision variable must be known and fixed.

(ii) Divisibility (continuity): Another assumption of LPP is that the decision variables are continuous. This means a combination of outputs can be used with the fractional values along with the integer values.

(iii) Proportionality: This requires the contribution of each decision variable in both the objective function and the constraints to be directly proportional to the value of the variable.

(iv) Additivity: The value of objective function and the total amount of each resources used must be equal to the sum of the respective individual contributions (profit or cost) by decision variables.

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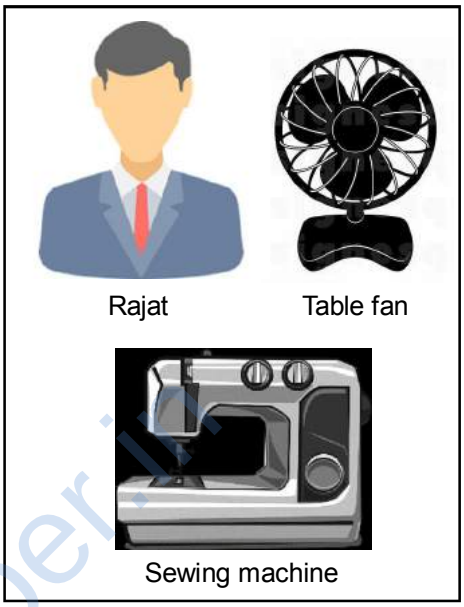
8.2 MATHEMATICAL FORMULATION A LINEAR PROGRAMMING PROBLEM

Let us take an example to understand how LPP is used to solve real-life problems.



Rajat wishes to purchase a number of table-fans and sewing machines. He has Rs.57600 to invest and has available space for at most 20 items. A table-fan costs Rs. 360 and a sewing machine costs Rs.240. Rajat wishes to sell one table-fan at a profit of Rs.22 and a sewing machine at a profit of Rs. 18.

Now, Rajat is in confusion as to how many table-fans and sewing machines should he purchase from the available money to get the maximum profit, assuming that he can sell all the items which he buys.

To maximize the profit, let us suppose that Rajat purchases x number of table-fans and y number of sewing machines which are the decision variables for the LPP

Clearly, we can assume that $x \geq 0$ and $y \geq 0$, which are sometimes also referred to as trivial constraints

Since Rajat has space for at most 20 items.

Therefore,

Total number of table-fans + Total number of sewing machine should be less than or equal to 20.

$$\Rightarrow x + y \leq 20 \ldots\ldots(i)$$

Also, we are given that a table-fan costs Rs. 360 and a sewing machine costs Rs. 240.

$\therefore$ Total cost of x table-fans and y sewing machine is (360x+240y)

Since he has only Rs. 57600 to invest.

$\therefore$ Total cost of x number of table-fan and y number of sewing machine should be less than or equal to 5760.

$$\Rightarrow \quad 360x + 240y \leq 57600 \ldots..(ii)$$

Since Rajat can sell all the items that he can buy and the profit on a table-fan is Rs.22 and Rs.18. on a sewing machine

$\therefore$ Total profit on x table-fans and y sewing machine is Rs. (22x +18y)

Let Z denote the total profit, which is to be maximized in this case

Therefore, the linear objective function $Z = 22x + 18y$

The above situation gives the description of the type of a Linear programming Problem.

Hence the given LPP can be mathematically formulated as:

(Objective function) To maximize $Z = 22x + 18y$

Subject to constraints:

$x \geq 0, y \geq 0$

$x + y \leq 20$

$360x + 240y \leq 57600$

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8.3 TYPES OF LINEAR PROGRAMMING PROBLEMS

The application of LPP can be found in various daily life situations.

Some of the important LP problems we shall study are:

1. Manufacturing Problem
2. Diet problem
3. Transportation Problem
4. Assignment Problem

Manufacturing Problem - These problems involve the production and sale of different products by a company. The production of the products requires optimization of labour force, machine hours, raw material, storage space, etc. Different products are produced to satisfy the aforementioned constraints and the investment available.

Diet Problem - Very often the dieticians and nutritionists are required to prepare health and diet charts. The objective of these diet charts is to include all the important kinds of nutrients that are required by the human body to stay healthy–at a reasonable cost. Thus, in the diet problems, a minimum amount of available nutrients, thereby minimizing the cost of such a diet plan.

Transportation Problem - These problems are related to the study of the efficient transportation routes i.e. how efficiently the product from different sources of production is transported to the different markets, such as the total transportation cost is minimized. Analysis of such problem is very crucial for big companies with several production plants and a widespread area to cater to. In this type of problem, constraints mean the specific supply and demand patterns and objective function means the transportation cost should be minimized.

Assignment problem:-

This type of problems are related with the completion of a particular task /assignment of a company by choosing a certain number of employees to complete the assignment within the required deadline, given that a single person works on only one job within the assignment.

In this type of problem, the number of employees, the work- hours of each employee etc. are considered as constraints and the total assignment to be done is treated as objective function.

Example 1

A furniture manufacture makes two products: chairs and tables. Processing of these products is done on two machines A and B. A Chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs. 2 and Rs.10, respectively. Formulate this problem as a linear programming problem to maximize the total profit of the manufacturer.

Solution:

The given problem can be tabulated as follows for convenience:

Machine	Chair	Table	Available time
A	2 hours	5 hours	16 hours
B	6 hours	0	30 hours
Profit per unit	Rs. 2	Rs. 10	

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Let x and y number of chairs and tables be produced respectively.

Then total profit to be maximized, $Z = 2x + 10y$

Since the number of chairs and tables cannot be negative.

$$x \geq 0 \text{ and } y \geq 0$$

It is given that a chair requires 2 hours on machine A and a table requires 5 hours on machine A Therefore, it must be less than or equal to the total time available on machine A.

$$2x + 5y \leq 16$$

Similarly, for machine B,

$$6x \leq 30 \text{ Or } x \leq 5$$

Hence the mathematical form of the given LPP is as follows:

Maximize $Z = 2x + 10y$

Subject to the constraints:

$$x \geq 0, y \geq 0$$
$$2x + 5y \leq 16$$
$$x \leq 5$$

Example 2:

A small manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry shop, and then sent to the machine shop for finishing. The number of man-hours of labor required in each shop for the production of each unit of A and B, the number of man hours the firm has available per week are as follows:

Gadget	Foundry	Machine-shop
A	10	5
B	6	4
Firm's capacity per week	1000	600

The profit on the sale of gadget A is Rs. 30 per unit as compared with Rs. 20 per unit of gadget B. Formulate this problem as LPP to maximize the total profit

Solution:

Let x and y number of weekly production of gadgets A and B.

Therefore, $Z = 30x + 20y$ (Since total profit is Z)

Since the number of weekly productions of gadgets, A and B cannot be negative.

$$x \geq 0, \textit{ and } y \geq 0$$

It is given that 10 and 6 man-hours of labor required in foundry shop for the production of each unit of gadgets A and B.

Therefore, Total man-hours of labor required in foundry shop for the production of each unit of gadgets A and B is (10x + 6y).

But firm's total capacity per week is 1,000 man-hours of labor.

So, Total man-hours of labor required in foundry shop for the production of each unit of gadgets A and B is less than or equal to 1000.

$10x + 6y \leq 1000$

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$\Rightarrow 5x + 3y \leq 500$

Similarly, for finishing,

$$5x + 4y \leq 600$$

Hence the mathematical form of the given LPP is as follows:

Maximize $Z = 30x + 20y$

Subject to the constraints:

$$x \geq 0, y \geq 0$$

$5x + 3y \leq 500$

And $5x + 4y \leq 600$,

Example 3

A firm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrients constituents (call them X, Y and Z), It is necessary to buy two additional products, say A and B. One unit of product A contains 36 units of nutrient X, 3 units of nutrient Y and 20 units of nutrient Z. One unit of product B contains 6 units of nutrient X, 12 units of nutrient Y and 10 units of nutrient Z. The minimum requirement of nutrients X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs ₹ 20 per unit and product B costs ₹ 40 per unit. Formulate the above as a linear programming problem to minimize total cost.

Solution: Let x and y number of units of product A and B.

Therefore, Total cost = 20x + 40y.

Now, according to the question,

Nutrient constituents	Nutrient content in product of nutrient.		Minimum amount
	A	B	
X	36	6	108
Y	03	12	36
Z	20	10	100
Cost of product	Rs. 20	Rs. 40	

Making use of above information, the appropriate mathematical formulation of the linear programming problem is:

Minimize $Z = 20x + 40y$.

Subject to the constraints: $x \geq 0, y \geq 0$

$36x + 6y \geq 108 \quad \Rightarrow 6x + y \geq 18$

$3x + 12y \geq 36 \quad \Rightarrow x + 4y \geq 12$

$20x + 10y \geq 100 \quad \Rightarrow 2x + y \geq 10$

Example 4

There is a factory located at each of the two places P and Q .From these locations, a certain commodity is derived to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5 ,5 and 4 units of the commodity while the production capacity of the factories at P and Q are 8 and 6 units respectively. The cost of transportation per unit is given below:

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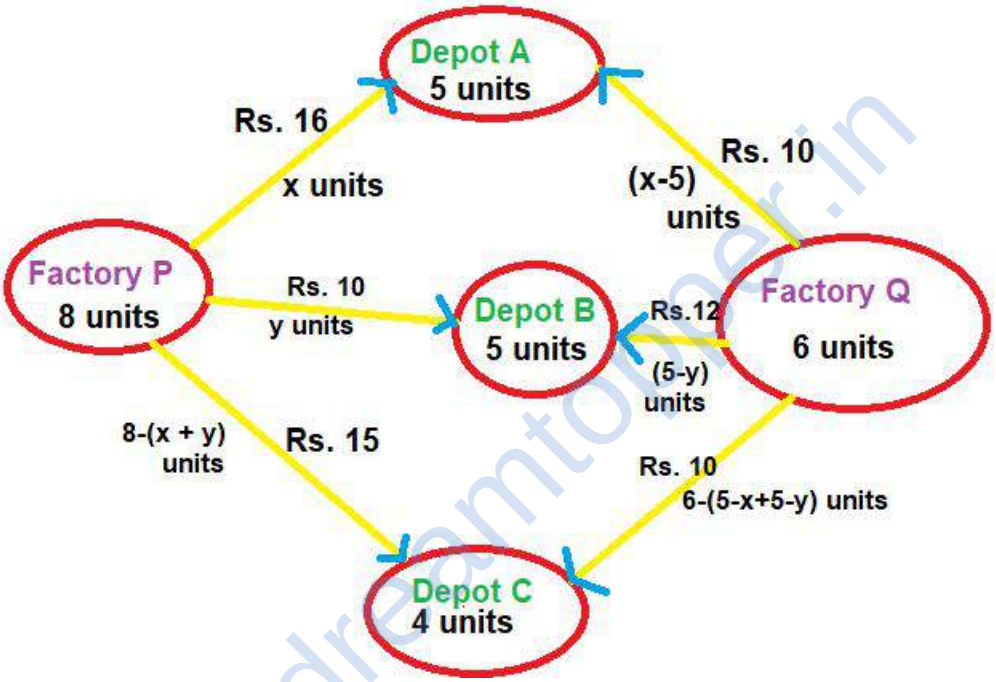
From/to	Costs (in Rs)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. Formulate above as a linear programming problem.

Solution: The above given problem can be represented in diagrammatically as follows:



Let the factory at P transports x units of commodity to depot at A and y units to depot at B.

Since the requirements are always non negative quantities. Therefore, $x \geq 0$, and $y \geq 0$

Also, the factory at P has the capacity of 8 units of the commodity.

Therefore, the left over (8-x-y) units will be transported to depot at C

Clearly, $8 - x - y \geq 0$

$\Rightarrow \quad x + y \leq 8$

Since the weekly requirement of the depot at A is 5 units of the commodity and x units are transported from the factory at P.

Therefore, the remaining quantity of $(5 - x)$ units are to be transported from the factory at Q.

Similarly, $(5 - y)$ units of the commodity will be transported from the factory at Q to the depot at B.

But the factory at Q has the capacity of 6 units only, therefore the remaining units

6-(5-x+5-y) =x+y-4 units will be transported to the depot at C.

As the requirements of the depots at A, B and C are always non negative.

$x - 5 \geq 0, \quad 5 - y \geq 0, \quad \textit{and} \quad x + y - 4 \geq 0$

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$\Rightarrow$ $x \leq 5$,

$y \leq 5$,

and $x + y \geq 4$

The transportation cost from the factory at P to the factory at A, B and C are respectively Rs.16x, 10y and 15(8- x -y).

Similarly, the transportation cost from the factory at Q to the depots at A, B and C are respectively Rs.10 (5-x), 12(5-y) and10(x + y - 4).

Therefore, the total transportation cost Z is given by:

$Z = 16x+10y+15(8-x-y) +10(x-5) +12(5-y) +10(x+y-4)$

$= x-7y+190$

Hence, the above LPP can be stated mathematically as follows:

Minimize $Z = x - 7y + 190$

Subject to the constraints:

$x \geq 0, y \geq 0$

$x + y \leq 8$

$x + y \geq 4$

$x, y \leq 5$

Example 5

A company has two groups of inspectors namely, group A and B, who are assigned to do a quality inspection work. It is required that at least 1800 pieces are inspected for 8-hour day. It is known that inspectors of group A can check pieces at the rate of 25 per hour with an accuracy of 98%, while inspectors of group B can check at the rate of 15 pieces per hour with an accuracy of 95%. The inspectors of group A and B are paid Rs 40 and Rs 30 per hour respectively to do the work. Each time an error is caused by the any inspector, it costs a loss of Rs 20 to the company. The company has 8 inspectors in group A and 10 in group B. The company wants to determine the optimal assignment of Inspectors to minimise total inspection cost. Formulate an LPP

Solution:

Let an inspector of group A inspect for x number of hours and each inspector of group B inspect for y number of hours

The data of the given problem can be summarized as follows:

	Group A Inspector	**Group B Inspector**
Number of Inspectors	8	10
Rate of checking per hour	25 pieces	15 pieces
Inaccuracy in checking	1-0.98=0.02	1-0.95=0.05
Cost of Inaccuracy in checking	Rs.20	Rs.20
Wage rate per hour	Rs.40	Rs. 30

Hourly costs of each **Group** A and **Group** B inspectors are given by:

Group A Inspector: Rs. (40+20 ×0.02×25) = Rs. 50

Group B Inspector: Rs. (30+20 ×0.05×15) = Rs. 45

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Using the above information, the appropriate LPP is

Minimize $Z = 8 \times 50x + 10 \times 45y = 400x + 450y$

Subject to the constraints:

$x \geq 0, y \geq 0$

$4x + 3y \geq 120$

$x \leq 8, y \leq 10$

8.4 SOLVING A LINEAR PROGRAMMING PROBLEM

In this section we are going to learn how to solve an LPP. Let us understand a few terms used while solving it.

Solution: The set of values of decision variables x_j (j = 1, 2,….,n) which satisfy the constraints of an LP problem is said to constitute solution to that LP problem.

Feasible Solution: The set of values of decision variables x_j (j = 1, 2,….,n) which satisfy all the constraints and non-negativity condition of an LP problem is said to constitute feasible solution to that LP problem.

In other way, a solution that also satisfies the non-negativity restrictions of a LPP, is called a feasible solution.

Infeasible Solution: The set of values of decision variables x_j (j = 1, 2,….,n) which do not satisfy all the constraints and non-negativity condition of an LP problem is said to constitute the infeasible solution to that LP problem.

Feasible region: Feasible region is the common region determined by all the constraints including non-negative constraints of a LPP and every point in this region is the feasible solution of the given LPP.

Optimal Feasible Solution: A feasible solution of a LPP that optimizes (maximizes or minimizes) the objective function is called the optimal solution of the LPP. At times, an LPP can have no solution or more than one optimal solution.

Theorem 1: Let R be the feasible region for a linear programming problem and let

$Z = ax + by$ be the objective function.

When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point

(A corner point of a feasible region is a point in the region which is the intersection of two boundary lines)

Theorem 2 : Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function.

(i) If R is **bounded,** then the objective function Z has both a **maximum** and a **minimum value** on R and each of these occurs at a corner point of R.

(ii) If R is **unbounded, then** maximum or minimum value of objective function may not exit. However, if it exits then it must occur at the corner point of the feasible region.

An LPP can be solved using many methods. In the next section we shall learn to solve a given LPP using graphical method

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8.5 GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEM:

In the previous sections we learnt how to formulate a linear programming problem in mathematical form, the next step is to solve the problem to get the optimal solution for the given LPP.

In this unit we shall focus on solving a linear programming problem with only two variables using a graphical method as the graphical method provides a pictorial representation of the solution process and a great deal of insight into the basic concept. So, in this chapter we shall focus on the graphical methods involving two variables only.

The following methods are used to solve LP problems graphically:

(i) Corner - Point Method

(ii) Iso -Profit or Iso - cost method

8.5.1 CORNER - POINT METHOD

In this method, the coordinates of all corner (extreme) points of the feasible region are determined and the value of the objective function at these points are computed because the mathematical theory of LP states that an optimal solution to any LP problem always lie at one of the corner points of the feasible region.

This method consists of the following steps:

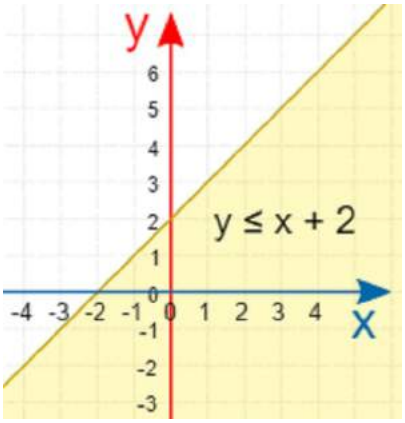
(i) Formulate the given LPP in mathematical form.

(ii) Draw X-axis and Y- axis on the graph paper, the non -negativity restrictions i.e., $x \geq 0$, $y \geq 0$ imply that the values of the variables x and y can lie only in first quadrant.

Feasible region will always be in the first quadrant

(iii) Plot the inequality constraints on the graph and decide the area of feasible region according to the inequality sign of constraints.

To determine the region represented by an inequations replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation



(iv) Shade the common region of the graph that satisfies all the constraints. The common region is called the feasible region of the given LPP. Any point on or inside the feasible region is the feasible solution of the given LPP. The feasible region can be bounded (closed) or unbounded (open) as shown below:

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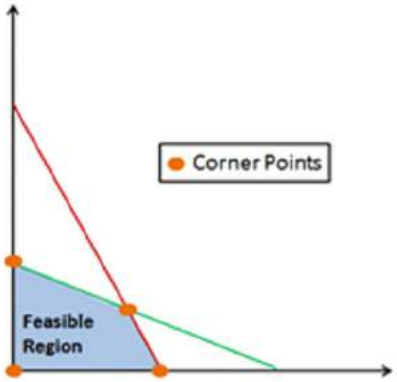
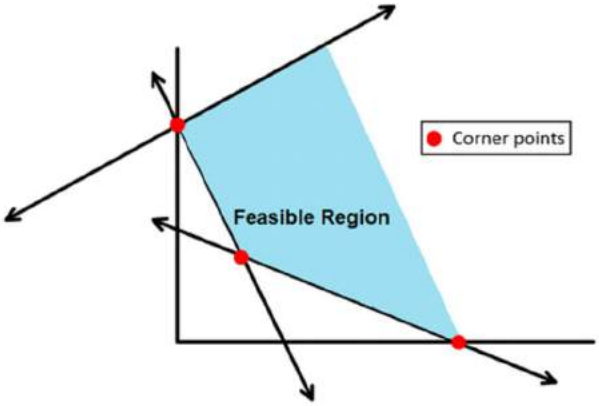


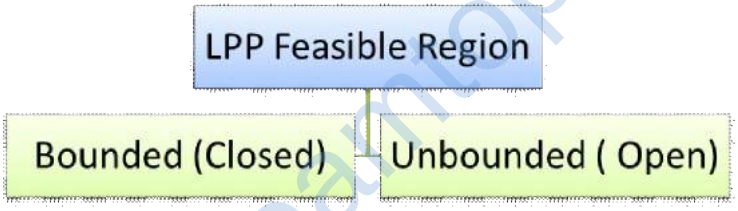


Bounded (closed) Feasible region Unbounded (Open) Feasible region

(v) Now determine the coordinates of corner points of the feasible region

(vi) Now evaluate the objective function Z at each corner point of the feasible region. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given LP problem.

Now let us discuss the two possibilities of feasible region in detail:



Case(i) If the feasible region of a LPP is bounded:

In this case the objective function has both a maximum value and a minimum value at a corner point of the given feasible region. For example:

Corner Points	Z = x + 2y	
A (2, 3)	8	Minimum
B (1, 7)	15	
C (4, 9)	22	Maximum

Case (ii)- If the feasible region of a LPP is unbounded:

In this case the objective function has both a maximum value and a minimum value at a corner point of the given feasible region.

Corner Points	Z = x + 2y	
A (1, 5)	11	
B (3, 5)	13	
C (5, 8)	21	Maximum
D (0, 2)	4	Minimum

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In order to check whether Z (objective function) has maximum or minimum values respectively, we proceed as follows:

1) Draw the line ax+by = M and find the open half plane $ax + by > M$.

   If the open half plane represented by $ax + by > M$, has no point common with the unbounded feasible region, then M is the maximum value of Z, otherwise Z has no maximum value.

2) Draw the line ax+by =m and find the open half plane $ax + by < m$.

   If the open half plane represented by $ax + by < m$, has no point common with the unbounded feasible region, then m is the minimum value of Z, otherwise Z has no minimum value.

   We shall now illustrate these steps of Corner Point Method by considering some examples:

Example 6

Solve the following Linear Programming Problem Graphically.

Maximize Z = 5x + 3y

Subject to constraints:

$3x + 5y \leq 15$

$5x + 2y \leq 10$

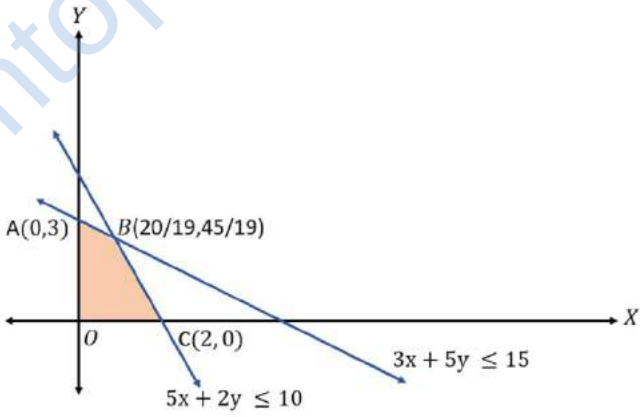
$x \geq 0$, and $y \geq 0$

Solution:

By plotting the given linear inequalities, we can see that the inequality $3x + 5y \leq 15$ meets the co-ordinates axes at points (5,0) and A(0,3) respectively.

Also the inequality $5x + 2y \leq 10$ meets the co-ordinates axes at points C(2,0) and (0,5) respectively.

As shown in graph (i) the shaded bounded region OABCO represents the common region of the above inequations. This region is the feasible region of the given LPP.



Graph (i)

The coordinates of the vertices (corner point) of the shaded bounded feasible region are O (0, 0), A (0, 3), B (20/19, 45/19) and C (2, 0).

These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously. The value of the objective function as these points are given in the following table:

Corner Points	Coordinates	Objective Function Z = 5x + 3y
O	(0,0)	0
A	(0,3)	9
B	(20/19,45/19)	235/19
C	(2,0)	10

Clearly, Z is the maximum at P (20/19, 45/19)

Hence, x = 20/19, y = 45/19 is the optimal solution of the given LPP.

The optimal maximum value of Z is 235 /19 when x = 20/19 and y = 45/19

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Example 7

Solve the following Linear Programming Problem Graphically.

Maximize Z = 2x + 4y

Subject to constraints:

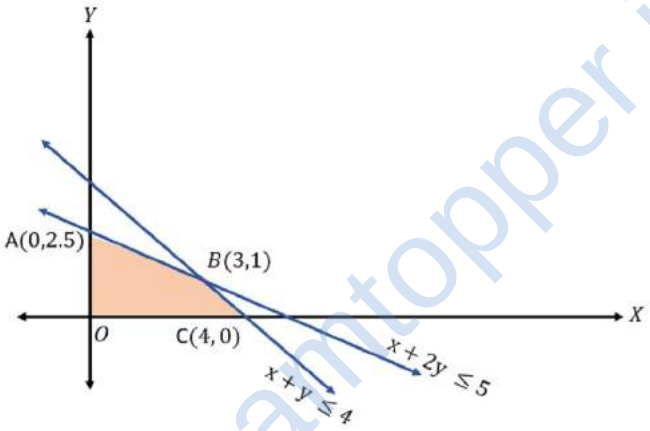
$x + 2y \leq 5$

$x + y \leq 4$

$x \geq 0$ and $y \geq 0$

Solution: By plotting the given linear inequalities, we can see that the inequality $x + 2y \leq 5$ meets the co-ordinates axes at the point (5,0) and A(0,2.5) respectively.

Similarly, The inequality $x + y \leq 4$ meets the co-ordinates axes at the point C (4, 0) and (0, 4) respectively.



Graph (ii)

As shown in the graph above, the shaded bounded region OABCO represents the common region of the above inequation. This region is the feasible region of the given LPP.

The coordinates of the vertices (corner point) of the shaded feasible region are O (0, 0), A (0, 2.5), B (3, 1) and C (4, 0).

The value of the objective function as these corner points are given in the following table:

Corner Points	Coordinates	Objective Function Z = 2x + 4y
O	(0,0)	0
A	(0,2.5)	10 (Max.)
B	(3,1)	10 (Max.)
C	(4,0)	8

Clearly, Z has maximized at two corner points A (0, 2.5) and B (3, 1).

Hence, any point on the line segment joining points A and B will give the maximum value Z = 10 of the objective function.

The optimal maximised value of Z is 10 when x = 0 and y = 2.5 or when x = 3 and y = 1

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Example 8:

Minimise Z = x + 2y

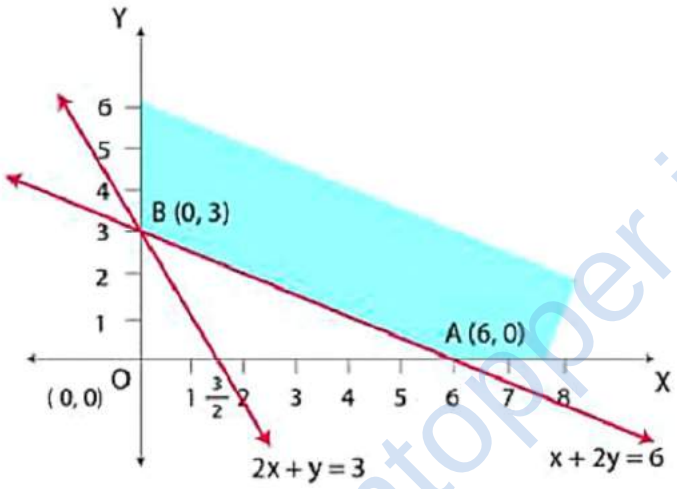
Subject to constraints:

$x \geq 0$ and $y \geq 0$

$2x + y \geq 3$

$x + 2y \geq 6$

Show that minimum Z has more than two optimal soutions



Graph (iii)

Solution: By plotting the given linear inequalities, we can see that the inequality $x + 2y \geq 6$ meets the co-ordinates axes at ~~the~~ points B (0, 3) and A(6,0) respectively

Also, the inequality $2x + y \geq 3$ meets the co-ordinates axes at points (0, 3) and (3/2, 0) respectively

As shown in the graph above, the shaded feasible region is unbounded.

The coordinates of the vertices (corner point) of the shaded feasible region are A (6, 0), and B (0, 3)

The value of Z at the corner points are as follows:

Corner Points	Coordinates	Objective function Z = x + 2y
A	(6, 0)	6 min
B	(0, 3)	6 min

There are no distinct maximum or minimum values of Z as the value of Z at points A and B are same

Therefore, all the points lying on the line joining the points A and B will minimise and maximise the objective function at many more points than A and B

Hence the minimum value of Z occurs for more than two corner points, i.e., all the points lying on the line segment AB will minimize the objective function

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Example 9

Solve the following Linear Programming Problem Graphically.

Maximize Z = 6x + y

Subject to constraints:

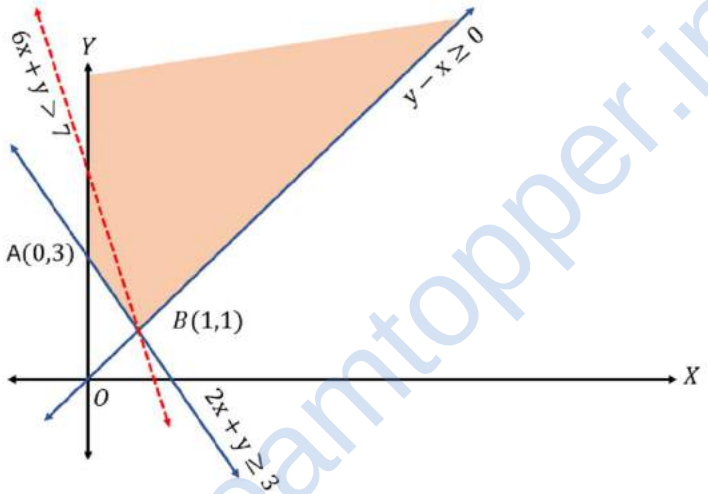
$2x + y \geq 3$

$y - x \geq 0$

$x \geq 0$, and $y \geq 0$

Solution: By plotting the given linear inequalities, we can see that the inequality $2x + y \geq 3$ meets the co-ordinates axes at the point (1.5, 0) and (0, 3) respectively.

Similarly, the inequality $y - x \geq 0$ meets the co-ordinate axis at the point O (0, 0) respectively.



Graph (iv)

As shown in the graph above, the shaded feasible region is unbounded.

The coordinates of the vertices (corner point) of the shaded feasible region are A (0, 3), and B (1, 1).

The value of the objective function as these points are given in the following table:

Corner Points	Coordinates	Objective Function Z=6x + y
A	(0,3)	3
B	(1,1)	7 (Max.)

From this table, we find that 7 is the maximum value of Z at the corner point B (1, 1)

As the feasible region is unbounded.

Therefore, 7 may or may not be the maximum value of Z.

To decide this issue, we graph the inequality $6x + y > 7$.

Plot this inequation on the same graph and check whether the resulting open half plane has points in common with the feasible region or not.

As shown in the figure the red line representing the inequality $6x + y > 7$ is passing through corner point B(1, 1) but lies in the feasible region

Hence the given LP problem has no solution and Z cannot be maximized for any values of x and y.

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Example 10:

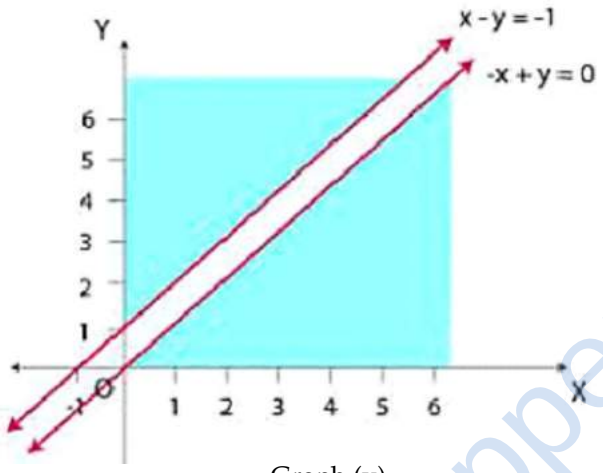
Maximize Z = x + y

Subject to constraints: $x, y \geq 0$

$x - y \leq -1$

$x \geq y$

Solution:



Graph (v)

Plotting the graph, we can see that there is no possible feasible region for the given constraints Hence the given LPP has no solution and Z cannot be maximized

Example 11

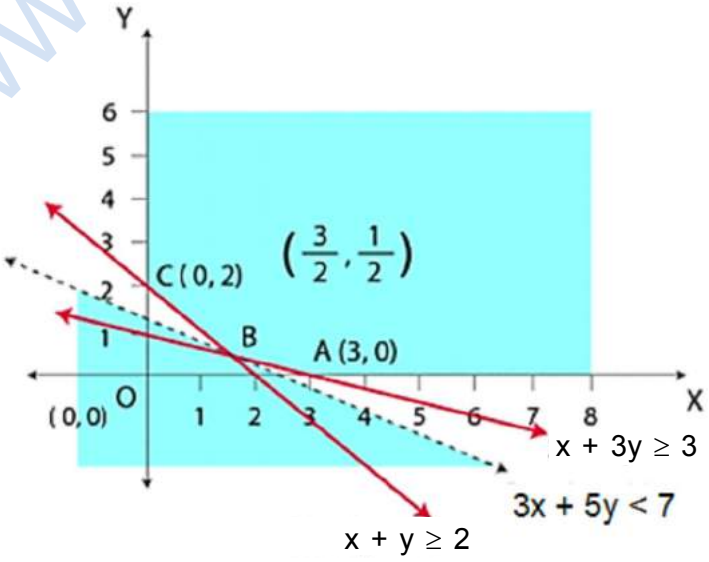
Minimize Z = 3x + 5y

Subject to constraints: $x, y \geq 0$

$x + 3y - 3 \geq 0$

$x + y - 2 \geq 0$

Solution : The feasible region determined by the system of constraints, $x + 3y \geq 3$, $x + y \geq 2$, and $x, y \geq 0$ is given below:



Graph (vi)

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Here, the feasible region is unbounded.

The corner points of the feasible region are A (3, 0), B (3 / 2, 1 / 2) and C (0, 2)

The values of Z at these corner points are given below:

Corner Points	Coordinates	Objective Function Z = 3x + 5y
A	(03, 0)	9
B	(3/2, 1/2)	7 (Min.)
C	(0, 2)	10

As we wish to minimize Z, we are going to draw graph of Z = 3x + 5y < 7 and check whether the resulting half plane has any common points with the feasibe region or not

As the inequality, Z – 3x + 5y < 7 passes through a corner point B (3/2, 1/2) without interfering the feasible region

That means, the corner point B (3/2, 1/2) minimizes Z and the minimum value of Z is 7. When x = 3/2, y = 1/2.

8.5.2 ISO-PROFIT/ ISO-COST METHOD:

In this section we are going to learn another method to solve a given LP problem. Iso-Profit method is another way to find the optimal solution by using the slope of the objective function line (or equation).

An iso-profit (or cost) line is a collection of points which designate solution with same value of objective function. By assigning various values to Z, we get different Profit (cost) lines. Graphically, many such lines can be plotted parallel to each other .

The steps of iso-profit (cost) function method are as follows.

1. Formulate the given LPP in mathematical form
2. Identify the feasible region and extreme (corner) points of the feasible region .(As discussed in Corner-Point method)
3. Give some convenient values to Z and draw the line so obtained in xy- plane.
4. If the objective function is to be maximized, then draw lines parallel to the line in step 3.
5. Obtain a line which is farthest from the origin and has at least one point common to the feasible region.
6. If the objective function is to be minimized, then draw lines parallel to the line in step 3 and obtain a line which is nearest to the origin and has at least one point common to the feasible region.
7. Find the co-ordinate of the common point obtained in step 4. The point so obtained determine the optimal solution and the value of the objective function at these points give the optimal solution.

Example 12

Solve the following Linear Programming Problem Graphically.

Maximize Z = 15x + 10y

Subject to 4x + 6y ≤ 360

3x ≤ 180

5y ≤ 200

x ≥ 0, and y ≥ 0

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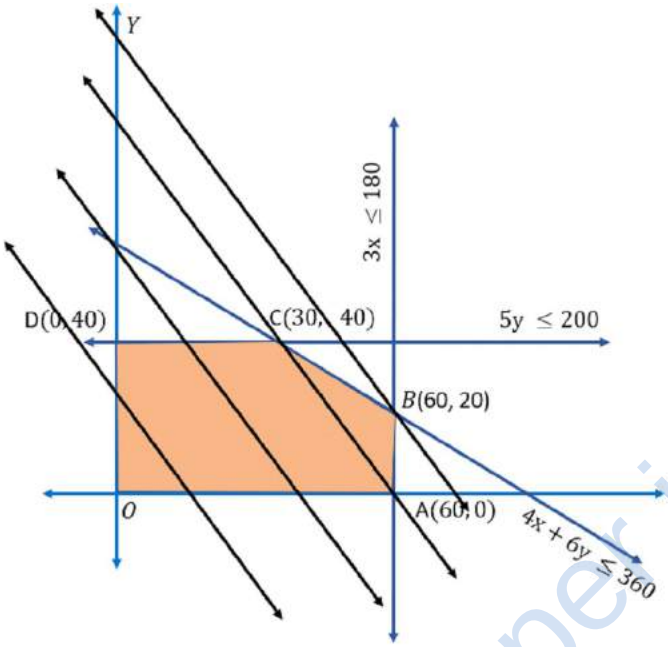
Solution:



Graph (vii)

To begin with, equality constraints are considered equations, as shown in the above figure.

The bounded feasible area is formed by considering the area to the lower left side of each equation (towards origin). A family of lines that represents various levels of objective function is drawn (black lines in figure).

For choosing arbitrary value for Z, we can use the LCM (a, b) for the objective function Z = ax + by

These lines are called iso- profit lines.

Let us select an arbitrary value of Z as 300

Hence, the iso-profit function equation becomes 15x + 10 y = 300.

This equation can be plotted in the same manner as the equality constraints were plotted. This line is then moved upward until it first intersects a corner in the feasible region (corner B).

The coordinates of corner point B can be read from the graph or can be computed as the intersection of the two linear equations.

The coordinates x = 60 and y = 20 of corner point B satisfy the given constraints and the total profit obtained is Z = 1100.

Example 13

Solve the following Linear Programming Problem Graphically.

Minimize $Z = 18x + 10y$

Subject to $4x + y \geq 20$

$2x + 3y \geq 30$

and $x \geq 0$ and $y \geq 0$

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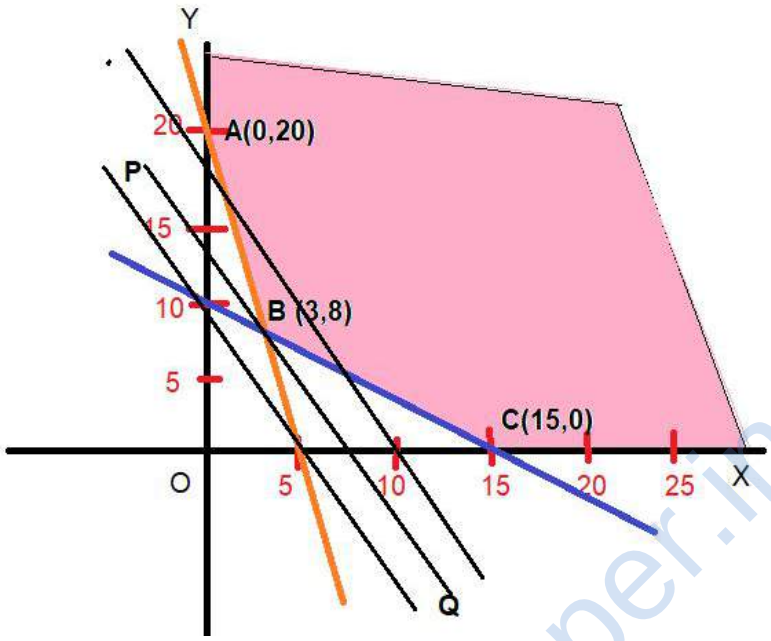
Solution: As shown in the graph below, the feasible region of the LPP is unbounded



Graph (viii)

Give a value, say 180 equal to (2 times LCM of 18 and 10) to Z to obtain the line 18x + 10y =180. This line meets the co-ordinate axes at (10, 0) and (0, 18).

Join these points by black line. Move this line parallel to itself in the decreasing direction towards the origin so that it passes through only one point of the feasible region. clearly PQ is such a line passing through the vertex B of the feasible region. The coordinates of B are obtained by solving the lines 4x + y = 20 and 2x + 3y = 30.

Solving these equations, we get x = 3 and y = 8.

Putting x = 3 and y = 8 in the objective function Z = 18x + 10y, we get Z= 134

The minimum value of Z is 134 at x = 3 and y = 8 .

8.6 CHECK YOUR PROGRESS

1) To maintain his health, a person must fulfill certain minimum daily requirements for several kinds of nutrients. Assuming that there are only three kinds of nutrients -calcium, protein and Calories and the person's diet consist of only two food items 1 and 2, whose price and nutrient contents are shown in the table below:

Nutrients	Food (per lb)	Food II (per lb)	Minimum daily requirement
Calcium	10	4	20
Protein	5	5	20
Calories	2	6	13
Price (in Rs.)	0.60	1.00	

What combination of two food items will satisfy the daily requirement and entail the least cost? Formulate this problem as a LPP.

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2) Vitamins A and B are found in two different foods F1 and F2 One unit of food F1 contains two units of Vitamin A and 3 units of Vitamin B, one unit of food F2 contains 4 units of Vitamin A and 2 units of Vitamin B, one unit of food F1 and F2 cost ₹ 5 and ₹ 2.5 respectively. The minimum daily requirements for a person of Vitamin A and B are 40 and 50 units respectively. Assuming that anything in excess of daily minimum requirement of Vitamin A and B is not harmful, find out the optimum mixture of food F1 and F2 at the minimum cost which meets the daily minimum requirement of Vitamin A and B. Formulate this problem as an LPP.

3) A brick manufacturer has two depots, A and B with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in Rs. of transporting 1,000 bricks to the builders from the depots are given below:

From\ To	P	Q	R
A	40	20	30
B	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum? Formulate the above problem as linear programming problem.

4) A Cooperative Society of farmers has 50 hectares of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as ₹ 10,500 and ₹ 9,000 respectively. To control weeds, a liquor herbicide has to be used for crops X and Y at rate of 20 liters and 10 liters per hectare. Further, no more than 800 liters of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximize the total profit of the society? Formulate the above problem as linear programming problem.

5) A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1,500 pieces must be inspected in an 8-hour day. Grade I inspector. can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92 %. Wages of grade I Inspector are Rs. 5 per hour while those of Grade II Inspector are Rs. 4 per hour. Any error made by Inspector costs Rs. 3 to the company. If there are, in all, 10 grade I inspectors and 15 Grade II inspectors in the company, find the optimal assignment of inspectors that minimizes the daily inspection cost. Formulate the above problem as linear programming problem.

6) Solve the following Linear Programming Problem graphically:

i. Maximize $Z = 3x + 2y$

Subject to the constraints: $-2x + y \leq 1$

$x \leq 2,$

$x + y \leq 3,$ and $x \geq 0, y \geq 0$

ii. Minimize $Z = 5x - 2y$

Subject to the constraints: $2x + 3y \geq 1,$

and $x \geq 0, y \geq 0$

iii. Minimize $Z = -x + 2y$

Subject to the constraints: $-x + 3y \leq 10,$

$x + y \leq 6$

$x - y \leq 2$

and $x \geq 0, y \geq 0$

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iv. Maximize $Z = -x + 2y$

Subject to the constraints: $-0.5x + y \leq 2$,

$x - y \leq -1$

and $x \geq 0, y \geq 0$

v. Maximize $Z = 5x + 4y$

Subject to the constraints: $x - 2y \leq 1$,

$x + 2y \leq 6$

$x - y \geq 3$

and $x \geq 0, y \geq 0$

Solve the following Linear Programming Problem graphically by using Iso-cost method:

7. Minimize $Z = 4x - 2y$

Subject to the constraints: $x + y \leq 14$,

$2x + y \leq 24$,

$3x + 2y \geq 14$,

and $x \geq 0, y \geq 0$

8. Maximize $Z = 3x + 9y$

Subject to the constraints: $x + 4y \leq 8$,

$x + 2y \leq 4$,

and $x \geq 0, y \geq 0$

9. Maximize, $Z = 3x + 2y$

Subject to the constraints: $-2x + y \leq 1$,

$x + y \leq 3$,

$x \leq 2$,

and $x \geq 0, y \geq 0$

CHECK YOUR PROGRESS ANSWERS

1) Minimize $Z = 0.60x + 1.00y$

Subject to the constraints: $10x + 4y \geq 20$

$5x + 5y \geq 20$

$2x + 6y \geq 13$

and $x \geq 0, y \geq 0$

2) Minimize $Z = 5x + 2.5y$

Subject to the constraints $2x + 4y \geq 0$

$3x + 2y \geq 50$

and $x \geq 0, y \geq 0$

3) Minimize $Z = 30x - 30y + 1800$

Subject to the constraints: $x + y \leq 30$

$x \leq 15, y \leq 20, \quad x + y \geq 15$

and $x \geq 0, y \geq 0$

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4) Maximize Z = 10500x + 9000y

Subject to the constraints: $x + y \leq 50$

$2x + y \leq 80$

and $x \geq 0, y \geq 0$

5) Minimize Z = 59.20x + 58.88y

Subject to the constraints: $x, y \geq 0$

$x \leq 10, y \leq 15$

$160x + 112y \geq 1500$

6) i. $x = 2, y = 1$, max. $z = 8$

ii. $x = 0, y = \frac{1}{3}$, min $z = -\frac{2}{3}$

iii. $x = 2, y = 0$, min. $z = -2$

iv. multiple optimal solutions, max. Z = 4 at (0, 2) and (2, 3) and infinite points on the line segment joining them.

v. No solution (unbounded solution)

7) $x = 8, y = 6$ and Min. Z = 20

8) $x = 0, y = 2$ Max. Z = 18

9) $x = 2, Y = 1$, Max. Z = 8

8.7 UNIT SUMMERY

1. Linear programming problem deals with the optimization (Minimization or Maximization) of a linear function of a number of variables subject to a number of conditions on the variables in the form of linear inequations or equations in variables involved.
2. A Linear programming problem (LPP) consists of three important components:
   (i) Decision variables
   (ii) The Objective function
   (iii) The Linear Constraints
3. The decision variables refer to the limitations or the activities that are competing with one another for sharing the available resources
4. A linear objective is a real valued function, represented as Z = ax + by, where a, b are arbitrary constants, where Z is to be maximized or minimized
5. the conditions $x \geq 0, y \geq 0$ are called non-negative restrictions on the decision variables
6. Some of the important LP problem we shall study are:
   a. Manufacturing Problem
   b. Diet problem
   c. Transportation Problem
   d. Assignment Problem
7. The set of values of decision variables x_j (j=1,2,….,n) which satisfy the constraints of an LP problem is said to constitute solution to that LP problem.

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The set of values of decision variables x_j (j = 1,2,….,n) which satisfy all the constraints and non-negativity condition of an LP problem is said to constitute feasible solution to that LP problem.

8. The set of values of decision variables x_j (j=1,2,….,n) which do not satisfy all the constraints and non-negativity condition of an LP problem is said to constitute the infeasible solution to that LP problem.
9. Feasible region is the common region determined by all the constraints including non-negative constraints of a LPP and every point in this region is the feasible solution of the given LPP.
10. A feasible solution of a LPP that optimizes (maximizes or minimizes) the objective function is called the optimal solution of the LPP.
11. An LPP can have no solution or more than one optimal solution.
12. **Theorem:** Let R be the feasible region for a linear programming problem and let Z = $ax + by$ be the objective function.

    When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point

    (A corner point of a feasible region is a point in the region which is the intersection of two boundary lines)
13. **Theorem:** Let R be the feasible region for a linear programming problem, and let Z = $ax + by$ be the objective function.

    If R is **bounded,** then the objective function Z has both a **maximum** and a **minimum value** on R and each of these occurs at a corner point of R.

    If R is **unbounded, then** maximum or minimum value of objective function may not exit. However, if it exits then it must occur at the corner point of the feasible region
14. There are two methods to solve an LPP graphically:
    i. Corner-Point Method
    ii. Iso-Profit or Iso-cost method

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