7

Integrals

Short Answer Type Questions

Verify the following

Q. 1 $\int \frac{2x-1}{2x+3} dx = x - \log|(2x+3)^2| + C$

Sol. Let
$$I = \int \frac{2x-1}{2x+3} dx = \int \frac{2x+3-3-1}{2x+3} dx$$
$$= \int 1 dx - 4\int \frac{1}{2x+3} dx = x - \int \frac{4}{2\left(x+\frac{3}{2}\right)} dx$$
$$= x - 2\log + \left|\left(x+\frac{3}{2}\right)\right| C' = x - 2\log\left|\left(\frac{2x+3}{2}\right)\right| + C'$$
$$= x - 2\log|(2x+3)| + 2\log 2 + C' \qquad \left[\because \log\frac{m}{n} = \log m - \log n\right]$$
$$= x - \log|(2x+3)^2| + C \qquad [\because C = 2\log 2 + C']$$

Q. 2 $\int \frac{2x+3}{x^2+3x} dx = \log|x^2+3x| + C$

Sol. Let
$$I = \int \frac{2x+3}{x^2+3x} dx$$

Put $x^2 + 3x = t$

$\Rightarrow$ $(2x+3)\, dx = dt$

$\therefore$ $I = \int \frac{1}{t} dt = \log|t| + C$

$= \log|(x^2+3x)| + C$

Q. 3 $\int \frac{(x^2 + 2)d}{x + 1} x$

Thinking Process

First of all divided numerator by denominator, then use the formula $\int \frac{1}{x} dx = \log |x|$ *to get the solution.*

Sol. Let

$$I = \int \frac{x^2 + 2}{x + 1} dx$$

$$= \int \left(x - 1 + \frac{3}{x + 1} \right) dx$$

$$= \int (x - 1)\, dx + 3\int \frac{1}{x + 1} dx$$

$$= \frac{x^2}{2} - x + 3\log|(x + 1)| + C$$

Q. 4 $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} dx$

Sol. Let

$$I = \int \left(\frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} \right) dx$$

$$= \int \left(\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} \right) dx \qquad [\because a\log b = \log b^a]$$

$$= \int \left(\frac{x^6 - x^5}{x^4 - x^3} \right) dx \qquad [\because e^{\log x} = x]$$

$$= \int \left(\frac{x^3 - x^2}{x - 1} \right) dx = \int \frac{x^2(x - 1)}{x - 1} dx$$

$$= \int x^2 dx = \frac{x^3}{3} + C$$

Q. 5 $\int \frac{(1 + \cos x)}{x + \sin x} dx$

Sol. Consider that,

$$I = \int \frac{(1 + \cos x)}{(x + \sin x)} dx$$

Let $x + \sin x = t \Rightarrow (1 + \cos x) dx = dt$

$\therefore$ $I = \int \frac{1}{t} dt = \log|t| + C$

$$= \log|(x + \sin x)| + C$$

Q. 6 $\int \frac{dx}{1+\cos x}$

Thinking Process

$\cos x = 2\cos^2\frac{x}{2} - 1$ and also use formula i.e., $\int \sec^2 x = \tan x + C$ to solve the above problem.

Sol. Let

$$I = \int \frac{dx}{1+\cos x} = \int \frac{dx}{1 + 2\cos^2\frac{x}{2} - 1}$$

$$= \frac{1}{2}\int \frac{1}{\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\cdot\tan\frac{x}{2}\cdot 2 + C = \tan\frac{x}{2} + C \qquad \left[\because \int \sec^2 x\, dx = \tan x\right]$$

Q. 7 $\int \tan^2 x \sec^4 x\, dx$

Thinking Process

Use the formula $\sec^2 x = 1 + \tan^2 x$ and put $\tan x = t$ to solve this problem.

Sol. Let $I = \int \tan^2 x \sec^4 x\, dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore$ $I = \int t^2 (1 + t^2)\, dt = \int (t^2 + t^4)\, dt$

$$= \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$$

Q. 8 $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\,dx$

Sol. Let $I = \int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\,dx = \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}}\,dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}}\,dx = \int 1\, dx = x + C$$

Q. 9 $\int \sqrt{1+\sin x}\, dx$

Sol. Let $I = \int \sqrt{1+\sin x}\, dx$

$$= \int \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx \qquad \left[\because \sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1\right]$$

$$= \int \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}\,dx = \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx$$

$$= -\cos\frac{x}{2}\cdot 2 + \sin\frac{x}{2}\cdot 2 + C = -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + C$$

Q. 10 $\int \frac{x}{\sqrt{x}+1}dx$

Sol. Let $I = \int \frac{x}{\sqrt{x}+1}dx$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow$ $dx = 2\sqrt{x}\,dt$

$\therefore$ $I = 2\int \left(\frac{x\sqrt{x}}{t+1}\right)dt = 2\int \frac{t^2 \cdot t}{t+1}dt = 2\int \frac{t^3}{t+1}dt$

$$= 2\int \frac{t^3+1-1}{t+1}dt = 2\int \frac{(t+1)(t^2-t+1)}{t+1}dt - 2\int \frac{1}{t+1}dt$$

$$= 2\int (t^2-t+1)\,dt - 2\int \frac{1}{t+1}dt$$

$$= 2\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|(t+1)|\right] + C$$

$$= 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log|(\sqrt{x}+1)|\right] + C$$

Q. 11 $\int \sqrt{\frac{a+x}{a-x}}\,dx$

Thinking Process

Here, put $x = a\cos 2\theta$ *and also use the formula i.e.,* $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, *to get the solution.*

Sol. Let $I = \int \sqrt{\frac{a+x}{a-x}}\,dx$

Put $x = a\cos 2\theta$

$\Rightarrow$ $dx = -a \cdot \sin 2\theta \cdot 2 \cdot d\theta$

$\therefore$ $I = -2\int \sqrt{\frac{a + a\cos 2\theta}{a - a\cos 2\theta}} \cdot a\sin 2\theta\, d\theta$

$$\left[\because \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1}\frac{x}{a} \Rightarrow \theta = \frac{1}{2}\cos^{-1}\frac{x}{a}\right]$$

$$= -2a\int \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\sin 2\theta\, d\theta = -2a\int \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\sin 2\theta\, d\theta$$

$$= -2a\int \cot\theta \cdot \sin 2\theta\, d\theta = -2a\int \frac{\cos\theta}{\sin\theta} \cdot 2\sin\theta \cdot \cos\theta\, d\theta$$

$$= -4a\int \cos^2\theta\, d\theta = -2a\int (1+\cos 2\theta)d\theta$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= -2a\left[\frac{1}{2}\cos^{-1}\frac{x}{a} + \frac{1}{2}\sqrt{1-\frac{x^2}{a^2}}\right] + C$$

$$= -a\left[\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{1-\frac{x^2}{a^2}}\right] + C$$

Alternate Method

Let $$I = \int \sqrt{\frac{a+x}{a-x}}\,dx = \int \sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}}\,dx$$

$$= \int \frac{(a+x)}{\sqrt{a^2 - x^2}}dx$$

$$I = \int \frac{a}{\sqrt{a^2-x^2}} + \int \frac{x}{\sqrt{a^2-x^2}}dx$$

∴ $$I = I_1 + I_2 \qquad \ldots(i)$$

Now, $$I_1 = \int \frac{a}{\sqrt{a^2-x^2}} = a\sin^{-1}\left(\frac{x}{a}\right) + C_1$$

and $$I_2 = \int \frac{x}{\sqrt{a^2-x^2}}dx$$

Put $$a^2 - x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$$

∴ $$I_2 = -\int \frac{t}{t}dte = -\int 1dt$$

$$= -t + C_2 = -\sqrt{a^2 - x^2} + C_2$$

∴ $$I = a\sin^{-1}\left(\frac{x}{a}\right) + C_1 - \sqrt{a^2-x^2} + C_2 \qquad [\because t^2 = a^2 - x^2]$$

$$I = a\sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2} + C \qquad [\because C = C_1 + C_2]$$

Q. 12 $\int \frac{x^{1/2}}{1+x^{3/4}}dx$

Sol. Let $$I = \int \frac{x^{1/2}}{1+x^{3/4}}dx$$

Put $$x = t^4 \Rightarrow dx = 4t^3dt$$

∴ $$I = 4\int \frac{t^2(t^3)}{1+t^3}dt = 4\int \left(t^2 - \frac{t^2}{1+t^3}\right)dt$$

$$I = 4\int t^2dt - 4\int \frac{t^2}{1+t^3}dt$$

$$I = I_1 - I_2$$

$$I_1 = 4\int t^2 dt = 4\cdot\frac{t^3}{3} + C_1 = \frac{4}{3}x^{3/4} + C_1$$

Now, $$I_2 = 4\int \frac{t^2}{1+t^3}dt$$

Again, put $$1 + t^3 = z \Rightarrow 3t^2dt = dz$$

⇒ $$t^2dt = \frac{1}{3}dz = \frac{4}{3}\int \frac{1}{z}dz$$

$$= \frac{4}{3}\log|z| + C_2 = \frac{4}{3}\log|(1+t^3)| + C_2$$

$$= \frac{4}{3}\log|(1+x^{3/4})| + C_2$$

∴ $$I = \frac{4}{3}x^{3/4} + C_1 - \frac{4}{3}\log|(1+x^{3/4})| - C_2$$

$$= \frac{4}{3}x^{3/4} - \log|(1+x^{3/4}) + C \qquad [\because C = C_1 - C_2]$$

Q. 13 $\int \frac{\sqrt{1+x^2}}{x^4}dx$

Sol. Let
$$I = \int \frac{\sqrt{1+x^2}}{x^4}dx = \int \frac{\sqrt{1+x^2}}{x}\cdot\frac{1}{x^3}dx$$
$$= \int \sqrt{\frac{1+x^2}{x^2}}\cdot\frac{1}{x^3}dx = \int \sqrt{\frac{1}{x^2}+1}\cdot\frac{1}{x^3}dx$$

Put
$$1+\frac{1}{x^2} = t^2 \Rightarrow \frac{-2}{x^3}dx = 2t\,dt$$

$\Rightarrow$
$$-\frac{1}{x^3} = t\,dt$$

$\therefore$
$$I = -\int t^2 dt = -\frac{t^3}{3} + C = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2} + C$$

Q. 14 $\int \frac{dx}{\sqrt{16-9x^2}}$

Thinking Process

First of all concert the expression in form of $\frac{1}{\sqrt{a^2-x^2}}$, *then use the formula,*

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Sol. Let $I = \int \frac{dx}{\sqrt{16-9x^2}} = \int \frac{dx}{\sqrt{(4)^2-(3x)^2}}dx = \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$

Q. 15 $\int \frac{dt}{\sqrt{3t-2t^2}}$

Sol. Let
$$I = \int \frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left(t^2-\frac{3}{2}t\right)}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t^2-2\cdot\frac{1}{2}\cdot\frac{3}{2}t\right)+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t-\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$
$$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}}$$
$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-\frac{3}{4}}{\frac{3}{4}}\right) + C = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4t-3}{3}\right) + C$$

Q. 16 $\int \frac{3x-1}{\sqrt{x^2+9}}dx$

Thinking Process

First of all convert to the given integral into two parts, then by using formula i.e., $\int \frac{1}{\sqrt{a^2+x^2}} = \log|x+\sqrt{a^2+x^2}|+C$, *get the desired result.*

Sol. Let

$$I = \int \frac{3x-1}{\sqrt{x^2+9}}dx$$

$$I = \int \frac{3x}{\sqrt{x^2+9}}dx - \int \frac{1}{\sqrt{x^2+9}}dx$$

$$I = I_1 - I_2$$

Now, $I_1 = \int \frac{3x}{\sqrt{x^2+9}}$

Put $x^2 + 9 = t^2 \Rightarrow 2xdx = 2t\,dt \Rightarrow xdx = tdt$

$\therefore$ $I_1 = 3\int \frac{t}{t}dt$

$$= 3\int dt = 3t + C_1 = 3\sqrt{x^2+9} + C_1$$

and $I_2 = \int \frac{1}{\sqrt{x^2+9}}dx = \int \frac{1}{\sqrt{x^2+(3)^2}}dx$

$$= \log|x + \sqrt{x^2+9}| + C_2$$

$\therefore$ $I = 3\sqrt{x^2+9} + C_1 - \log|x + \sqrt{x^2+9}| - C_2$

$= 3\sqrt{x^2+9} - \log|x + \sqrt{x^2+9}| + C$ $\quad [\because C = C_1 - C_2]$

Q. 17 $\int \sqrt{5-2x+x^2}\,dx$

Thinking Process

First of all convert the given expression into $\sqrt{x^2+a^2}$ *form, then use the formula i.e.,* $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C.$

Sol. Let $I = \int \sqrt{5-2x+x^2}\,dx = \int \sqrt{x^2-2x+1+4}\,dx$

$$= \int \sqrt{(x-1)^2+(2)^2}\,dx = \int \sqrt{(2)^2+(x-1)^2}\,dx$$

$$= \frac{x-1}{2}\sqrt{2^2+(x-1)^2} + 2\log|x-1+\sqrt{2^2+(x-1)^2}| + C$$

$$= \frac{x-1}{2}\sqrt{5-2x+x^2} + 2\log|x-1+\sqrt{5-2x+x^2}| + C$$

Q. 18 $\int \frac{x}{x^4 - 1} dx$

Sol. Let $I = \int \frac{x}{x^4 - 1} dx$

Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$

$\therefore$ $I = \frac{1}{2} \int \frac{dt}{t^2 - 1} = \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C$ $\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$

$= \frac{1}{4} [\log |x^2 - 1| - \log |x^2 + 1|] + C$

Q. 19 $\int \frac{x^2}{1 - x^4} dx$

Thinking Process

Here, use $\int \frac{1}{1 + x^2} dx = \tan^{-1} x + C$ *and* $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{1+x}{1-x} \right| + C$, *to solve this problem.*

Sol. Let $I = \int \frac{x^2}{1 - x^4} dx$

$= \int \frac{\left(\frac{1}{2} + \frac{x^2}{2} - \frac{1}{2} + \frac{x^2}{2} \right)}{(1 - x^2)(1 + x^2)} dx$ $[\because a^2 - b^2 = (a+b)(a-b)]$

$= \int \frac{\frac{1}{2}(1 + x^2) - \frac{1}{2}(1 - x^2)}{(1 - x^2)(1 + x^2)} dx$

$= \int \frac{\frac{1}{2}(1 + x^2)}{(1 - x^2)(1 + x^2)} dx - \frac{1}{2} \int \frac{(1 - x^2)}{(1 - x^2)(1 + x^2)} dx$

$= \frac{1}{2} \int \frac{1}{1 - x^2} dx - \frac{1}{2} \int \frac{1}{1 + x^2} dx = \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + C_1 - \frac{1}{2} \tan^{-1} x + C_2$

$= \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C$ $[\because C = C_1 + C_2]$

Q. 20 $\int \sqrt{2ax - x^2} dx$

Sol. Let $I = \int \sqrt{2ax - x^2} dx = \int \sqrt{-(x^2 - 2ax)} dx$

$= \int \sqrt{-(x^2 - 2ax + a^2 - a^2)} dx = \int \sqrt{-(x-a)^2 - a^2} dx$

$= \int \sqrt{a^2 - (x-a)^2} dx$

$= \frac{x-a}{2} \sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$

$= \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$

Q. 21 $\int \frac{\sin^{-1} x}{(1-x^2)^{3/4}} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/4}} dx = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

and $x = \sin t \Rightarrow 1 - x^2 = \cos^2 t$

$\Rightarrow \cos t = \sqrt{1-x^2}$

$\therefore I = \int \frac{t}{\cos^2 t} dt = \int t \cdot \sec^2 t dt$

$= t \cdot \int \sec^2 t dt - \int \left(\frac{d}{dt} t \cdot \int \sec^2 t \, dt \right) dt$

$= t \cdot \tan t - \int 1 \cdot \tan t \, dt$

$= t \tan t + \log |\cos t| + C$ $\quad [\because \int \tan x dx = -\log |\cos x| + C]$

$= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} + \log |\sqrt{1-x^2}| + C$

Q. 22 $\int \frac{(\cos 5x + \cos 4x)}{1 - 2\cos 3x} dx$

Sol. Let $I = \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx = \int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1 - 2\left(2\cos^2 \frac{3x}{2} - 1\right)} dx$

$\left[\because \cos C + \cos D = 2\cos \frac{C+D}{2} . \cos \frac{C-D}{2} \text{ and } \cos 2x = 2\cos^2 x - 1\right]$

$\therefore I = \int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{3 - 4\cos^2 \frac{3x}{2}} dx = -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{4\cos^2 \frac{3x}{2} - 3} dx$

$= -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{4\cos^3 \frac{3x}{2} - 3\cos \frac{3x}{2}} dx$ $\quad \left[\text{multiply and divide by } \cos \frac{3x}{2}\right]$

$= -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos 3 \cdot \frac{3x}{2}} dx = -\int 2\cos \frac{3x}{2} \cdot \cos \frac{x}{2} dx$

$= -\int \left\{ \cos \left(\frac{3x}{2} + \frac{x}{2} \right) + \cos \left(\frac{3x}{2} - \frac{x}{2} \right) \right\} dx$

$= -\int (\cos 2x + \cos x) dx$

$= -\left[\frac{\sin 2x}{2} + \sin x \right] + C$

$= -\frac{1}{2} \sin 2x - \sin x + C$

Q. 23 $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

Thinking Process

Use $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ *and* $\sec^2 x = 1 + \tan^2 x$*,* $\text{cosec}^2 x = 1 + \cot^2 x$ *to solve the above problem.*

Sol. Let

$$I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \frac{\sin^4 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \tan^2 x \, dx + \int \cot^2 x \, dx - \int 1 dx$$

$$= \int (\sec^2 x - 1) dx + \int (\text{cosec}^2 x - 1) dx - \int 1 dx$$

$$= \int \sec^2 x dx + \int \text{cosec}^2 x dx - 3\int dx$$

$$I = \tan x - \cot x - 3x + C$$

Q. 24 $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

Sol. Let

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$$

Put

$$x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt$$

$\therefore$

$$I = \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C$$

$$= \frac{2}{3} \sin^{-1} \frac{x^{3/2}}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \sqrt{\frac{x^3}{a^3}} + C$$

Q. 25 $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

Thinking Process

Apply the formula, $\cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2}$ *and* $\cos x = 1 - 2\sin^2 \frac{x}{2}$ *to solve it.*

Sol. Let

$$I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx = \int \frac{2 \sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{1 - 1 + 2\sin^2 \frac{x}{2}} dx$$

$$= 2\int \frac{\sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx = \int \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} dx$$

$$= \int \frac{3\sin\frac{x}{2} - 4\sin^3\frac{x}{2}}{\sin\frac{x}{2}} dx \qquad [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$= 3\int dx - 4\int \sin^2\frac{x}{2} dx = 3\int dx - 4\int \frac{1-\cos x}{2} dx$$

$$= 3\int dx - 2\int dx + 2\int \cos x\, dx$$

$$= \int dx + 2\int \cos x\, dx = x + 2\sin x + C = 2\sin x + x + C$$

Q. 26 $\int \frac{dx}{x\sqrt{x^4 - 1}}$

Sol. Let $I = \int \frac{dx}{x\sqrt{x^4-1}}$

Put $x^2 = \sec\theta \Rightarrow \theta = \sec^{-1} x^2$

$\Rightarrow$ $2x\, dx = \sec\theta \cdot \tan\theta\, d\theta$

$\therefore$ $I = \frac{1}{2}\int \frac{\sec\theta\cdot\tan\theta}{\sec\theta\tan\theta} d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$

$= \frac{1}{2}\sec^{-1}(x^2) + C$

Q. 27 $\int_0^2 (x^2 + 3) dx$

Thinking Process

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f\{a+(n-1)h\}]$, *where* $h = \frac{b-a}{n} \to 0$ *as* $n \to \infty$.

Sol. Let $I = \int_0^2 (x^2 + 3)\, dx$

Here, $a = 0, b = 2$ and $h = \frac{b-a}{n} = \frac{2-0}{n}$

$\Rightarrow$ $h = \frac{2}{n} \Rightarrow nh = 2 \Rightarrow f(x) = (x^2 + 3)$

Now, $\int_0^2 (x^2+3)\, dx = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}]$ ...(i)

$\because$ $f(0) = 3$

$\Rightarrow$ $f(0+h) = h^2 + 3,\ f(0+2h) = 4h^2 + 3 = 2^2h^2 + 3$

$f[0 + (n-1)h] = (n^2 - 2n + 1)h + 3 = (n-1)^2 h + 3$

From Eq. (i),

$$\int_0^2 (x^2+3)dx = \lim_{h\to 0} h[3 + h^2 + 3 + 2^2h^2 + 3 + 3^2h^2 + 3 + \ldots + (n-1)^2h^2 + 3]$$

$$= \lim_{h\to 0} h[3n + h^2\{1^2 + 2^2 + \ldots + (n-1)^2\}]$$

$$= \lim_{h\to 0} h\left[3n + h^2\left(\frac{(n-1)(2n-2+1)(n-1+)}{6}\right)\right] \qquad \left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0} h\left[3n + h^2\left(\frac{(n^2-n)(2n-1)}{6}\right)\right]$$

$$= \lim_{h\to 0} h\left[3n + \frac{h^2}{6}(2n^3 - n^2 - 2n^2 + n)\right]$$

$$= \lim_{h\to 0}\left[3nh + \frac{2n^3h^3 - 3n^2h^2\cdot h + nh\cdot h^2}{6}\right]$$

$$= \lim_{h\to 0}\left[3\cdot 2 + \frac{2\cdot 8 - 3\cdot 2^2\cdot h + 2\cdot h^2}{6}\right] = \lim_{h\to 0}\left[6 + \frac{16 - 12h + 2h^2}{6}\right]$$

$$= 6 + \frac{16}{6} = 6 + \frac{8}{3} = \frac{26}{3}$$

Q. 28 $\int_0^2 e^x dx$

Sol. Let $I = \int_0^2 e^x\, dx$

Here, $a = 0$ and $b = 2$

$\therefore$ $h = \frac{b-a}{n}$

$\Rightarrow$ $nh = 2$ and $f(x) = e^x$

Now, $\int_0^2 e^x dx = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}]$

$\therefore$ $I = \lim_{h\to 0} h[1 + e^h + e^{2h} + \ldots + e^{(n-1)h}]$

$$= \lim_{h\to 0} h\left[\frac{1\cdot (e^h)^n - 1}{e^h - 1}\right] = \lim_{h\to 0} h\left(\frac{e^{nh} - 1}{e^h - 1}\right)$$

$$= \lim_{h\to 0} h\left(\frac{e^2 - 1}{e^h - 1}\right)$$

$$= e^2 \lim_{h\to 0}\frac{h}{e^h - 1} - \lim_{h\to 0}\frac{h}{e^h - 1} \qquad \left[\because \lim_{h\to 0}\frac{h}{e^h - 1} = 1\right]$$

$$= e^2 - 1 = e^2 - 1$$

Evaluate the following questions.

Q. 29 $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Sol. Let $I = \int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x}{1 + e^{2x}} dx$

Put $e^x = t$

$\Rightarrow$ $e^x\, dx = dt$

$\therefore$ $I = \int_1^e \frac{dt}{1 + t^2} = [\tan^{-1} t]_1^e$

$$= \tan^{-1} e - \tan^{-1} 1$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

Q. 30 $\int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x}\,dx$

Sol. Let

$$I = \int_0^{\pi/2} \frac{\tan x\,dx}{1+m^2\tan^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2\cdot\frac{\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2\sin^2 x}{\cos^2 x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x\,dx}{1-\sin^2 x + m^2\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x(1-m^2)}\,dx$$

Put $\sin^2 x = t$

$\Rightarrow$ $2\sin x\cos x\,dx = dt$

$\therefore$ $$I = \frac{1}{2}\int_0^1 \frac{dt}{1-t(1-m^2)}$$

$$= \frac{1}{2}\left[-\log|1-t(1-m^2)|\cdot\frac{1}{1-m^2}\right]_0^1$$

$$= \frac{1}{2}\left[-\log|1-1+m^2|\cdot\frac{1}{1+m^2} + \log|1|\cdot\frac{1}{1-m^2}\right]$$

$$= \frac{1}{2}\left[-\log|m^2|\cdot\frac{1}{1-m^2}\right] = \frac{2}{2}\cdot\frac{\log m}{(m^2-1)}$$

$$= \log\frac{m}{m^2-1}$$

Q. 31 $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

Thinking Process

First of all convert the given function into $\frac{1}{\sqrt{a^2-x^2}}$ *form, then apply the formula i.e.,*

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C.$$

Sol. Let

$$I = \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} = \int_1^2 \frac{dx}{\sqrt{2x-x^2-2+x}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-(x^2-3x+2)}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left[x^2 - 2\cdot\frac{3}{2}x + \left(\frac{3}{2}\right)^2 + 2 - \frac{9}{4}\right]}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}}$$

$$= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= [\sin^{-1}(2x - 3)]_1^2 = \sin^{-1}1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \sin(-\theta) = -\sin\theta\right]$$

$$= \pi$$

Q. 32 $\int_0^1 \frac{x}{\sqrt{1 + x^2}}\, dx$

Sol. Let $I = \int_0^1 \frac{x}{\sqrt{1 + x^2}}\, dx$

Put $1 + x^2 = t^2$

$\Rightarrow$ $2x\, dx = 2t dt$

$\Rightarrow$ $x\, dx = t dt$

$\therefore$ $I = \int_1^{\sqrt{2}} \frac{t dt}{t}$

$= [t]_1^{\sqrt{2}} = \sqrt{2} - 1$

Q. 33 $\int_0^\pi x \sin x \cos^2 x\, dx$

Thinking Process

Here, use the property i.e., $\int_0^a f(x)dx = \int_0^a (a - x)dx$ *and*

$\sin(\pi - x) = \sin x, \cos(\pi - x) = \cos x$.

Sol. Let $I = \int_0^\pi x \sin x \cos^2 x\, dx$ ...(i)

and $I = \int_0^\pi (\pi - x)\sin(\pi - x)\cos^2(\pi - x)dx$

$\Rightarrow$ $I = \int_0^\pi (\pi - x)\sin x \cos^2 x\, dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \pi \sin x \cos^2 x\, dx$$

Put $\cos x = t$

$\Rightarrow$ $-\sin x\, dx = dt$

As $x \to 0$, then $t \to 1$
and $x \to \pi$, then $t \to -1$

$\therefore$ $$I = -\pi \int_1^{-1} t^2\, dt \Rightarrow I = -\pi \left[\frac{t^3}{3}\right]_1^{-1}$$

$\Rightarrow$ $$2I = -\frac{\pi}{3}[-1-1] \Rightarrow 2I = \frac{2\pi}{3}$$

$\therefore$ $$I = \frac{\pi}{3}$$

Q. 34 $\displaystyle\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol. Let $$I = \int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

Put $x = \sin\theta$

$\Rightarrow$ $dx = \cos\theta\, d\theta$

As $x \to 0$, then $\theta \to 0$

and $x \to \frac{1}{2}$, then $\theta \to \frac{\pi}{6}$

$\therefore$ $$I = \int_0^{\pi/6} \frac{\cos\theta}{(1+\sin^2\theta)\cos\theta}\, d\theta = \int_0^{\pi/6} \frac{1}{1+\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos^2\theta\,(\sec^2\theta + \tan^2\theta)}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+\tan^2\theta+\tan^2\theta}\, d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+2\tan^2\theta}\, d\theta$$

Again, put $\tan\theta = t$

$\Rightarrow$ $\sec^2\theta\, d\theta = dt$

As $\theta \to 0$, then $t \to 0$

and $\theta \to \frac{\pi}{6}$, then $t \to \frac{1}{\sqrt{3}}$

$\therefore$ $$I = \int_0^{1/\sqrt{3}} \frac{dt}{1+2t^2} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$$

$$= \frac{1}{2}\cdot\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{2}}}\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}[\tan^{-1}(\sqrt{2}\,t)]_0^{1/\sqrt{3}}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - 0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

Long Answer Type Questions

Q. 35 $\int \frac{x^2}{x^4 - x^2 - 12} dx$

Thinking Process

Use $\frac{px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{x-b}$, *where* $a \neq b$, *then compare the coefficient of* x *to get the value of A and B.*

Sol. Let
$$I = \int \frac{x^2}{x^4 - x^2 - 12} dx$$
$$= \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx$$
$$= \int \frac{x^2 dx}{x^2(x^2 - 4) + 3(x^2 - 4)}$$
$$= \int \frac{x^2 dx}{(x^2 - 4)(x^2 + 3)}$$

Now, $\frac{x^2}{(x^2 - 4)(x^2 + 3)}$ [let $x^2 = t$]

$\Rightarrow$ $\frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$

$\Rightarrow$ $t = A(t+3) + B(t-4)$

On comparing the coefficient of t on both sides, we get

$A + B = 1$ ...(i)

$\Rightarrow$ $3A - 4B = 0$ ...(ii)

$\Rightarrow$ $3(1 - B) - 4B = 0$

$\Rightarrow$ $3 - 3B - 4B = 0$

$\Rightarrow$ $7B = 3$

$\Rightarrow$ $B = \frac{3}{7}$

If $B = \frac{3}{7}$, then $A + \frac{3}{7} = 1$

$\Rightarrow$ $A = 1 - \frac{3}{7} = \frac{4}{7}$

$$\frac{x^2}{(x^2 - 4)(x^2 + 3)} = \frac{4}{7(x^2 - 4)} + \frac{3}{7(x^2 + 3)}$$

$\therefore$
$$I = \frac{4}{7}\int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7}\int \frac{1}{x^2 + (\sqrt{3})^2} dx$$
$$= \frac{4}{7} \cdot \frac{1}{2 \cdot 2} \log\left|\frac{x-2}{x+2}\right| + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + C$$
$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

Q. 36 $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$

Sol. Let $$I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$$

Now, $$\frac{x^2}{(x^2+a^2)(x^2+b^2)}$$ [let $x^2 = t$]

$$= \frac{t}{(t+a^2)(t+b^2)} = \frac{A}{(t+a^2)} + \frac{B}{(t+b^2)}$$

$$t = A(t+b^2) + B(t+a^2)$$

On comparing the coefficient of t, we get

$$A + B = 1 \quad \text{...(i)}$$

$$b^2A + a^2B = 0 \quad \text{...(ii)}$$

$\Rightarrow$ $$b^2(1-B) + a^2B = 0$$

$\Rightarrow$ $$b^2 - b^2B + a^2B = 0$$

$\Rightarrow$ $$b^2 + (a^2 - b^2)B = 0$$

$\Rightarrow$ $$B = \frac{-b^2}{a^2-b^2} = \frac{b^2}{b^2-a^2}$$

From Eq. (i), $$A + \frac{b^2}{b^2-a^2} = 1$$

$\Rightarrow$ $$A = \frac{b^2-a^2-b^2}{b^2-a^2} = \frac{-a^2}{b^2-a^2}$$

$\therefore$ $$I = \int \frac{-a^2}{(b^2-a^2)(x^2+a^2)}dx + \int \frac{b^2}{b^2-a^2}\cdot\frac{1}{x^2+b^2}dx$$

$$= \frac{-a^2}{(b^2-a^2)}\int \frac{1}{x^2+a^2}dx + \frac{b^2}{b^2-a^2}\int \frac{1}{x^2+b^2}dx$$

$$= \frac{-a^2}{b^2-a^2}\cdot\frac{1}{a}\tan^{-1}\frac{x}{a} + \frac{b^2}{b^2-a^2}\cdot\frac{1}{b}\tan^{-1}\frac{x}{b}$$

$$= \frac{1}{b^2-a^2}\left[-a\tan^{-1}\frac{x}{a} + b\tan^{-1}\frac{x}{b}\right]$$

$$= \frac{1}{a^2-b^2}\left[a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right]$$

Q. 37 $\int_0^{\pi} \frac{x}{1+\sin x}$

Sol. Let $$I = \int_0^{\pi} \frac{x}{1+\sin x}dx \quad \text{...(i)}$$

and $$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi-x)}dx = \int_0^{\pi}\frac{\pi-x}{1+\sin x}dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$= \pi\int_0^{\pi}\frac{(1-\sin x)\,dx}{(1+\sin x)(1-\sin x)}$$

$$= \pi\int_0^{\pi} \frac{(1-\sin x)\,dx}{\cos^2 x}$$

$$= \pi\int_0^{\pi}(\sec^2 x - \tan x\cdot\sec x)\,dx$$

$$= \pi\int_0^{\pi}\sec^2 x\,dx - \pi\int_0^{\pi}\sec x\,x\cdot\tan x\,dx$$

$$= \pi[\tan x]_0^{\pi} - \pi[\sec x]_0^{\pi}$$

$$= \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi[\tan\pi - \sec\pi - \tan 0 - \sec 0]$$

$$\Rightarrow \quad 2I = \pi[0 + 1 - 0 + 1]$$

$$2I = 2\pi$$

$$\therefore \quad I = \pi$$

Q. 38 $\displaystyle\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Thinking Process

Apply $\dfrac{px+q}{(x-a)(x-b)(x-c)} = \dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$, *then get the values of A, B and C and use* $\displaystyle\int\frac{1}{x}dx = \log|x| + C.$

Sol. Let $\displaystyle I = \int \frac{(2x-1)}{(x-1)(x+2)(x-3)}dx$

Now, $\dfrac{2x-1}{(x-1)(x+2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-3)}$

$\Rightarrow \quad 2x - 1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

Put $x = 3$, then

$6 - 1 = C(3-1)(3+2)$

$\Rightarrow \quad 5 = 10C \Rightarrow C = \dfrac{1}{2}$

Again, put $x = 1$, then

$2 - 1 = A(1+2)(1-3)$

$\Rightarrow \quad 1 = -6A \Rightarrow A = -\dfrac{1}{6}$

Now, put $x = -2$, then

$-4 - 1 = B(-2-1)(-2-3)$

$\Rightarrow \quad -5 = 15B \Rightarrow B = -\dfrac{1}{3}$

$$\therefore \quad I = -\frac{1}{6}\int\frac{1}{x-1}dx - \frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{2}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{6}\log|(x-1)| - \frac{1}{3}\log|(x+2)| + \frac{1}{2}\log|(x-3)| + C$$

$$= -\log|(x-1)|^{1/6} - \log|(x+2)|^{1/3} + \log|(x-3)|^{1/2} + C$$

$$= \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C$$

Q. 39 $\int e^{\tan^{-1} x}\left(\dfrac{1 + x + x^2}{1 + x^2}\right)dx$

Sol. Let $I = \int e^{\tan^{-1} x}\left(\dfrac{1+x+x^2}{1+x^2}\right)dx$

$$= \int e^{\tan^{-1} x}\left(\frac{1+x^2}{1+x^2} + \frac{x}{1+x^2}\right)dx$$

$$= \int e^{\tan^{-1} x}dx + \int \frac{x\, e^{\tan^{-1} x}}{1+x^2}dx$$

$$I = I_1 + I_2 \quad \ldots\text{(i)}$$

Now, $I_2 = \int \dfrac{x\, e^{\tan^{-1} x}}{1+x^2}dx$

Put $\tan^{-1} x = t \Rightarrow x = \tan t$

$\Rightarrow \dfrac{1}{1+x^2}dx = dt$

$\therefore$ $I = \int \underset{\text{I}}{\tan t} \cdot \underset{\text{II}}{e^t}\, dt$

$$= \tan t \cdot e^t - \int \sec^2 t \cdot e^t dt + C$$

$$= \tan t \cdot e^t - \int (1 + \tan^2 t)\, e^t dt + C \qquad [\because \sec^2\theta = 1 + \tan^2\theta]$$

$$I_2 = \tan t \cdot e^t - \int (1+x^2)\frac{e^{\tan^{-1} x}}{1+x^2}dx + C$$

$$I_2 = \tan t \cdot e^t - \int e^{\tan^{-1} x}dx + C$$

$\therefore$ $I = \int e^{\tan^{-1} x}dx + \tan t \cdot e^t - \int e^{\tan^{-1} x}dx + C$

$$= \tan t \cdot e^t + C$$

$$= x\, e^{\tan^{-1} x} + C$$

Q. 40 $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\, dx$

Thinking Process

First of all put $x = \tan^2\theta$ and convert the given expression into two parts, then use the formulae for integration by part i.e., $\int \text{I} \cdot \text{II} dx = \text{I}\int \text{II} dx - \int\left(\dfrac{d}{dx}\text{I}\int \text{II} dx\right)dx$

Sol. Let $I = \int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\, dx$

Put $x = a\tan^2\theta$

$\Rightarrow$ $dx = 2a\tan\theta\sec^2\theta\, d\theta$

$\therefore$ $I = \int \sin^{-1}\sqrt{\dfrac{a\tan^2\theta}{a + a\tan^2\theta}}(2a\tan\theta \cdot \sec^2\theta)\, d\theta$

$$= 2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta \cdot \sec^2\theta\, d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta \cdot \sec^2\theta\, d\theta$$

$$= 2a\int \underset{\text{I}}{\theta} \cdot \tan\theta \underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$= 2a\left[\theta \cdot \int \tan\theta \cdot \sec^2\theta\, d\theta - \int \left(\frac{d}{d\theta}\theta \cdot \int \tan\theta \cdot \sec^2\theta\, d\theta\right) d\theta\right]$$

$$\left[\begin{array}{ll} \text{Put} & \tan\theta = t \\ \Rightarrow & \sec\theta \cdot \tan\theta \cdot d\theta = dt \\ \Rightarrow \int \tan\theta \sec^2\theta\, d\theta & = \int t\, dt \end{array}\right]$$

$$= 2a\left[\theta \cdot \frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2} d\theta\right]$$

$$= a\theta \tan^2\theta - a\int (\sec^2\theta - 1)\, d\theta$$

$$= a\theta \cdot \tan^2\theta - a\tan\theta + a\theta + C$$

$$= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + C$$

Q. 41 $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$

Sol. Let

$$I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$$

$$= \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^2\sqrt{1+\cos x}}\, dx$$

$$= \int_{\pi 3}^{\pi 2} \frac{1}{(1-\cos^2 x)}\, dx = \int_{\pi 3}^{\pi 2} \frac{1}{\sin^2 x}\, dx$$

$$= \int_{\pi 3}^{\pi 2} \operatorname{cosec}^2 x\, dx = [-\cot x]_{\pi/3}^{\pi/2}$$

$$= -\left[\cot\frac{\pi}{2} - \cot\frac{\pi}{3}\right] = -\left[0 - \frac{1}{\sqrt{3}}\right] = +\frac{1}{\sqrt{3}}$$

Alternate Method

Let

$$I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx = \int_{\pi/3}^{\pi/2} \frac{\left(2\cos^2\frac{x}{2}\right)^{1/2}}{\left(2\sin^2\frac{x}{2}\right)^{5/2}}\, dx$$

$$= \frac{\sqrt{2}}{4\sqrt{2}}\int_{\pi/3}^{\pi/2} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx \quad = \frac{1}{4}\int_{\pi/3}^{\pi/2} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\, dx$$

Put $\sin\frac{x}{2} = t$

$\Rightarrow$ $\cos\frac{x}{2} \cdot \frac{1}{2}dx = dt$

$\Rightarrow$ $\cos\frac{x}{2}dx = 2dt$

As $x \to \frac{\pi}{3}$, then $t \to \frac{1}{2}$

and $x \to \frac{\pi}{2}$, then $t \to \frac{1}{\sqrt{2}}$

$\therefore$
$$I = \frac{2}{4}\int_{1/2}^{1\sqrt{2}} \frac{dt}{t^5} = \frac{1}{2}\left[\frac{t^{-5+1}}{-5+1}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} - \frac{1}{\left(\frac{1}{2}\right)^4}\right]$$

$$= -\frac{1}{8}(4-16) = \frac{12}{8} = \frac{3}{2}$$

Note *If we integrate the trigonometric function in different ways [using different identities] then, we can get different answers.*

Q. 42 $\int e^{-3x}\cos^3 x\,dx$

Sol. Let
$$I = \int \underset{\text{II}}{e^{-3x}}\ \underset{\text{I}}{\cos^3 x}\,dx$$

$$= \cos^3 x \int e^{-3x}\,dx - \int\left(\frac{d}{dx}\cos^3 x \int e^{-3x}\,dx\right)dx$$

$$= \cos^3 x \cdot \frac{e^{-3x}}{-3} - \int(-3\cos^2 x)\sin x \cdot \frac{e^{-3x}}{-3}dx$$

$$= -\frac{1}{3}\cos^3 x e^{-3x} - \int \cos^2 x \sin x e^{-3x} dx$$

$$= -\frac{1}{3}\cos^3 x\, e^{-3x} - \int(1-\sin^2 x)\sin x\, e^{-3x}dx$$

$$= -\frac{1}{3}\cos^3 x e^{-3x} - \int \sin x\, e^{-3x}dx + \int \underset{\text{I}}{\sin^3 x}\ \underset{\text{II}}{e^{-3x}}dx$$

$$= -\frac{1}{3}\cos^3 x\, e^{-3x} - \int \sin x\, e^{-3x}dx + \sin^3 x \cdot \frac{e^{-3x}}{-3} - \int 3\sin^2 x \cos x \cdot \frac{e^{-3x}}{-3}dx$$

$$= -\frac{1}{3}\cos^3 x\, e^{-3x} - \int \sin x\, e^{-3x}dx - \frac{1}{3}\sin^3 x\, e^{-3x} + \int(1-\cos^2 x)\cos x\, e^{-3x}dx$$

$$I = -\frac{1}{3}\cos^3 x\, e^{-3x} - \int \underset{\text{I}}{\sin x}\ \underset{\text{II}}{e^{-3x}} - \frac{1}{3}\sin^3 x\, e^{-3x} + \int \cos x\, e^{-3x}dx - \int \cos^3 x\, e^{-3x}dx$$

$$2I = \frac{e^{-3x}}{3}[\cos^3 x + \sin^3 x] - \left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3}dx\right] + \int \cos x\, e^{-3x}dx$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] + \frac{1}{3}\sin x \cdot e^{-3x} - \frac{1}{3}\int \cos x \cdot e^{-3x}dx + \int \cos x\, e^{-3x}dx$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] + \frac{1}{3}\sin x\, e^{-3x} + \frac{2}{3}\int \cos x\, e^{-3x}dx$$

Now, let $I_1 = \int \underset{\text{I}}{\cos x}\, \underset{\text{II}}{e^{-3x}} dx$

$$I_1 = \cos x \cdot \frac{e^{-3x}}{-3} - \int (-\sin x) \cdot \frac{e^{-3x}}{-3} dx$$

$$I_1 = \frac{-1}{3} \cos x \cdot e^{-3x} - \frac{1}{3} \int \sin x \cdot e^{-3x} dx$$

$$= -\frac{1}{3} \cos x \cdot e^{-3x} - \frac{1}{3}\left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3} dx\right]$$

$$= -\frac{1}{3} \cos x \cdot e^{-3x} + \frac{1}{9} \sin x \cdot e^{-3x} - \frac{1}{9} \int \cos x \cdot e^{-3x} dx$$

$$I_1 + \frac{1}{9} I_1 = -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x}$$

$$\left(\frac{10}{9}\right) I_1 = -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x}$$

$$I_1 = \frac{-3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \sin x$$

$$2I = -\frac{1}{3} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{1}{3} \sin x \cdot e^{-3x} - \frac{3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \cdot \sin x + C$$

∴ $$I = -\frac{1}{6} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{13}{30} e^{-3x} \cdot \sin x - \frac{3}{10} e^{-3x} \cdot \cos x + C$$

$$\left[\because \sin 3x = 3\sin x - 4\sin^3 x \text{ and } \cos 3x = 4\cos^3 x - 3\cos x\right]$$

$$= \frac{e^{-3x}}{24} [\sin 3x - \cos 3x] + \frac{3e^{-3x}}{40} [\sin x - 3\cos x] + C$$

Q. 43 $\int \sqrt{\tan x}\, dx$

Sol. Let $I = \int \sqrt{\tan x}\, dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

∴ $$I = \int t \cdot \frac{2t}{\sec^2 x} dt = 2\int \frac{t^2}{1 + t^4} dt$$

$$= \int \frac{(t^2 + 1) + (t^2 - 1)}{(1 + t^4)} dt$$

$$= \int \frac{t^2 + 1}{1 + t^4} dt + \int \frac{t^2 - 1}{1 + t^4} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 - \left(-\frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2} + \int \frac{1 + \left(-\frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} dt$$

Put $\quad u = t - \frac{1}{t} \Rightarrow du = \left(1 + \frac{1}{t^2}\right) dt$

and $\quad v = t + \frac{1}{t} \Rightarrow dv = \left(1 - \frac{1}{t^2}\right) dt$

$\therefore$

$$I = \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + C$$

Q. 44 $\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

Sol. Let

$$I = \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$$

Divide numerator and denominator by $\cos^4 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^4 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$$

Put $\quad \tan x = t$

$\Rightarrow \quad \sec^2 x \, dx = dt$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{2}$, then $t \to \infty$

$$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2 t^2)^2}$$

Now, $\quad \frac{1 + t^2}{(a^2 + b^2 t^2)^2}$ $\quad$ [let $t^2 = u$]

$$\frac{1 + u}{(a^2 + b^2 u)^2} = \frac{A}{(a^2 + b^2 u)} + \frac{B}{(a^2 + b^2 u)^2}$$

$\Rightarrow \quad 1 + u = A(a^2 + b^2 u) + B$

On comparing the coefficient of x and constant term on both sides, we get

$$a^2 A + B = 1 \quad \ldots\text{(i)}$$

and $\quad b^2 A = 1 \quad \ldots\text{(ii)}$

$\therefore \quad A = \frac{1}{b^2}$

Now, $\quad \frac{a^2}{b^2} + B = 1$

$\Rightarrow \quad B = 1 - \frac{a^2}{b^2} = \frac{b^2 - a^2}{b^2}$

$\therefore$

$$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2 t^2)^2}$$

$$= \frac{1}{b^2} \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} + \frac{b^2 - a^2}{b^2} \int_0^{\infty} \frac{dt}{(a^2 + b^2 t^2)^2}$$

$$= \frac{1}{b^2}\int_0^\infty \frac{dt}{b^2\left(\frac{a^2}{b^2}+t^2\right)} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\left(\frac{tb}{a}\right)\right]_0^\infty + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4}\cdot\frac{1}{a^3b}\right)$$

$$= \frac{1}{ab^3}[\tan^{-1}\infty - \tan^{-1}0] + \frac{\pi}{4}\cdot\frac{b^2-a^2}{(a^3b^3)}$$

$$= \frac{\pi}{2ab^3} + \frac{\pi}{4}\cdot\frac{b^2-a^2}{(a^3b^3)}$$

$$= \pi\left(\frac{2a^2+b^2-a^2}{4a^3b^3}\right) = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

Q. 45 $\int_0^1 x\log(1+2x)\,dx$

Thinking Process

Use formula for integration by part i.e., $\int \mathrm{I}\cdot\mathrm{II}\,dx = \mathrm{I}\int\mathrm{II}\,dx - \int\left(\frac{d}{dx}\mathrm{I}\int\mathrm{I}\,dx\right)dx$ *and also use* $\int\frac{1}{x} = \log|x| + C.$

Sol. Let

$$I = \int_0^1 x\log(1+2x)dx$$

$$= \left[\log(1+2x)\frac{x^2}{2}\right]_0^1 - \int\frac{1}{1+2x}\cdot 2\cdot\frac{x^2}{2}dx$$

$$= \frac{1}{2}[x^2\log(1+2x)]_0^1 - \int\frac{x^2}{1+2x}dx$$

$$= \frac{1}{2}[1\log 3 - 0] - \left[\int_0^1\left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x}\right)dx\right]$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_0^1\frac{x}{1+2x}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1\frac{\frac{1}{2}(2x+1-1)}{(2x+1)}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2}-0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1\frac{1}{1+2x}dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\log|(1+2x)|]_0^1$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3$$

$$= \frac{3}{8}\log 3$$

Q. 46 $\int_0^{\pi} x \log \sin x \, dx$

Thinking Process

First of all use property of definite integral i.e., $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, *then use* $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

Sol. Let

$$I = \int_0^{\pi} x \log \sin x \, dx \quad \ldots\text{(i)}$$

$$I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x)\, dx$$

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx \quad \ldots\text{(ii)}$$

$$2I = \pi \int_0^{\pi} \log \sin x \, dx \quad \ldots\text{(iii)}$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \quad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx\right]$$

$$I = \pi \int_0^{\pi/2} \log \sin x \, dx \quad \ldots\text{(iv)}$$

Now,

$$I = \pi \int_0^{\pi/2} \log \sin(\pi/2 - x)\, dx \quad \ldots\text{(v)}$$

On adding Eqs. (iv) and (v), we get

$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$2I = \pi \int_0^{\pi/2} \log \sin x \cos x \, dx$$

$$= \pi \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2} dx$$

$$2I = \pi \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$2I = \pi \int_0^{\pi/2} \log \sin 2x \, dx - \pi \int_0^{\pi/2} \log 2 \, dx$$

Put $2x = t \Rightarrow dx = \frac{1}{2} dt$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{2}$, then $t \to \pi$

$\therefore$
$$2I = \frac{\pi}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi^2}{2} \log 2$$

$\Rightarrow$
$$2I = \frac{\pi}{2} \int_0^{\pi} \log \sin x \, dx - \frac{\pi^2}{2} \log 2$$

$\Rightarrow$
$$2I = I - \frac{\pi^2}{2} \log 2 \quad \text{[from Eq. (iii)]}$$

$\therefore$
$$I = -\frac{\pi^2}{2} \log 2 = \frac{\pi^2}{2} \log\left(\frac{1}{2}\right)$$

Q. 47 $\int_{\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$ ...(i)

$$I = \int_{-\pi/4}^{\pi/4} \log\left\{\sin\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)\right\} dx$$

$$= \int_{-\pi/4}^{\pi/4} \log\{\sin(-x) + \cos(-x)\}\, dx$$

and $I = \int_{-\pi/4}^{\pi 4} \log(\cos x - \sin x)\, dx$ ...(ii)

From Eqs. (i) and (ii),

$$2I = \int_{-\pi/4}^{\pi 4} \log\cos 2x\, dx$$

$$2I = \int_{0}^{\pi/4} \log\cos 2x\, dx \quad \text{...(iii)}$$

$$\left[\because \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x), \text{ if } f(-x) = f(x)\right]$$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{4}$, then $t \to \frac{\pi}{2}$

$$2I = \frac{1}{2}\int_0^{\pi/2} \log\cos t\, dt \quad \text{...(iv)}$$

$$2I = \frac{1}{2}\int_0^{\pi/2} \log\cos\left(\frac{\pi}{2} - t\right) dt \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow 2I = \frac{1}{2}\int_0^{\pi/2} \log\sin t\, dx \quad \text{...(v)}$$

On adding Eqs. (iv) and (v), we get

$$4I = \frac{1}{2}\int_0^{\pi/2} \log\sin t \cos t\, dt$$

$$\Rightarrow 4I = \frac{1}{2}\int_0^{\pi/2} \log\frac{\sin 2t}{2}\, dt$$

$$\Rightarrow 4I = \frac{1}{2}\int_0^{\pi/2} \log\sin 2x\, dx - \frac{1}{2}\int_0^{\pi/2} \log 2\, dx$$

$$\Rightarrow 4I = \frac{1}{2}\int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - 2x\right) dx - \log 2 \cdot \frac{\pi}{4}$$

$$\Rightarrow 4I = \frac{1}{2}\int_0^{\pi/2} \log\cos 2x\, dx - \frac{\pi}{4}\log 2$$

$$\Rightarrow 4I = \int_0^{\pi/4} \log\cos 2x\, dx - \frac{\pi}{4}\log 2 \quad \left[\because \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx\right]$$

$$\Rightarrow 4I = 2I - \frac{\pi}{4}\log 2 \quad \text{[from Eq. (iii)]}$$

$$\therefore I = -\frac{\pi}{8}\log 2 = \frac{\pi}{8}\log\left(\frac{1}{2}\right)$$

Objective Type Questions

Q. 48 $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$ is equal to

(a) $2(\sin x + x\cos\theta) + C$ (b) $2(\sin x - x\cos\theta) + C$

(c) $2(\sin x + 2x\cos\theta) + C$ (d) $2(\sin x - 2x\cos\theta) + C$

Thinking Process

Use formula $\cos 2\theta = 2\cos^2\theta - 1$ *to get simplest form, then apply* $\int \cos x\, dx = \sin x + C$.

Sol. *(a)* Let

$$I = \int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2\theta + 1)}{\cos x - \cos\theta} dx$$

$$= 2\int \frac{(\cos x + \cos\theta)(\cos x - \cos\theta)}{(\cos x - \cos\theta)} dx$$

$$= 2\int (\cos x + \cos\theta)\, dx$$

$$= 2(\sin x + x\cos\theta) + C$$

Q. 49 $\frac{dx}{\sin(x-a)\sin(x-b)}$ is equal to

(a) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$ (b) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

(c) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$ (d) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

Sol. *(c)* Let

$$I = \int \frac{dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int [\cot(x-b) - \cot(x-a)]\, dx$$

$$= \frac{1}{\sin(b-a)} [\log|\sin(x-b)| - \log|\sin(x-a)|] + C$$

$$= \operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$$

Q. 50 $\int \tan^{-1}\sqrt{x}\, dx$ is equal to

(a) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$ (b) $x\tan^{-1}\sqrt{x} - \sqrt{x} + C$
(c) $\sqrt{x} - x\tan^{-1}\sqrt{x} + C$ (d) $\sqrt{x} - (x+1)\tan^{-1}\sqrt{x} + C$

Thinking Process

Use formula for integration by part i.e., $\int \text{I}\cdot\text{II}\,dx = \text{I}\int \text{II}\,dx - \int\left(\frac{d}{dx}\text{I}\int\text{II}\,dx\right)dx$

Sol. *(a)* Let
$$I = \int 1\cdot\tan^{-1}\sqrt{x}\,dx$$
$$= \tan^{-1}\sqrt{x}\cdot x - \frac{1}{2}\int\frac{1}{(1+x)}\cdot\frac{2}{\sqrt{x}}dx$$
$$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int\frac{2}{\sqrt{x}(1+x)}dx$$

Put $x = t^2 \Rightarrow dx = 2t\,dt$

$\therefore$
$$I = x\tan^{-1}\sqrt{x} - \int\frac{t}{t(1+t^2)}dt$$
$$= x\tan^{-1}\sqrt{x} - \int\frac{t^2}{1+t^2}dt$$
$$= x\tan^{-1}\sqrt{x} - \int\left(1-\frac{1}{1+t^2}\right)dt$$
$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}t + C$$
$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$$
$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$$

Q. 51 $\int\frac{x^9}{(4x^2+1)^6}dx$ is equal to

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5} + C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5} + C$
(c) $\frac{1}{10x}(1+4)^{-5} + C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5} + C$

Sol. *(d)* Let
$$I = \int\frac{x^9}{(4x^2+1)^6}dx = \int\frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6}dx$$
$$= \int\frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$$

Put $4+\frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3}dx = dt$

$\Rightarrow \frac{1}{x^3}dx = -\frac{1}{2}dt$

$\therefore$
$$I = -\frac{1}{2}\int\frac{dt}{t^6} = -\frac{1}{2}\left[\frac{t^{-6+1}}{-6+1}\right] + C$$
$$= \frac{1}{10}\left[\frac{1}{t^5}\right] + C = \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5} + C$$

Q. 52 If $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$, then

(a) $a = \frac{-1}{10}, b = \frac{-2}{5}$ (b) $a = \frac{1}{10}, b = -\frac{2}{5}$

(c) $a = \frac{-1}{10}, b = \frac{2}{5}$ (d) $a = \frac{1}{10}, b = \frac{2}{5}$

Thinking Process

Use method of partial fraction i.e., $\frac{1}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

to solve the above problem.

Sol. *(c)* Given that, $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$

Now, $I = \int \frac{dx}{(x+2)(x^2+1)}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow$ $1 = A(x^2+1) + (Bx+C)(x+2)$

$\Rightarrow$ $1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$

$\Rightarrow$ $1 = (A+B)x^2 + (2B+C)x + A + 2C$

$\Rightarrow$ $A + B = 0, A + 2C = 1, 2B + C = 0$

We have, $A = \frac{1}{5}, B = -\frac{1}{5}$ and $C = \frac{2}{5}$

$\therefore$ $$\int \frac{dx}{(x+2)(x^2+1)} = \frac{1}{5}\int \frac{1}{x+2}dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1}dx$$

$$= \frac{1}{5}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{1+x^2}dx + \frac{1}{5}\int \frac{2}{1+x^2}dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log|1+x^2| + \frac{2}{5}\tan^{-1}x + C$$

$\therefore$ $b = \frac{2}{5}$ and $a = \frac{-1}{10}$

Q. 53 $\int \frac{x^3}{x+1}$ is equal to

(a) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$ (b) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

(c) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$ (d) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

Sol. *(d)* Let $I = \int \frac{x^3}{x+1}dx$

$$= \int \left((x^2 - x + 1) - \frac{1}{(x+1)}\right)dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

Q. 54 $\int \frac{x+\sin x}{1+\cos x}\,dx$ is equal to

(a) $\log|1+\cos x|+C$
(b) $\log|x+\sin x|+C$
(c) $x-\tan\frac{x}{2}+C$
(d) $x\cdot\tan\frac{x}{2}+C$

Sol. *(d)* Let

$$I=\int\frac{x+\sin x}{1+\cos x}dx$$
$$=\int\frac{x}{1+\cos x}dx+\int\frac{\sin x}{1+\cos x}dx$$
$$=\int\frac{x}{2\cos^2 x/2}dx+\int\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}dx$$
$$=\frac{1}{2}\int x\sec^2 x/2\,dx+\int\tan x/2\,dx$$
$$=\frac{1}{2}\left[x\cdot\tan x/2\cdot 2-\int\tan\frac{x}{2}\cdot 2\,dx\right]+\int\tan\frac{x}{2}dx$$
$$=x\cdot\tan\frac{x}{2}+C$$

Q. 55 If $\frac{x^3dx}{\sqrt{1+x^2}}=a(1+x^2)^{3/2}+b\sqrt{1+x^2}+C$, then

(a) $a=\frac{1}{3}, b=1$
(b) $a=\frac{-1}{3}, b=1$
(c) $a=\frac{-1}{3}, b=-1$
(d) $a=\frac{1}{3}, b=-1$

Sol. *(d)* Let $I=\int\frac{x^3}{\sqrt{1+x^2}}dx=a(1+x^2)^{3/2}+b\sqrt{1+x^2}+C$

$\because$ $$I=\int\frac{x^3}{\sqrt{1+x^2}}dx=\int\frac{x^2\cdot x}{\sqrt{1+x^2}}dx$$

Put $1+x^2=t^2$

$\Rightarrow$ $2x\,dx=2t\,dt$

$\therefore$ $$I=\int\frac{t(t^2-1)}{t}dt=\frac{t^3}{3}-t+C$$
$$=\frac{1}{3}(1+x^2)^{3/2}-\sqrt{1+x^2}+C$$

$\therefore$ $a=\frac{1}{3}$ and $b=-1$

Q. 56 $\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

Sol. *(a)* Let

$$I=\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x}=\int_{-\pi/4}^{\pi/4}\frac{dx}{2\cos^2 x}$$
$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx=\int_0^{\pi/4}\sec^2 x\,dx=[\tan x]_0^{\pi/4}=1$$

Q. 57 $\int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$ is equal to

(a) $2\sqrt{2}$
(b) $2(\sqrt{2}+1)$
(c) 2
(d) $2(\sqrt{2}-1)$

Sol. *(d)* Let

$$I = \int_0^{\pi/2} \sqrt{1-\sin 2x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\, dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2}\, dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= 2\sqrt{2} - 2 = 2(\sqrt{2}-1)$$

Q. 58 $\int_0^{\pi/4} \cos x e^{\sin x} dx$ is equal to

(a) $e+1$
(b) $e-1$
(c) e
(d) $-e$

Sol. *(b)* Let

$$I = \int_0^{\pi/2} \cos x\, e^{\sin x} dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

As $x \to 0$, then $t \to 0$

and $x \to \pi/2$, then $t \to 1$

$\therefore$

$$I = \int_0^1 e^t dt = [e^t]_0^1$$

$$= e^1 - e^0 = e - 1$$

Q. 59 $\int \frac{x+3}{(x+4)^2} e^x dx$ is equal to

(a) $e^x\left(\frac{1}{x+4}\right) + C$
(b) $e^{-x}\left(\frac{1}{x+4}\right) + C$
(c) $e^{-x}\left(\frac{1}{x-4}\right) + C$
(d) $e^{2x}\left(\frac{1}{x-4}\right) + C$

Sol. *(a)* Let

$$I = \int \frac{x+3}{(x+4)^2} e^x\, dx$$

$$= \int \frac{e^x}{(x+4)} - \int \frac{e^x}{(x+4)^2} dx$$

$$= \int e^x \left(\frac{1}{(x+4)} - \frac{1}{(x+4)^2}\right) dx$$

$$= e^x \left(\frac{1}{x+4}\right) + C \qquad \left[\because \int e^x \{f(x) + f'(x)\}\, dx = e^x f(x) + C\right]$$

Fillers

Q. 60 If $\int_0^a \frac{1}{1+4x^2}\,dx = \frac{\pi}{8}$, then $a =$ ....

Sol. Let $I = \int_0^a \frac{1}{1+4x^2}\,dx = \frac{\pi}{8}$

Now,
$$\int_0^a \frac{1}{4\left(\frac{1}{4}+x^2\right)}\,dx = \frac{2}{4}[\tan^{-1}2x]_0^a$$
$$= \frac{1}{2}\tan^{-1}2a - 0 = \pi/8$$
$$\frac{1}{2}\tan^{-1}2a = \frac{\pi}{8}$$
$$\Rightarrow \quad \tan^{-1}2a = \pi/4$$
$$\Rightarrow \quad 2a = 1$$
$$\therefore \quad a = \frac{1}{2}$$

Q. 61 $\int \frac{\sin x}{3+4\cos^2 x}\,dx =$ ....

Sol. Let
$$I = \int \frac{\sin x}{3+4\cos^2 x}\,dx$$
Put
$$\cos x = t \Rightarrow -\sin x\,dx = dt$$
$$\therefore \quad I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4}\int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$
$$= -\frac{1}{4}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t}{\sqrt{3}} + C$$
$$= -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$$

Q. 62 The value of $\int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx$ is ....

Sol. We have,
$$f(x) = \int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx$$
$$f(-x) = \int_{-\pi}^{\pi} \sin^3(-2) - \cos^2(-x)\,dx$$
$$= -f(x)$$

Since, $f(x)$ is an odd function.

$$\therefore \quad \int_{-\pi}^{\pi} \sin^3 x\cos^2 x\,dx = 0$$