Unit 3a

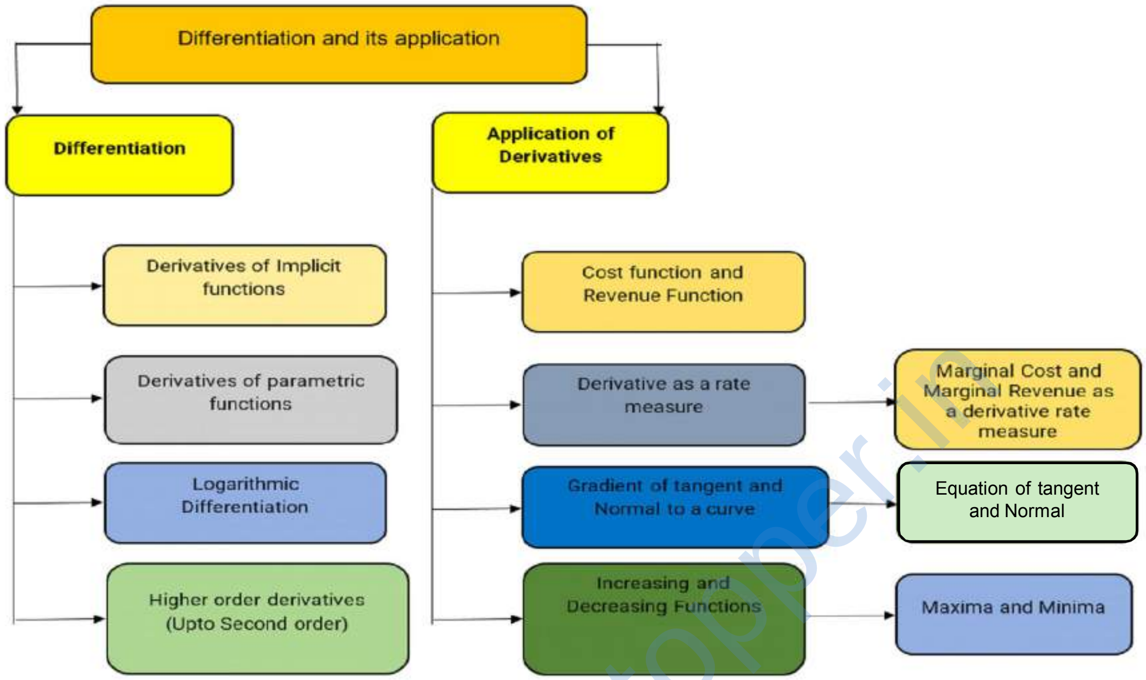
Differentiation and Its Applications

3.0 LEARNING OUTCOMES

After completion of this unit the students will be able to

- Find derivative of implicit functions, parametric functions
- Apply logarithmic differentiation in the functions of the type $[f(x)]^{g(x)}$
- Find second order derivative
- Define Cost function and Revenue function
- Understand derivatives as the rate of change of various quantities
- Define marginal cost and marginal revenue
- Understand the gradient of a tangent and normal to a curve at a given point
- Write the equation of tangent and normal to a curve at a given point
- Recognize whether a function is increasing or decreasing or none
- Determine the condition for an increasing or a decreasing function
- Find the maximum and minimum values of a function at a given point
- Determine turning points (critical points) of the graph of a function
- Find the values of local maxima and Local minima at a point
- Find the absolute maximum and Absolute minimum value of a function on a closed interval
- Apply the derivatives in real life problems

3.1 CONCEPT MAP



3.2 RECALL SOME STANDARD RESULTS OF DIFFERENTIATION

1. Derivatives of standard functions

i. $\frac{d(x^n)}{dx} = nx^{n-1}$

ii. $\frac{d}{dx}(a^x) = a^x \cdot \log a$

iii. $\frac{d}{dx}(e^x) = e^x$

iv. $\frac{d}{dx}(\log x) = \frac{1}{x}$

v. $\frac{d}{dx}(\text{constant}) = 0$

2. Basic rules of differentiation

i. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}(f(x))$, where 'k' is some real number.

ii. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$.

iii. $\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$, also called as product rule.

iv. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \cdot \frac{d}{dx}(g(x))}{(g(x))^2}$, also called as quotient rule.

3.3 DIFFERENTIATION OF IMPLICIT FUNCTIONS

The equation $3x^2 + xy + y = 0$ can be written as $y = \frac{-3x^2}{1+x}$ i.e. 'y' can be expilictly expressed in terms of 'x' only i.e. in terms of independnent variable 'x' . We have been finding derivatives in such cases. As, in above example

$$\frac{dy}{dx} = \frac{-6x \cdot (1+x) + 3x^2}{(1+x)^2} = -\frac{3x^2 + 6x}{(1+x)^2} \text{ or } \frac{-3x(x+2)}{(1+x)^2}$$

Let us consider the equation $x^3 + y^3 - 9xy = 0$. The graph of this equation is the union of the graphs of the functions $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$, as shown below, which are differentiable except at O and A. How do we get the derivative when we can not conveniently find the functions? Here, in this example, 'y' can not be explicitly expressed as a function of 'x'. Such functions are called implicit functions. We treat y as a differentiable function of x and differentiate both sides of the equation with respect to x using the differentiation rules.

The process by which we find derivative is called implicit differentiation. The equation above defines three functions and we find their derivatives implicitly without knowing explicit formula to work with.

The process by which we find $\frac{dy}{dx}$ is called implicit differentiation. The equation above defines three functions f_1, f_2, f_3 and we find their derivatives implicitly without knowing explicit formula to work with.

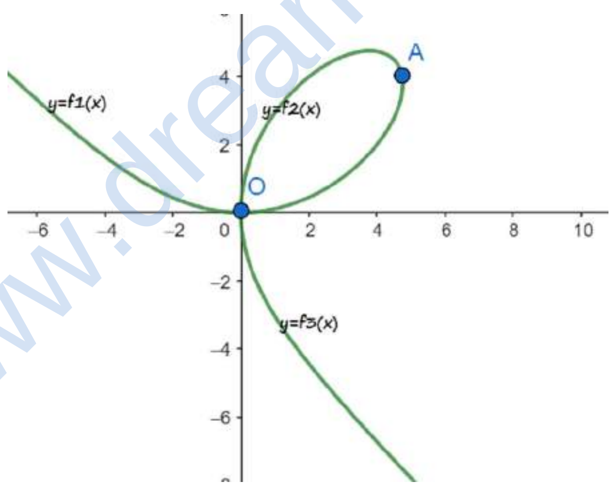


FIG. 1

The process by which we find $\frac{dy}{dx}$ is called implicit differentiation. The equation above defines three functions f_1, f_2, f_3 and we find their derivatives implicitly without knowing explicit formula to work with.

Example 1

If $y = f(x)$ is a real function, then find derivative of the following with respect to 'x'.

i. y^2 ii. $x^3 \cdot y^5$ iii. $\log(xy^2)$ iv. $\frac{x^2}{1+e^{xy}}$

Solution:

i. In order to differentiate y^2 with respect to 'x', we shall have to use chain rule, as y^2 depends on 'y' and 'y' depends on 'x'.

$$\therefore \frac{d}{dx}\left(y^2\right)=\frac{dy^2}{dy}\cdot\frac{dy}{dx}=2y\cdot\frac{dy}{dx}$$

ii. We shall use product rule to find the derivative of x^3y^5 with respect to 'x'

$$\frac{d}{dx}\left(x^3y^5\right)=\frac{dx^3}{dx}\cdot y^5+x^3\cdot\frac{dy^5}{dx}=3x^2y^5+x^3\left(5y^4\right)\frac{dy}{dx}=3x^2y^5+5x^3y^4\frac{dy}{dx}$$

iii. Both chain rule and the product rule will be used to diffrentiate log (xy^2)

$$\frac{d}{dx}\left(\log xy^2\right)=\frac{1}{xy^2}\cdot\frac{d}{dx}\left(xy^2\right)=\frac{1}{xy^2}\left(y^2+2xy\frac{dy}{dx}\right)=\frac{1}{x}+\frac{2}{y}\frac{dy}{dx}$$

But, one can also first simplify $\log(xy^2) = \log x + 2\log y$ and then find the derivative. (Try youself !)

iv. We shall use quotient rule, chain rule and product rule to find the derivative as follows

$$\frac{d}{dx}\left(\frac{x^2}{1+e^{xy}}\right)=\frac{2x\left(1+e^{xy}\right)-x^2\cdot e^{xy}\cdot\left(y+x\frac{dy}{dx}\right)}{\left(1+e^{xy}\right)^2}=\frac{2x+e^{xy}\left(2x-x^2y-x^3\frac{dy}{dx}\right)}{\left(1+e^{xy}\right)^2}$$

Example 2

Find $\frac{dy}{dx}$, when $x^3 + y^3 = xy$.

Solution. Differentiating with respect to 'x',

$$\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right)=\frac{d}{dx}(xy)$$

$$\Rightarrow 3x^2+3y^2\frac{dy}{dx}=y+x\frac{dy}{dx}$$

$$\Rightarrow\left(3y^2-x\right)\frac{dy}{dx}=y-3x^2$$

$$\Rightarrow\frac{dy}{dx}=\frac{y-3x^2}{3y^2-x}$$

Example 3

If $x^m\cdot y^n=(x+y)^{m+n}$, then show that $\frac{dy}{dx}=\frac{y}{x}$

Solution. We first take log of both sides of the equation

$$\log\left(x^m\cdot y^n\right)=\log\left(x+y\right)^{m+n}$$

$$\Rightarrow m\log x+n\log y=(m+n)\log(x+y)$$

Differentiatiing both sides with respect to 'x'.

$$\Rightarrow \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right)=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n(x+y)-(m+n)y}{y(x+y)}\right)=\frac{(m+n)x-m(x+y)}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right)=\frac{nx-my}{(x+y)x} \quad \therefore \frac{dy}{dx}=\frac{y}{x}$$

3.4 DIFFERENTIATION OF PARAMETRIC FUNCTIONS

It is sometimes convenient to represent the relation between the variables x and y by two equations $x = g(t)$, $y = f(t)$. For example, the equations $x = at^2$, $y = 2at$, where $t \in R$, a is a constant >0, are the parametric equations for the curve (rightward parabola) $y^2 = 4ax$, $a > 0$. The variable t is a parameter for the curve.

We can verify that the parametric equations represent the parabola as $y=2at \Rightarrow y^2=4a^2t^2 \Rightarrow$ $y^2=4a\left(at^2\right)\Rightarrow y^2=4ax$, i.e., the points $(at^2, 2at)$ satisfy the equation $y^2 = 4ax$, $a > 0$, where $t \in R$.

The equations $x = at^2$, $y = 2at$ define y as a composite function of x and are said to represent the function in parametric form.

If $x = g(t), y = f(t)$ represent a function in parametric form, then $y = f(\emptyset(x))$, where $t = \emptyset(x)$ is an inverse function with respect to the function $x = g(t)$. Using chain rule and applying $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$, the derivative of 'y' with respect to 'x' can be obtained as

$$\boxed{\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Example 4.

Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$

Solution. Differntiating, with repect to 't', $x=at^2 \Rightarrow \frac{dx}{dt}=2at$, and $y=2at \Rightarrow \frac{dy}{dt}=2a$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2a}{2at}=\frac{1}{t}$$

Example 5

Find $\left.\frac{dy}{dx}\right]_{t=1}$ if $x=\frac{1-t}{1+t}$, $xy = 2t^3$.

Solution. Differntiating with respect to 't', we get

$$x=\frac{1-t}{1+t} \Rightarrow \frac{dx}{dt}=\frac{(-1)(1+t)-(1-t)}{(1+t)^2}=\frac{-2}{(1+t)^2} \text{ and } y=2t^3 \Rightarrow \frac{dy}{dt}=6t^2$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=6t^2\times\frac{(1+t)^2}{-2}=-3t^2(1+t)^2 \Rightarrow \left.\frac{dy}{dx}\right]_{t=1}=-3(2)^2=-12$$

3.5 LOGARITHMIC DIFFERENTIATION

We know that $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ where 'n' is any real number and $\frac{d}{dx}\left(a^x\right)=a^x\log a$, where 'a' is any positive real number, other than 1. Both of these formulae can not be used in the differentiation of the functions of the type $(f(x))^{g(x)}$ like x^x, $\left(\frac{1+x}{1-x}\right)^{x^2}$ etc

$$\therefore \quad \frac{d\left(x^x\right)}{dx}\neq x\cdot x^{x-1} \text{ or } \frac{d\left(x^x\right)}{dx}\neq x^x\cdot\log x$$

In the functions of the type $(f(x))^{g(x)}$ we use logarithm to find the derivative of the function as shown the following examples.

Example 6

Differentiate the following with respect to 'x'

i. $y = x^x$ ii. $y = x^y$

Solution:

i. $y = x^x$, Taking log of both sides, we get $\log y = x\cdot\log x$

Differentiating both sides with respect to 'x', we get

$$\frac{1}{y}\frac{dy}{dx}=1.\log x + x\left(.\frac{1}{x}\right) \Rightarrow \frac{dy}{dx}=y(\log x+1)=x^x(\log x+1)$$

ii. $y = x^y$, Taking log of both sides, we get $\log y = y\cdot\log x$

Differentiating both sides with respect to 'x', we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{dy}{dx}\log x+\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}-\log x\right)=\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}\cdot\frac{y}{(1-y\log x)}=\frac{y^2}{x(1-y\log x)}$$

Example 7.

If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

Solution. Let $u = x^y$, $v = y^x$ $\quad \therefore x^y + y^x = a^b \Rightarrow u + v = a^b$, where $u = x^y$; $v = y^x$ Differntiating both sides of the equation $u + v = a^b$ with respect to 'x', we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \text{— (i)}$$

$$u = x^y \Rightarrow \log u = y \log x \Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log x + \frac{y}{x} \Rightarrow \frac{du}{dx} = x^y\left(\frac{dy}{dx}\log x + \frac{y}{x}\right) \quad \text{—(ii)}$$

$$v = y^x \Rightarrow \log v = x \log y \Rightarrow \frac{1}{v}\frac{dv}{dx} = 1 \cdot \log y + \frac{x}{y}\frac{dy}{dx} \Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \quad \text{—(iii)}$$

Substituting $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (ii) and (iii) in (i), we get

$$\Rightarrow x^y\left(\frac{dy}{dx}\log x + \frac{y}{x}\right) + y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx}\left(x^y \log x + xy^{x-1}\right) = -\left(x^{y-1}y + y^x \log y\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{y-1}y + y^x \log y}{x^y \log x + xy^{x-1}}$$

3.6 SECOND AND HIGHER ORDER DERIVATIVES

Let y = f (x) be a differentiable function of 'x', then

i. Derivative of f′(x) with respect to 'x' = $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = f''(x)$, is the second order derivative of 'y' or f(x).

ii. Derivative of f″(x) with respect to 'x' = $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y''' = f'''(x)$, is the third order derivative of 'y' or f(x).

Similarly, we can find the other higher order derivatives of $y = f(x)$.

Note: *For derivatives higher than three we do not use primes, instead we write the, n^{th} order derivatives as*

$$\frac{d^ny}{dx^n} = y_n = f^n(x)$$

Example 8

Find $\frac{d^2y}{dx^2}$ for the following functions

i. $y = x$ ii. $y = \log x$ iii. $y = \sqrt{x^2 - 1}$

Solution:

i. $y = x$, differentiate with respect to 'x', $\frac{dy}{dx} = 1$,

Differentiating again with respect to 'x', we get $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$ i.e. $\frac{d^2y}{dx^2} = 0$

ii. $y = \log x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$, Differentiating again with respect to 'x', we get

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

iii. $y = \sqrt{x^2 - 1} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x^2 - 1\right)^{1/2} = \frac{1}{2}\left(x^2 - 1\right)^{-1/2} \cdot \frac{d}{dx}\left(x^2 - 1\right) = \frac{2x}{2\sqrt{x^2 - 1}} = \frac{x}{\sqrt{x^2 - 1}}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1 \cdot \sqrt{x^2 - 1} - x \cdot \frac{x}{\sqrt{x^2 - 1}}}{x^2 - 1} = \frac{x^2 - 1 - x^2}{\left(x^2 - 1\right)\sqrt{x^2 - 1}} = \frac{-1}{\left(x^2 - 1\right)^{3/2}}$$

Example 9

If $x = \frac{t^2}{1 + t}$ and $y = \frac{t}{1 + t}$, find y_2.

Solution. $x = \frac{t^2}{1 + t} \Rightarrow \frac{dx}{dt} = \frac{2t(1 + t) - t^2}{(1 + t)^2} = \frac{2t + t^2}{(1 + t)^2}$

$$y = \frac{t}{1 + t} \Rightarrow \frac{dy}{dt} = \frac{1 \cdot (1 + t) - t \cdot 1}{(1 + t)^2} = \frac{1}{(1 + t)^2}$$

$$y_1 = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{(1 + t)^2} \cdot \frac{(1 + t)^2}{2t + t^2} = \frac{1}{2t + t^2}$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dx}\left(\frac{1}{2t + t^2}\right) = \frac{d}{dt}\left(\frac{1}{2t + t^2}\right) \cdot \frac{dt}{dx} = -\frac{2 + 2t}{\left(2t + t^2\right)^2} \cdot \frac{(1 + t)^2}{2t + t^2} = \frac{-2(1 + t)^3}{t^3(t + 2)^3}$$

Exercise 3.1

1. Find $\frac{dy}{dx}$ from the following

   i. $x^3 + y^3 = 3axy$
   ii. $e^{xy} - axy = a$
   iii. $3x^3 - 5x^2y + 2xy^2 + 4y^3 = 0$
   iv. $x^{1/3} + y^{1/3} = a^{2/3}$
   v. $x = y\log(xy)$

2. Find $\frac{dy}{dx}$ from the following parametric equations

   i. $x = at$, $y = \frac{a}{t}$
   ii. $x = t \cdot \log t$, $y = \frac{\log t}{t}$
   iii. $x = \frac{a(1-t^2)}{1+t^2}$, $y = \frac{2bt}{1+t^2}$

3. Find $\frac{dy}{dx}$ from the following equations

   i. $x^y = y^x$
   ii. $x^y = e^{x-y}$
   iii. $(x-y)e^{x/(x-y)} = 7$
   iv. $y = x^{\log x}$

4. Find $\frac{d^2y}{dx^2}$ from the following

   (i) $y = x\log x$
   (ii) $y = x^2e^x$
   (iii) $y = \log(\log x)$
   (iv) $y = 3e^{2x} + 2e^{3x}$

5. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, show that $(1+x^2)\frac{dy}{dx} + 1 = 0$.

6. If $y^{1/m} + y^{-1/m} = 2x$ then prove that $(x^2-1)y_1^2 = m^2y^2$.

7. If $y = \log\left(x + \sqrt{a^2 + x^2}\right)$, show that $(a^2 + x^2)y_2 + xy_1 = 0$.

8. If $y = \left(x + \sqrt{x^2 + 1}\right)^p$, prove that $(x^2 + 1)y_2 + xy_1 - p^2y = 0$.

3.7 COST AND REVENUE FUNCTION

Any manufacturing company has to deal with two types of costs, the one which varies with the cost of raw material, direct labour cost, packaging etc. is the variable cost. The variable cost is dependent on production output. As the production output increases (decreases) the variable cost will also increase (decrease). The other one is the fixed cost, fixed costs are the expenses that remain the same irrespective of production output. Whether a firm makes sales or not, it must pay its fixed costs.

Cost Function: If V(x) is the variable cost of producing 'x' units and 'k' the fixed cost then, the total cost C(x) is given by

$$\boxed{C(x) = V(x) + k}$$

Revenue Function: if R is the total revenue a company receives by selling 'x' units at price 'p' per unit produced by it then the revenue function is given by

$$\boxed{R(x) = p.x}$$

Note: Generally, it is assumed that a company sells the number of units it produces.

Example 10

A company produces 'x' units in a year and the variable cost is $V(x) = x^2 - 2x$. Also, the company spends a fixed cost of Rs15,000 on commissions and rent, then

(i) Find the total cost function C(x)

(ii) If 'p' the price per unit is given by p = 5–x then find its revenue function.

Solution:

(i) The variable cost, $V(x) = x^2 - 2x$

$\therefore$ The total cost function is $C(x) = V(x) + 15000 = x^2 - 2x + 15000$

(ii) The revenue function is given by

R = px

$$\Rightarrow R = (5 - x) \cdot x = 5x - x^2$$

3.8 DERIVATIVE AS RATE OF CHANGE OF QUANTITIES

In science, business and economics there are variables one depending on the other such as distance and time, cost and production, revenue and production, price and demand etc. In all these examples we are interested in the rate at which one variable changes with respect to other to know the micro details of relation between these variables.

In class XI we have discussed that if y = f(x) is a real function then,

$$\boxed{\frac{dy}{dx} = \text{Rate (or instantaneous rate) of change of 'y' with respect to 'x'}}$$

Example 11

A boy is blowing air into a spherical balloon and thus the radius r of the balloon is changing, then find the rate of change of surface area of the balloon with respect to the radius r. Also find the rate of change of surface area when r = 2cm.

Solution. Let Area of balloon be A at the radius r then, $A = 4\pi r^2$

Rate of change of surface area of balloon with respect to the radius is

$$\frac{dA}{dr} = 8\pi r \text{ and } \left.\frac{dA}{dr}\right]_{r=2} = 8\pi r\Big]_{r=2} = 8\pi(2) = 16\pi \text{cm}^2/\text{cm}.$$

3.8.1 RELATED RATES

In related rate problems the quantities change with respect to time, you may be given rate of change of one variable with respect to time and rate of change of other variable has to be found with respect to time.

> Recall: If $y = f(x)$, then using chain rule, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot \frac{dx}{dt}$

Example 12

Find the rate of change of volume of a sphere with respect to its surface area when the radius is 5 m.

Solution: For the radius r, the volume V and the surface area S of the sphere is given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$.

As V and S both are functions of radius r, we will use chain rule to find derivative of V with respect to S

Since, $$\frac{dv}{dr} = \frac{4}{3}\left(3\pi r^2\right) = 4\pi r^2, \frac{dS}{dr} = 8\pi r$$

$$\therefore \frac{dV}{dS} = \frac{\frac{dV}{dr}}{\frac{dS}{dr}} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2} \Rightarrow \left.\frac{dV}{dS}\right]_{r=5} = \frac{5}{2} m^3/m^2$$

Example 13

A cylindrical vessel of radius 0.5 m is filled with oil at the rate of $0.25\pi m^3/min$. Find the rate at which the surface of the oil is rising.

Solution: The rate at which the surface area rises is the rate of change of height of oil in the vessel with respect to time. Let r be the radius, h be the height and V the volume of the oil at time t.

$$\text{Then } V = \pi r^2 h = \frac{\pi}{4}h \quad \text{as} \quad r = 0.5 = \frac{1}{2}$$

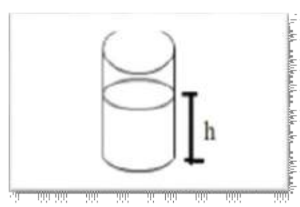


FIG. 2

As we are given rate of change in volume with respect to time t, therefore differentiating V with respect to t, we get

$$\frac{dV}{dt} = \frac{\pi}{4}\frac{dh}{dt} \Rightarrow 0.25\pi = \frac{\pi}{4}\frac{dh}{dt},$$

as we are given $\frac{dV}{dt} = 0.25\pi m^3$

$$\Rightarrow \frac{dh}{dt} = 1 m/\min ute$$

Example 14

A boy of height 1m is walking towards a lamp post of height 5 meters at the rate of 0.5 m/sec. Then find the rate at which the length of the shadow of the boy is decreasing.

Solution: Let RT = y, be the length of the shadow of the boy when he is x m away from the lamp post at time t.

$\therefore \frac{dx}{dt} = -0.5 m/sec$, as the boy is moving towards lamp post the distance between the boy and the lamp post is decreasing and hence the rate of change of the distance will be negative.

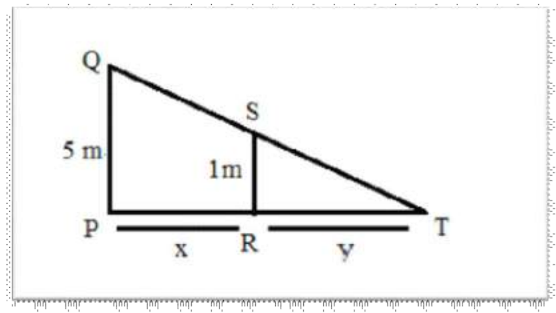


FIG. 3

$$\frac{PQ}{RS} = \frac{PT}{RT} \ (\because \Delta PQT \sim \Delta RST)$$

$\Rightarrow \frac{5}{1} = \frac{x+y}{y} \Rightarrow 4y = x$, differentiating with respect to t

$\Rightarrow 4\frac{dy}{dt} = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{-0.5}{4} = \frac{-1}{8} m/sec$, hence the shadow of the boy is decreasing at the rate of $\frac{1}{8} m/sec$.

3.8.2 MARGINAL COST AND MARGINAL REVENUE

Marginal cost and marginal revenue are the instantaneous rate of change of cost and revenue with respect to output i.e. rate of change of C(x) (or C), the cost function and R(x) (or R), the revenue function, with respect to production output 'x'.

Therefore, the Marginal cost (MC) and the Marginal revenue (MR) are given by

$$\boxed{MC \text{ (Marginal cost)} = C'(x) = \frac{dC}{dx}}$$

$$\boxed{MR \text{ (Marginal revenue)} = R'(x) = \frac{dR}{dx}}$$

Note: Marginal cost is an important factor in economics theory because a company that needs to maximize its profits will produce up to the point where marginal cost (MC) equals marginal revenue (MR). Beyond that point, the cost of producing an additional unit will exceed the revenue generated

Example 15

A toy manufacturing firm assesses its variable cost to be 'x' times the sum of 30 and 'x', where 'x' is the number of toys produced, also the cost incurred on storage is ₹ 1500. Find the total cost function and the marginal cost when 20 toys are produced.

Solution: The total cost function C(x) is given by,

$$C(x) = x(x+30) + 1500 = x^2 + 30x + 1500$$

The marginal cost MC is given by,

$$MC = \frac{dC}{dx} = 2x + 30$$

Marginal cost of producing 20 toys is $MC(20) = \left.\frac{dC}{dx}\right]_{x=20} = 2(20) + 30 = 70$

$\therefore$ The marginal cost of producing 20 toys is ₹ 70.

Example 16

The price per unit of a commodity produced by a company is given by $p = 30 - 2x$ and 'x' is the quantity demanded. Find the revenue function R, the marginal revenue when 5 commodities are in demand (or produced).

Marginal revenue ₹10 means revenue is increased by ₹10 if an additional unit beyond 5 is in demand or sold

Solution. The revenue function R (or R(x)) is given by,

$$R = px = (30 - 2x)\,x = 30x - 2x^2$$

$\therefore$ The marginal revenue $\Rightarrow MR = \frac{dR}{dx} = 30 - 4x$

The marginal revenue of producing 5 commodities is,

$$\left.\frac{dR}{dx}\right]_{x=5} = 30 - 4(5) = 10$$

$\therefore$ The marginal revenue when 5 commodities are in demand is ₹ 10.

Exercise 3.2

1. Find the rate of change of circumference of a circle with respect to the radius r.
2. Find the rate of change of lateral surface area of a cube with respect to side x, when x = 4cm.
3. If the rate of change of volume of a sphere is equal to the rate of change of its radius, then find its radius. Also find its surface area.
4. The volume of a cone changes at the rate 40 cm^3/sec. If height of the cone is always equal to its diameter, then find the rate of change of radius when its circular base area is $1m^2$.
5. For what values of x is the rate of increase of total cost function $C(x) = x^3 - 5x + 5x + 8$ is twice the rate of increase of x?
6. The radius of the base of a cone is increasing at the rate of 3cm/minute and the altitude is decreasing at the rate of 4cm/minute. Find the rate of change of lateral surface area when the radius is 7cm and the altitude 24 cm.
7. A ladder 10 meters long rests with one end against a vertical wall, the other on the floor. The lower end moves away from the wall at the rate of 2 meters / minute. Find the rate at which the upper end falls when its base is 6 meters away from the wall.
8. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3 /min. When the thickness of ice is 5 cm, find the rate at which the thickness of ice decreases.
9. A stationery company manufactures 'x' units of pen in a given time, if the cost of raw material is square of the pens produced, cost of transportation is twice the number of pens produced and the property tax costs ₹ 5000. Then,
   (i) Find the cost function C(x).

(ii) Find the cost of producing 21st pen.

(iii) The marginal cost of producing 50 pens.

10. A firm knows that the price per unit 'p' for one of its product is linear. It also knows that it can sell 1400 units when the price is ₹ 4 per unit, and it can sell 1800 units at a price of ₹ 2 per unit. Find the price per unit if 'x' units are sold (or demanded). Also find the revenue function and the marginal revenue function.

3.9 SLOPE (OR GRADIENT) OF TANGENT AND NORMAL

Consider the graph of a curve y = f(x) as shown in the Fig 4. Let A (x, f(x)) be a point on the graph and B(x + Δx, f(x + Δx)) be a neighbouring point on the curve. Then,

The slope of the line (secant) AB = $\tan\theta = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$,

where θ is the angle of inclination of the line AB.

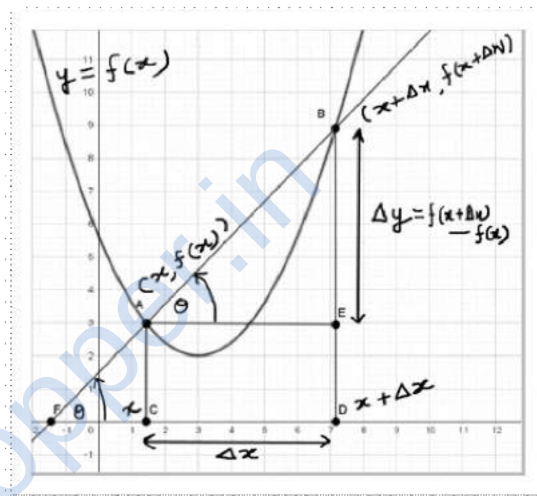


FIG. 4

The limiting position of the secant AB, when the point B moves along the curve and tends to coincide with the point A (i.e., when $\Delta x \to 0$), if the limiting position exists, is the tangent to the curve at the point A. Hence,

The slope of the tangent (non-vertical) to the curve at the point A is given by $\tan\varphi = \lim_{\theta\to\varphi} \tan\theta = \lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \dfrac{dy}{dx}$, where φ is the angle of inclination of the tangent line at the point A.

Please note that in the figure Δx chosen is positive, whereas Δx could be negative and Δx has to approach 0 from both sides.

Therefore, we conclude that,

Slope (or gradient) of a non-vertical tangent at a point $A(x_0, y_0)$ = $\left.\dfrac{dy}{dx}\right]_{A(x_0, y_0)}$

Normal to a curve at a point:

If y = f(x) is a real function then normal to the curve at a point $A(x_0, y_0)$, i.e., at $((x_0, f(x_0))$ on it is a line perpendicular to the tangent at the given point on the curve. As shown in the adjoining figure 5.

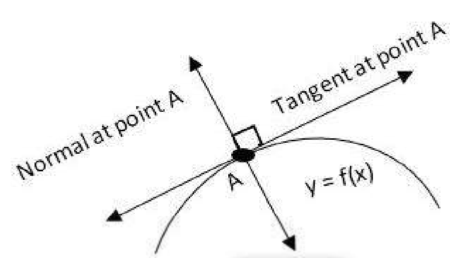


FIG. 5

Slope of a normal line to the curve at a point $A(x_0, y_0)$ = $\dfrac{-1}{\left.\dfrac{dy}{dx}\right]_{A(x_0,y_0)}}$, provided the normal is not perpendicular to the x-axis.

Example 17

Find slope of the tangent and normal at a point (2,6) to the curve $y = x^3 - x$

Solution: $y = x^3 - x \Rightarrow \dfrac{dy}{dx} = 3x^2 - 1$

$\therefore$ Slope of the tangent at (2,6) is given by

$$\left.\frac{dy}{dx}\right]_{(2,6)} = \left.\left(3x^2 - 1\right)\right]_{(2,6)} = 3(2)^2 - 1 = 11$$

And, slope of normal to the curve = $\dfrac{-1}{\left.\dfrac{dy}{dx}\right]_{(2,6)}} = \dfrac{-1}{11}$

Positive or negative slopes of tangents to cost (revenue) functions signifies the rising or declining cost (revenue) accordingly.

3.9.1 EQUATION OF TANGENT AND NORMAL TO A CURVE

Recall that equation of a line passing through (x_1, y_1) and having slope m is given by $(y - y_1) = m(x - x_1)$. Using this we can find equation of tangent and normal to a curve at a given point on it.

Let y = f(x) be a real function and $A(x_0, y_0)$ be a point on it then,

Equation of tangent to the curve at the point $A(x_0, y_0)$: $(y - y_0) = \left.\dfrac{dy}{dx}\right]_{(x_0,y_0)} (x - x_0)$

Equation of normal to the curve at the point $A(x_0, y_0)$: $(y - y_0) = \dfrac{-1}{\left.\dfrac{dy}{dx}\right]_{(x_0,y_0)}} (x - x_0)$

Example 18

Find the equation of the tangent and normal to the curve $x^{2/3} + y^{2/3} = 2$ at (1,1).

Solution. Differentiating with respect to 'x', we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

Equation of the tangent is

$$(y-1)=\left.\frac{dy}{dx}\right]_{(1,1)}(x-1)\Rightarrow(y-1)=-1(x-1)\Rightarrow x+y-2=0$$

Equation of normal is

$$(y-1)=\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,1)}}(x-1)\Rightarrow(y-1)=1(x-1)\Rightarrow y-x=0$$

The tangents help to geometrically measure price elasticity of demand curve

Example 19

Find the equation of the tangent and normal to the curve $f(x) = e^x + x^2 + 1$ at the point (0,2) on it.

Solution. $f'(x) = e^x + 2x$ then slope of tangent at the point (0,2) is $f'(0) = 1$ and slope of normal is $=\frac{-1}{f'(0)}=-1$

Therefore, equation of tangent is $(y-2) = 1(x-0) \Rightarrow x - y + 2 = 0$

And equation of normal is $(y - 2) = -1(x-0) \Rightarrow x + y - 2 = 0$

Example 20

Find the equation of the tangent to the curve $x^2 + 3y - 3 = 0$, which is parallel to the line $y = 4x - 5$.

Solution. Differentiating the given equation we get,

$$2x+3\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{-2x}{3}$$

Slope of the tangent to the curve = slope of the line $y = 4x - 5$

$$\frac{dy}{dx}=4\Rightarrow\frac{-2x}{3}=4\Rightarrow x=-6\Rightarrow(-6)^2+3y-3=0\Rightarrow y=\frac{-33}{3}=-11$$

Therefore, the point on the curve is (–6, –11) and equation of tangent is

$$(y + 11) = 4(x + 6) \Rightarrow 4x - y + 13 = 0$$

Example 21

Find the equation of the tangent to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

Solution. Putting y = 0 in the equation of the curve to get the points where it cuts the x-axis.

$$(x^3 - 1)(x - 2) = 0 \Rightarrow = 1, 2$$

Thus, the points of intersection of curve with x-axis are (1,0) and (2,0)

Differentiating the equation with respect to ‘x’,

$$\frac{dy}{dx}=3x^2(x-2)+(x^3-1)\Rightarrow\frac{dy}{dx}=-3\ at\ (1,0)\ and\ \frac{dy}{dx}=7\ at\ (2,0)$$

∴ The equations of the tangents are

$$y-0=\left(\frac{dy}{dx}\right)_{(1,0)}(x-1)\Rightarrow 3x+y-3=0$$

$$y-0=\left(\frac{dy}{dx}\right)_{(2,0)}(x-2)\Rightarrow 7x-y-14=0$$

Example 22

Find the equation of the tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3},0\right)$.

Solution. Note: The given point does not lie on the curve

Let us assume the tangent touches the curve at the point (h,k)

$$\therefore 3h^2-k^2=8 \qquad \text{——— (i)}$$

Differentiating the equation of the curve with respect to 'x', we get

$$6x-2y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{3x}{y}\Rightarrow\left.\frac{dy}{dx}\right]_{(h,k)}=\frac{3h}{k}$$

$\therefore$ Equation of the tangent is: $\left(y-\frac{4}{3}\right)=\frac{3h}{k}(x)$, also it passes through (h, k)

$$\Rightarrow\left(k-\frac{4}{3}\right)=\frac{3h}{k}\cdot h\Rightarrow 9h^2-3k^2+4k=0 \qquad \text{——— (ii)}$$

Replacing h^2 from (i) in (ii), we get,

$$9\left(\frac{k^2+8}{3}\right)-3k^2+4k=0\Rightarrow k=-6\Rightarrow h=\pm\sqrt{\frac{44}{3}}$$

$\therefore$ The equation of the tangents is $\left(y-\frac{4}{3}\right)=\pm\frac{3}{6}\times\sqrt{\frac{44}{3}}x\Rightarrow\pm\sqrt{33}x-3y+4=0$

Example 23

Prove that $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ for all $n \in N$, at the point (a,b).

Solution. $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2\Rightarrow b^nx^n+a^ny^n=2a^nb^n$ Differentiating with respect to 'x', we get

$$nb^nx^{n-1}+na^ny^{n-1}\cdot\frac{dy}{dx}=0\Rightarrow\left.\frac{dy}{dx}\right]_{(a,b)}=-\left.\frac{b^nx^{n-1}}{a^ny^{n-1}}\right]_{(a,b)}=-\frac{b}{a}$$

$\therefore$ Equation of the tangent at (a,b) is $(y-b)=-\frac{b}{a}(x-a)\Rightarrow\frac{x}{a}+\frac{y}{b}=2$

Example 24

Find the points on the curve $y = 4x^3 - 2x^5$, at which the tangent passes through the origin.

Solution. Let (h,k) be any such point on the curve

$$\therefore k = 4h^3 - 2h^5 \qquad \text{- (i)}$$

Differentiating the given equation with respect to 'x', we get

$$\frac{dy}{dx} = 12x^2 - 10x^4 \Rightarrow \left.\frac{dy}{dx}\right]_{(h,k)} = \left.12x^2 - 10x^4\right]_{(h,k)} = 12h^2 - 10h^4$$

The equation of the tangent through origin is,

$$(y - 0) = (12h^2 - 10h^4)(x - 0) \Rightarrow (k - 0) = (12h^2 - 10h^4)(h - 0)$$

As (h,k) lies on the tangent,

$$\Rightarrow k = 12h^3 - 10h^5 \qquad \text{(ii)}$$

Solving (i) and (ii),

$$4h^3 - 2h^5 = 12h^3 - 10h^5 \Rightarrow 8h^5 - 8h^3 = 0 \Rightarrow 8h^3(h^2 - 1) = 0 \Rightarrow h = 0, h = \pm 1$$

$\therefore$ The points on the curve are (0,0), (1,2) and (–1, –2).

Exercise 3.3

1. Find the slopes of the tangents and normal to the curves at the indicated points.
   i. $y = x^3 - x$ at $x = 1$.
   ii. $y = 3x^2 - 6x$ at $x = 2$.
   iii. $y = \frac{x-1}{x-2}, x \neq 2$ at $x = 10$.
   iv. $x^{2/3} + y^{2/3} = 2$ at (1,1).
2. Find the equations of the tangent and normal to the curves at the indicated points.
   i. $y = x^3 - 3x + 5$ at the point (2,7).
   ii. $x = at^2$, $y = 2at$ at $t = 2$
3. Find the equations of the tangents to the curve at points where the tangents to the curve $y = 2x^3 - 15x^2 + 36x - 21$ are parallel to x-axis.
4. Find the equation of the tangents to the curve $y = x^3 + 2x - 4$, which is perpendicular to the line $x + 14y + 3 = 0$.
5. Find the equation of the tangent and the normal to the curve $y = \frac{x-7}{x^2 - 5x + 6}$ at the point, where it cuts x-axis.
6. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1,2).
7. For the curve $y = x^2 + 3x + 4$, find all points at which the tangent passes through the origin.
8. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it crosses the y-axis.
9. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.
10. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

3.10 INCREASING AND DECREASING FUNCTIONS (MONOTONIC FUNCTIONS)

Increasing function: A real function y = f (x) is said to be an increasing function (abbreviated as↑) on an open interval (a,b) if the value of 'f' increases as 'x' increases or vice – versa. i.e., the graph of f(x) rises from left to right see Figure 6.

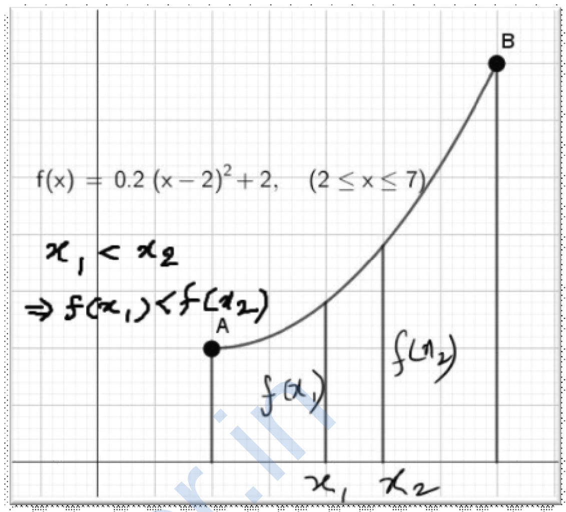


Fig. 6

Here, the open interval (a, b) includes (i) (a, b), where a, b are fixed real numbers such that a < b, (ii) (a, ∞), where a is a fixed real number, (iii) (-∞, b), where b is a fixed real number, (iv) (-∞, ∞) = R

Following is an analytical condition for a function to be increasing on (a, b)

$$\boxed{x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad \forall x_1, x_2 \in (a,b)}$$

Or

$$\boxed{x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \quad \forall x_1, x_2 \in (a,b)}$$

Decreasing function: A real function y = f(x) is said to be a decreasing function (abbreviated as↓) on an open interval (a,b) if the value of 'f' decreases as 'x' increases or vice – versa, i.e. the graph of f(x) falls from left to right see Figure 7

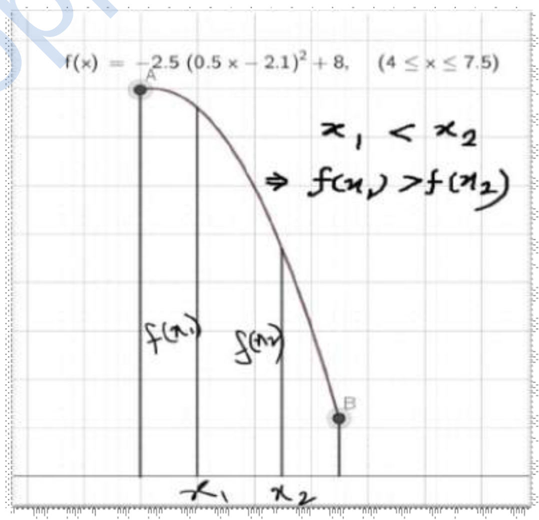


Fig., 7

Following is an analytical condition for a function to be decreasing on (a, b)

$$\boxed{x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \quad \forall x_1, x_2 \in (a,b)}$$

Or

$$\boxed{x_1 > x_2 \Rightarrow f(x_1) < f(x_2) \quad \forall x_1, x_2 \in (a,b)}$$

Note: 1) A function which is either increasing or decreasing on its domain (an open interval) is termed as a monotonic function

2) You may note that a function may be defined to be increasing/decreasing in any of the following intervals represented as I:

(a, b), (a, ∞), (– ∞, b), (– ∞, ∞), [a, b], [a, ∞), (– ∞, b]

Not all cases have been included here in this topic.

Example 25.

Show that $f(x) = x^2$ is an

(i) Increasing function on $(0, \infty)$

(ii) Decreasing function on $(-\infty, 0)$

Solution:

(i) Let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$

$$\Rightarrow x_1 \cdot x_1 < x_1 \cdot x_2 \Rightarrow {x_1}^2 < x_1 \cdot x_2$$

Also, $\Rightarrow x_1 \cdot x_2 < x_2 \cdot x_2 \Rightarrow x_1 \cdot x_2 < {x_2}^2$

From above we conclude that $x_1^2 < x_2^2 \Rightarrow f(x_1) < f(x_2)$

hence, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

i.e., f(x) is an increasing function over $(0, \infty)$

(ii) Let $x_1, x_2 \in (-\infty, 0)$ such that $x_1 < x_2$

$\Rightarrow x_1 \cdot x_1 > x_1 \cdot x_2 \Rightarrow {x_1}^2 > x_1 \cdot x_2$ $(\because x_1, x_2$ are negative numbers$)$

Also, $\Rightarrow x_1 \cdot x_2 > x_2 \cdot x_2 \Rightarrow x_1 \cdot x_2 > {x_2}^2$

From above we conclude that, ${x_1}^2 > {x_2}^2 \Rightarrow f(x_1) > f(x_2)$, hence, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

i.e., f(x) is a decreasing function over $(-\infty, 0)$

The term upward sloping is also sometimes used for an increasing function

The term downward sloping is also sometimes used for a decreasing function

Note: f(x) is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ therefore we say that f(x) is neither increasing nor decreasing on $(-\infty, \infty)$.

3.10.1 DERIVATIVE CONDITIONS FOR A MONOTONIC (INCREASING OR DECREASING) FUNCTION.

As discussed in section 3.4 that the derivative of a function at a given point on it is slope of the tangent to the curve at that point. It is an interesting fact to note that slope of tangents also determines the monotonicity of functions. It can be observed from the following figures.

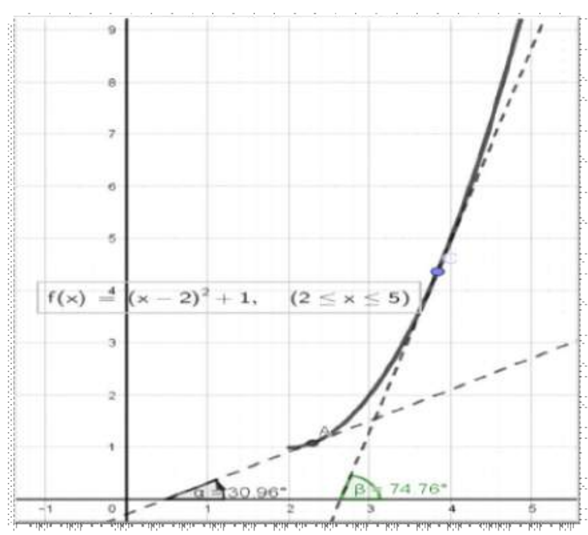


Fig. 8

In Fig.8 one can observe that the function is increasing in its domain and the tangents drawn to the curve are making acute angles of inclination at any point within the domain of the curve. You may encounter the graph of a certain increasing function in an open interval, where the angle of inclination of the tangent at a point is 0° or 90°. Please, check the graph of the functions $f(x) = x^3, g(x) = x^{\frac{1}{3}}$

DERIVATIVE TEST FOR INCREASING FUNCTIONS

A real function y = f(x) is an increasing function on (a,b) if

$$\boxed{f'(x) > 0, \forall x \in (a,b)}$$

Please note that the condition mentioned above is sufficient for a function f to be an increasing function in (a, b), but not necessary. The functions $f(x) = x^3, g(x) = x^{\frac{1}{3}}$ are examples of increasing functions over R, but $f'(0) = 0, g'(0)$ is not defined.

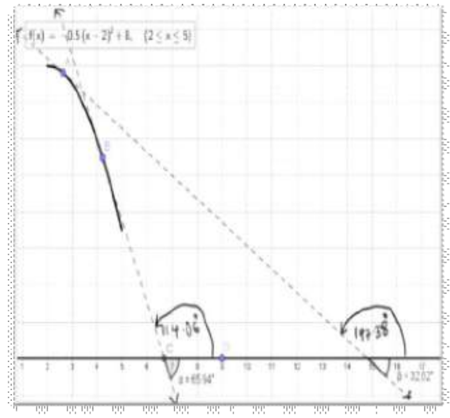


Fig. 9

In Fig.9, one can observe that the function is decreasing in its domain and the tangents drawn to the curve are making obtuse angles of inclination at any point within the domain of the curve. But, you may encounter some exceptions similar to what was observed above in case of an increasing function.

DERIVATIVE TEST FOR DECREASING FUNCTIONS

A real function y = f(x) is a decreasing function on (a,b) if

$$\boxed{f'(x) < 0, \forall x \in (a,b)}$$

The condition is sufficient for a function f to be a decreasing function in (a, b), but not necessary.

CRITICAL POINTS

Definition: An interior point of the domain of a function f where f′ is zero or undefined is a critical point of f.

Hence, the critical points are essentially interior points of the domain of the function f together with a second condition as mentioned above.

The word critical probably has been used because at this point an abrupt change in the behaviour of the graph of the function is noted.

i. The curve $f(x) = (x - 1)^2 + 2$, in the Fig.10, is turning at the point 'A' on it and tangent at 'A' is parallel to x-axis, i.e., slope of tangent at 'A' is 0, i.e., $f'(1) = 0$. The curve takes a smooth turn at the point (1, 2).

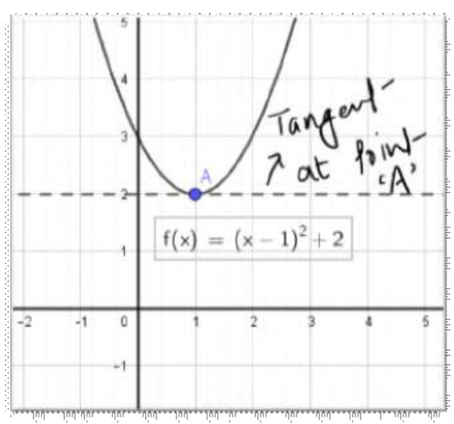


Fig. 10

1 is a critical point of the function.

ii. The graph of the function f (Fig. 11) is turning at the points B and C. f′(-2) = 0, f′(3) = 0.

–2 and 3 are the critical points of the function f. The curve takes a smooth turn at the points B and C.

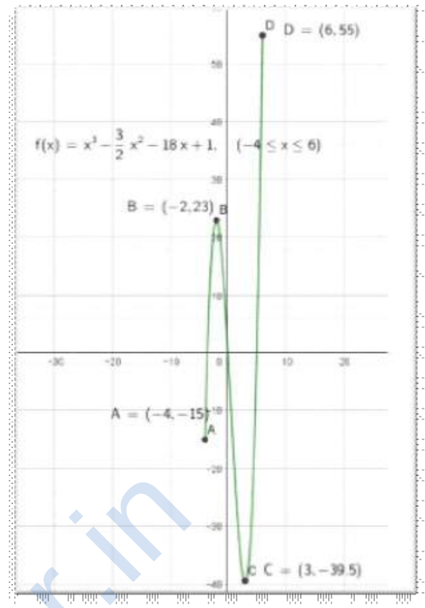


Fig. 11

iii. f′(0) does not exist. 0 is a critical point. Note that the graph of the function (Fig. 12) takes a sharp turn at (0, 0). There is a corner at the point (0, 0).

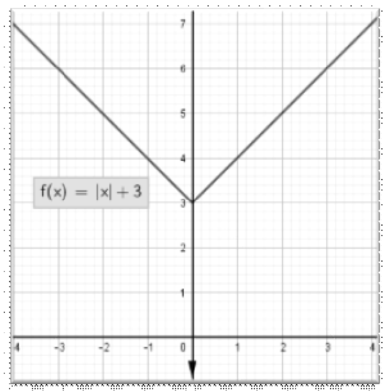


Fig. 12

iv. $g'(x) = \frac{1}{3x^{\frac{2}{3}}}$. g′(0) is not defined. 0 is a critical point.

Note that there is a vertical tangent at the point (0, 0), which is the y-axis itself and 0 is a point of inflexion. A point where the graph of a function (Fig. 13) has a tangent line and where the concavity changes is called a point of inflexion.

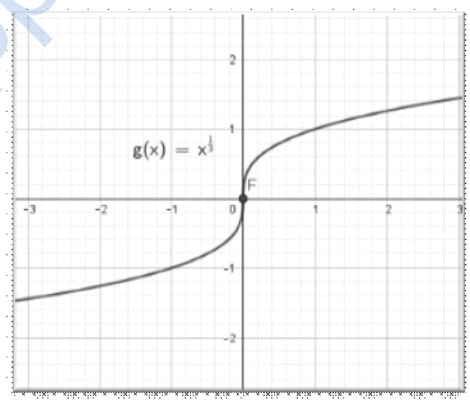


Fig. 13

v. h′(0)=0. 0 is a critical point. The tangent line at the point (0, 0) is the x-axis itself. 0 is a point of inflexion. (Fig. 14)

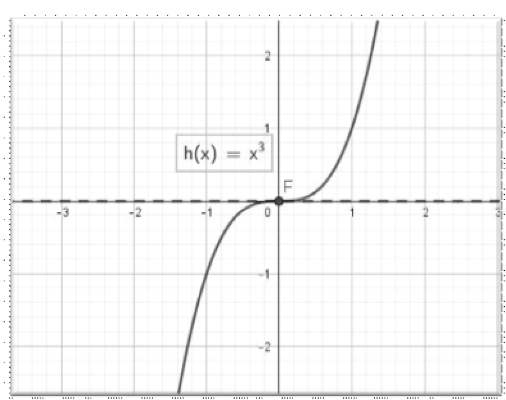


Fig. 14

vi. The following is the graph of the function $f(x)=x^{2/3}$

$f'(0)$ is not defined. 0 is a critical point. Note that at (0, 0), there is a cusp in the graph. (Fig. 15)

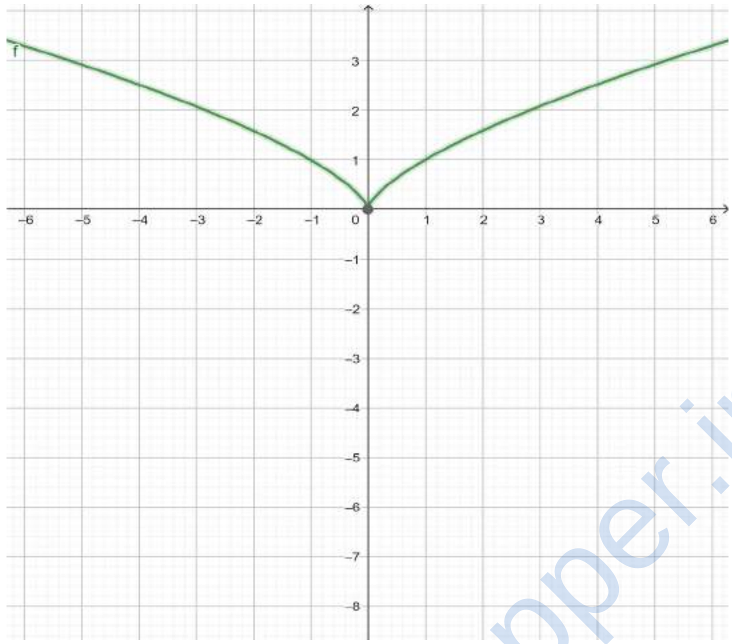


Fig. 15

vii. 0 is a point of discontinuity. The derivative of the function at 0 is not defined. 0 is a critical point. (Fig. 16)

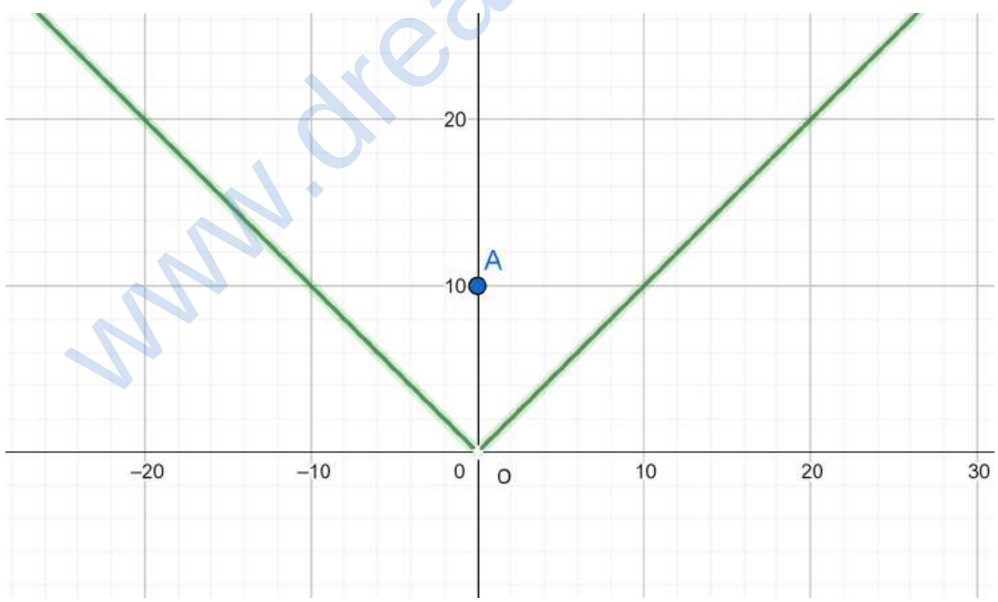


Fig. 16.

A function f(x) is defined in an interval I and c is an interior point of I.
If f(x) is continuous at x = c, $f'(c) = 0$, then c is a critical point.
If f(x) is continuous at x = c, $f'(c)$ is not defined, then c is a critical point.
If c is a point of discontinuity, then c is a critical point. Here also, $f'(c)$ is not defined.

Note: The conditions for the graph of a function to have a vertical tangent at a point, a corner at a point, a cusp at a point and a point of inflexion at a point may be explored in higher courses of mathematics.

Stationary points:

A Stationary point is the point where the derivative of the function is 0. It is essentially the point where the curve is momentarily at rest and then it either takes a smooth turn or becomes a point of inflexion. Stationary points are necessarily the interior points of the domain of the function.

Note:

1) *All stationary points are critical points, but not every critical point is stationary. In part i above, 1 is a stationary point. In part ii, -2 and 3 are stationary points. In part v, 0 is a stationary point.*

2) *If f(x) is differentiable in an open interval (a, b), even if the function is defined in the closed interval [a, b], the only critical points are the interior points of the domain of the function where f '(x) = 0.* ***These are stationary points too.***

Example 26

Find the critical point(s) of the following functions

i. $f(x)=12x^{\frac{4}{3}}-6x^{\frac{1}{3}}$ on $[-1,1]$

ii. $f(x)=\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x$

Solution:

i. $f'(x)=16x^{\frac{1}{3}}-\frac{2}{x^{\frac{2}{3}}}=\frac{16x-2}{x^{\frac{2}{3}}}$

$\therefore f'(x)=0 \Rightarrow x=\frac{1}{8}$ i.e., $x=\frac{1}{8}$ and x = 0, both are critical points of the function, as f′(0) is not defined.

ii. $f'(x)=x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$

$\therefore f'(x)=0 \Rightarrow x=1,2,3$ i.e., the function has three critical points.

Example 27

Using derivatives check whether the following functions are monotonic

i. $f(x) = x^2$ on $(0,\infty)$

ii. $f(x) = x^2$ on $(-\infty,0,)$

iii. $f(x) = x^2$ on $(-\infty,\infty)$

iv. $f(x)=x^{1/3}$

Solution:

i. $f(x)=x^2 \Rightarrow f'(x)=2x$ and $2x>0, \forall x \in (0,\infty)$, i.e., the function is increasing on $(0,\infty)$, hence it's a monotonic function.

ii. $f(x)=x^2 \Rightarrow f'(x)=2x$ and $2x<0, \forall x \in (-\infty,0)$,i.e., the function is decreasing on $(-\infty,0)$, hence it's a monotonic function.

iii. From above discussions in i and ii, we conclude that the function is neither increasing nor decreasing on $(-\infty, \infty)$. Hence the function is not monotonic on the given interval.

iv. $f'(x)=\dfrac{1}{3x^{2/3}}$, clearly $f'(x)$ is not defined at x = 0 which lies in the domain of the function.

Therefore x = 0 is a critical point of the function.

Also,

$f'(x)>0, \forall x\in(-\infty,0) \Rightarrow f(x)$ is an increasing function on $(-\infty, 0)$

$f'(x)>0, \forall x\in(0,\infty) \Rightarrow f(x)$ is an increasing function on $(0,\infty)$

Thus $f(x)$ is an increasing function on its domain, hence it is a monotonic function.

Example 28

Find the intervals in which the function $f(x)=\dfrac{x^4}{4}-2x^3+\dfrac{11}{2}x^2-6x$ is

(i) Increasing (ii) Decreasing.

Solution. $f'(x)=x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$

$\therefore f'(x)=0 \Rightarrow x=1,2,3$ are the critical points of the function, thus the domain R, i.e. $(-\infty,\infty)$ can be divided into four intervals to observe the increasing and decreasing behaviour of the function as follows.

Intervals	Sign of f′(x)	Conclusion
$(-\infty,1)$	$f'(x) < 0$	f is decreasing over $(-\infty,1)$
$(1,2)$	$f'(x) > 0$	f is increasing over $(1,2)$
$(2,3)$	$f'(x) < 0$	f is decreasing over $(2,3)$
$(3,\infty)$	$f'(x) > 0$	f is increasing over $(3,\infty)$

Example 29.

The cost function C(x) of a commodity is given as $C(x)=2x\left(\dfrac{x+3}{x+2}\right)+2$. Prove that the marginal cost falls as the output 'x' increases.

Solution. $C(x)=2x\left(\dfrac{x+3}{x+2}\right)+2 \Rightarrow C'(x)=2\cdot\dfrac{(2x+3)(x+2)-\left(x^2+3x\right)\cdot 1}{(x+2)^2}$

$$\Rightarrow C'(x)=2\frac{\left(x^2+4x+6\right)}{(x+2)^2} \Rightarrow MC=C'(x)=2\left[1+\frac{2}{(x+2)^2}\right]$$

$\Rightarrow \dfrac{d}{dx}(MC)=\dfrac{-8}{(x+2)^3}<0$, hence the marginal cost falls continuously as the output increases

Exercise 3.4

1. Find critical points of the following functions

   i. $f(x) = x^3 - 6x^2 + 9x - 10$ ii. $f(x) = \frac{\log x}{x}$, $x > 0$

   iii. $f(x) = 50\sqrt{x} - 0.5x - 1000$ iv. $f(x) = 5xe^{-x/3}$

2. Find the intervals in which the following functions are increasing or decreasing

   i. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$

   ii. $f(x) = (x + 2)^3 (x - 3)^3$ iii. $f(x) = x^2e^x$

3. Show that the function $f(x) = \log(1+x) + \frac{1}{1+x}$ increases on $(0,\infty)$.

4. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in (0,1).

5. A company finds that its total revenue may be determined by $R(x) = \left[240000 - (x-500)^2\right]$. Find when is the revenue function increasing and when decreasing?

6. The price 'p' per unit is given by the relation $x = \frac{1}{3}p^2 - 2p + 3$ where 'x' is the number of units sold then,

   i. Find the revenue function R.

   ii. Find the price interval for which the revenue is increasing and decreasing.

7. The total cost function of a manufacturing company is given by $C(x) = 2x\left(\frac{x+4}{x+3}\right) + 3$. Show that MC (Marginal Cost) falls continuously as the output 'x' increases.

8. The price 'p' per unit at which a company can sell all that it produces is given by $p = 29 - x$, where 'x' is the number of units produced. The total cost function $C(x) = 45 + 11x$. If $P(x) = R(x) - C(x)$, is the profit function then find the interval in which the profit is increasing and decreasing.

3.11 MAXIMA AND MINIMA

Let $y = f(x)$ be a real function defined on a set D. Then,

1. f(c) is the absolute minimum value of f(x) on D if $f(x) \geq f(c)$, $\forall\ x \in D$
2. f(c) is the absolute maximum value of f(x) on D if $f(x) \leq f(c)$, $\forall\ x \in D$
3. 'f' is said to have an absolute extremum value in its domain if there exists a point 'c' in the domain such that f(c) is either the absolute maximum value or the absolute minimum value of the function 'f' in its domain. The number f(c), in this case, is called an absolute extremum value of 'f' in its domain and the point 'c' is called a point of extremum.

Example 30

Find the maximum (absolute) and minimum (absolute) value of the following functions.

i. $f(x) = |x| + 3$ ii. $f(x) = 9x^2 + 12x + 2$

iii. $f(x) = 2x + 5, x \in (-2,4)$

Solution.

i. $0 \le |x| \Rightarrow 3 \le |x| + 3 \Rightarrow 3 \le f(x)$

$\therefore \text{Min}[f(x)] = 3$ but f(x) has no maximum value. We can observe this graphically in Fig.17.

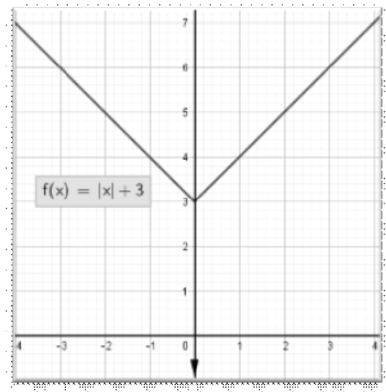


Fig. 17

Note: $f(x) = |x| + 3$ *is not a differentiable function on R but it has a minimum value (or an extreme value). Further if we restrict the domain to* [–2,1], *then f (x) has maximum value at x = –2. (Fig. 17)*

(ii) $f(x) = 9x^2 + 12x + 2 = (3x+2)^2 - 2$

$(3x+2)^2 \ge 0 \therefore f(x) \ge -2$. Hence, the minimum value of f(x) = -2, but $f(x)$ has no maximum value, see Fig.18.

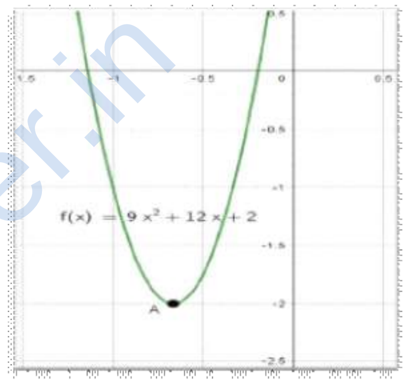


Fig. 18

Note: It may be observed from the graph that f(x) has no maximum value

iii. $x \in (-2, 4)$,

$\therefore -2 < x < 4 \Rightarrow -4 < 2x < 8$

$1 < 2x + 5 < 13 \Rightarrow 1 < f(x) < 13$

$\therefore f(x)$ has no maximum or minimum value, see Fig.19.

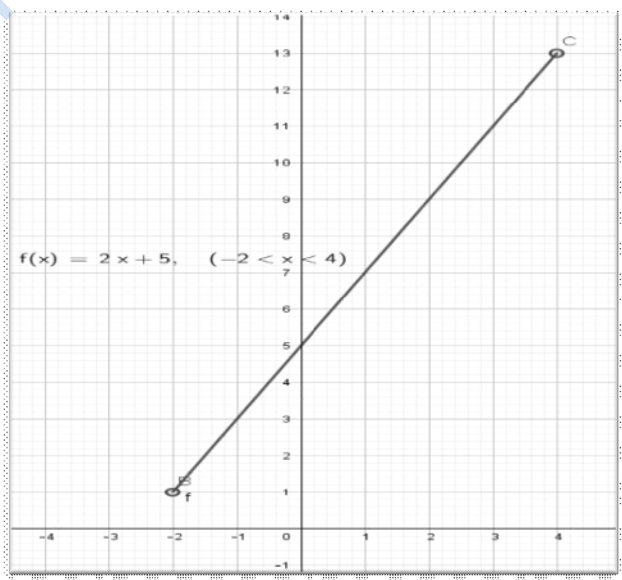


Fig. 19

Note:

(i) If we replace the domain by the closed interval, [–2,4] i.e. include x = –2 and x = 4 in the domain of 'f', then f(x) has extreme values i.e. the minimum and the maximum at x = –2 and x = 4.

LOCAL MAXIMA AND MINIMA

The graph of a continuous function in the adjoining Fig 20, defined in an open interval (a, b) [in this case (-3, 3)], is having three peek type points A, C and E and two valley type points B and D. Let's discuss about these points.

(i) The function or the curve is increasing on the left of each point A, C and E and decreasing on the right and each of the points gives a maximum value of the function in its neighbourhood such points are called as points of local maximum and the value of the function at these points is termed as the local maximum value (or relative maximum value) of the function.

(ii) The function or the curve is decreasing on the left of each point B, D and increasing on the right and each of the points gives a minimum value in its neighbourhood, such points are called as the points of local minimum and the value of the function at these points is termed as the local minimum value (or relative minimum value) of the function.

(iii) Tangents at all the five points A, B, C, D and E are parallel to x-axis i.e., slopes of the tangents at these five points are zero. Thus, these points are the critical points of the function.

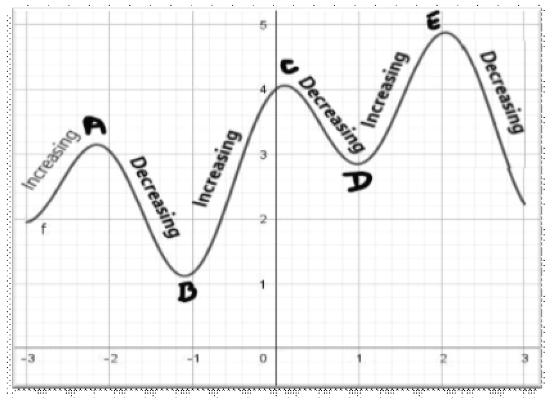


Fig. 20.

Note:

1) *The function f defined in an open interval will have a local maximum value or a local minimum value at a critical point only But Not every critical point is a point of local extremum.*

2) *A function f defined in a closed interval may have a local extremum value even at the boundary points (end points). No such examples have been taken in this course.*

Definition: Let 'f' be a real valued function and let 'c' be an interior point in the domain of 'f'. Then

(i) f is said to have a local maximum value at 'c', if there exists a positive real number h such that $f(c) > f(x) \forall x \in (c-h, c+h) - \{c\}$. c is called a point of local maximum and f(c) is called a local maximum value.

(ii) f is said to have a local minimum value at 'c', if there exists a positive real number h such that $f(c) < f(x) \forall x \in (c-h, c+h) - \{c\}$. c is called a point of local minimum and f(c) is called a local minimum value.

Geometrically, the above definition states that if x = c is a point of local maximum then f(x) will be increasing in the left neighbourhood of c, i.e., there exists h>0 such that f is increasing in (c–h, c).

And, f(x) will be decreasing in the right neighbourhood of c, i.e., there exists h>0 such that f is decreasing in (c, c+h), as shown in the Fig 21

The above will be the case when the function is continuous at an interior point c of the domain of the function.

Similarly, if x = c is a point of local minimum, then f(x) will be decreasing in the left neighbourhood of c, i.e., there exists h>0 such that f is decreasing in (c – h, c).

And, f(x) will be increasing in the right neighbourhood of c, i.e., there exists h>0 such that f is strictly increasing in (c, c+h), as shown in the Fig 22. The above will be the case when the function is continuous at an interior point c of the domain of the function.

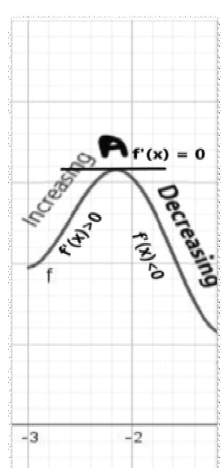


Fig. 21

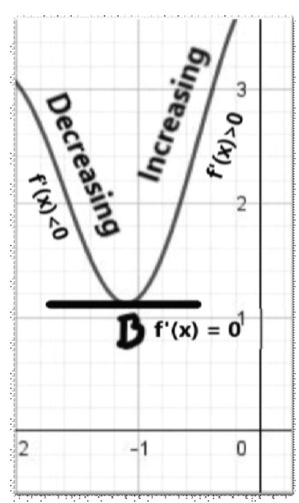


Fig. 22

A function may have a critical point where the derivative vanishes and it may not have any local extreme value at that point (or a local maximum or a local minimum value may not exist) for example for the function $h(x) = x^3$, $h'(x) = 3x^2$ and so $h'(0) = 0$. But '0' is neither a point of local maximum nor a point of local minimum, as the function is increasing on both sides of the point. See Fig 23.

Following is the working rule for finding points of local maxima or points of local minima using the first order derivatives.

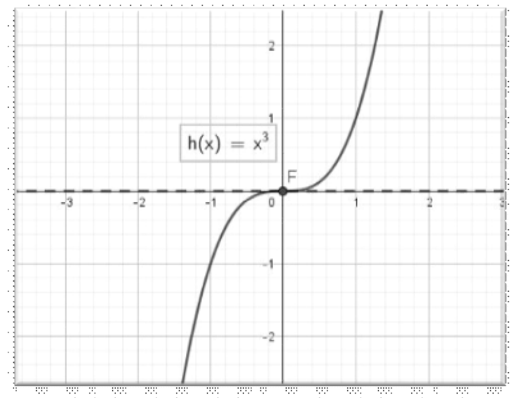


Fig. 23

FIRST DERIVATIVE TEST

Let 'f' be a function defined on an open interval (a,b). Let 'f' be continuous at a critical point $c \in (a,b)$ then,

(i) The point 'c' is a point of local minimum if there exists $h > 0$ such that $f'(x) < 0, \forall x \in (c-h, c)$ and $f'(x) > 0, \forall x \in (c, c+h)$

i.e. f(x) is decreasing in the left neighbourhood of 'c' and increasing in the right neighbourhood of 'c' (or f′(x) changes its sign from negative to positive as x increases through 'c'), also f(c) is a local minimum value of f(x).

(ii) The point 'c' is a point of local maximum if there exists $h > 0$ such that

$f'(x) > 0, \forall x \in (c-h, c)$ and $f'(x) < 0, \forall x \in (c, c+h)$

i.e. f(x) is increasing in the left neighbourhood of 'c' and decreasing in the right neighbourhood of 'c' (or f′(x) changes its sign from positive to negative as x increases through 'c'), also f(c) is a local maximum value of f(x).

(iii) If f′(x) does not change sign as x increases through 'c', then c is neither a point of local maximum nor a point of local minimum. Such a point is a point of inflexion.

Example 31

Find all the points of local maxima and local minima and the local maximum and local minimum values of the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$

Solution. First, we find the critical points of the function,

$$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3)$$

$f'(x) = 0 \Rightarrow x = 1, 2$ or 3 are the critical points of the function

We apply the first derivative test at each of the critical point,

Sign of $f'(x)$: 

At x = 1,

In the left neighbourhood of 1, i.e., for x's close to 1 and to the left of 1, $f'(x) < 0$ and in the right neighbourhood of 1, i.e., for x's close to 1 and to the right of 1, $f'(x) > 0$,

$f'(x)$ changes its sign from negative to positive as x increases through 1 $\therefore$ x = 1 is a point of local minimum and the local minimum value is f(1) = –8.

At x = 2,

In the left neighbourhood of 2, $f'(x) > 0$, and in the right neighbourhood of 2, $f'(x) < 0$,

$f'(x)$ changes its sign from positive to negative as x increases through 2 $\therefore$ x = 2 is a point of local maximum and the local maximum value is f(2) = –7

At x = 3,

In the left neighbourhood of 3, $f'(x) < 0$ and in the right neighbourhood of 3, $f'(x) > 0$,

$f'(x)$ changes its sign from negative to positive as x increases through 3 $\therefore$ x = 3 is a point of local minimum and the local minimum value is f(3) = –8

SECOND DERIVATIVE TEST

Let 'f' be a function defined on an open interval (a,b). Let 'f' be continuous at a critical point $c \in (a,b)$ then,

(i) The point 'c' is a point of local minimum if $f'(c) = 0$ and $f''(c) > 0$ and we say f(c) is a local minimum value of f(x)

(ii) The point 'c' is a local maximum if $f'(c) = 0$ and $f''(c) < 0$ and we say f(c) is a local maximum value of f(x).

Note: If f "(c) = 0, we say that the second derivative test fails and then we apply first derivative test to check whether 'c' is a point of local maximum, local minimum or a point of inflexion. If f'(x) does not change sign as x increases through c, then c is a point of inflexion.

Example 32

Use the second derivative test to find the local maxima and minima of $f(x) = \frac{4}{3}x^3 + 6x^2 + 8x + 7$.

Solution. $f'(x) = 4x^2 + 12x + 8 = 4(x+1)(x+2)$,

$f'(x) = 0 \Rightarrow x = -2, -1$

$\therefore$ The critical points are -2, -1

We apply second derivative test now,

$$f''(x) = 8x + 12$$

At x = –2, $f''(-2) = -16 + 12 = -4 < 0$

$\therefore$ x = –2 Is a point of local maxima and local maximum value is $f(-2) = \frac{-32}{3} + 24 - 16 + 7 = \frac{13}{3}$

At x = –1, $f''(-1) = -8 + 12 = 4 > 0$

$\therefore$ $x = -1$, is a point of local minima and local minimum value is $f(-1) = \frac{-4}{3} + 6 - 8 + 7 = \frac{11}{3}$

Example 33.

Use the second derivative test to find the local maxima and minima of $f(x)=x^3-3x^2+3x+5$, if any.

Solution: $f'(x)=3x^2-6x+3=3(x^2-2x+1)=3(x-1)^2$

$\therefore$ The critical points of f(x) are given by

$$f'(x)=0\Rightarrow 3(x-1)^2=0\Rightarrow x=1$$

At x = 1, $f''(1) = 0$

$\therefore$ the second derivative test fails,

We now apply first derivative test at $x=1$,

In the left neighbourhood of 1, $f'(x) > 0$,

In the right neighbourhood of 1, $f'(x) > 0$

$\therefore$ f′(x) does not change sign as x increases through 1, this means that $x=1$ is a point of inflexion. We can verify this from Fig.24 of the function.

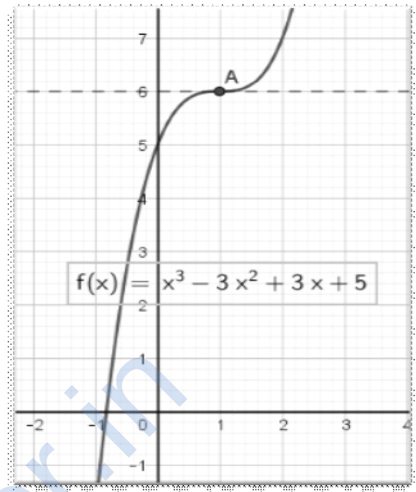


FIG. 24

MAXIMUM AND MINIMUM VALUES IN A CLOSED INTERVAL

If y = f (x) is real continuous function defined in a closed interval [a,b] then following is the step-by-step method to find maximum and minimum value of a function in the closed interval

Step 1: Find all critical points of the function by solving $f'(x) = 0$ in the open interval.

Step 2: If $x_1, x_2, \ldots, x_n$ are the 'n' critical points of the function, then we find n + 2 values of the function at the points $a, x_1, x_2, \ldots, x_n, b$, i.e., including the end points of the interval.

Step 3: The largest among $f(a), f(x_1), f(x_2), \ldots, f(x_n), f(b)$ is the absolute maximum value of the function and the least the absolute minimum value of the function.

Example 34

Find the absolute maximum and minimum value of the function

$$f(x)=x^3-\frac{3}{2}x^2-18x+1 \text{ on } [-4,6]$$

Solution: $f'(x)=3x^2-3x-18=3(x^2-x-6)=3(x+2)(x-3)$,

$\therefore$ The critical points of the function are given by

$$f'(x)=0\Rightarrow x=-2,3\ \in(-4,6)$$

The values of f(x) at x = –4, –2, 3, 6 are

$$f(-4)=(-4)^3-\frac{3}{2}(-4)^2-18(-4)+1=-64-24+72+1=-15$$

$f(-2) = 23$

$$f(3)=-\frac{79}{2}=-39.5=\text{the absolute minimum value of the function}$$

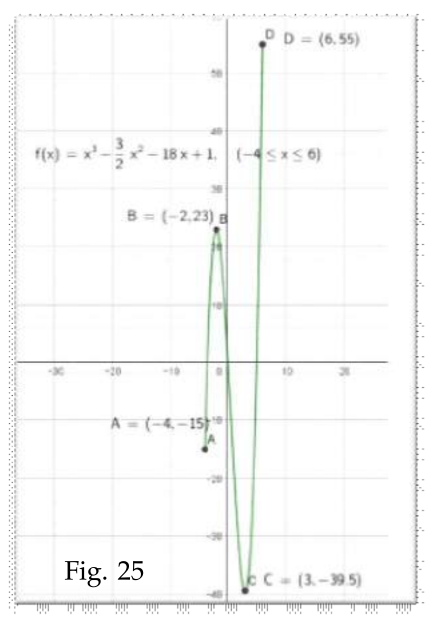


Fig. 25

$f(6) = 55$ = the absolute maximum value of the function

$\therefore$ the absolute minimum value of $f(x) = -39.5$ and the absolute maximum value of $f(x) = 55$,

The solution of above function can be observed from Fig. 25.

APPLIED PROBLEMS

Example 35

Find two positive numbers whose sum is 16 and whose product is as large as possible.

Solution. Let the two positive numbers be 'x' and 'y', then $x + y = 16 \Rightarrow y = 16 - x$

Let, $P = xy \Rightarrow P = x(16-x) = 16x - x^2$

$\frac{dP}{dx} = 16 - 2x$ Solving $\frac{dP}{dx} = 0$ for the critical points, we get $x = 8$

; $\left.\frac{d^2P}{dx^2}\right]_{x=8} = -2 < 0$,

$\therefore$ P has maximum value at $x = 8$; $y = 8$

Example 36

The production manager of a company plans to include 180 square centimetres of actual printed matter in each page of a book under production. Each page should have a 2.5 cm wide margin along the top and bottom and 2 cm wide margin along the sides. What are the most economical dimensions of each printed page?

Solution. Let 'x' and 'y' be the dimension of printed page, then $x \cdot y = 180$

If A is the area of each page of the book,

then $A = (x+4)(y+5) = xy + 5x + 4y + 20$

Using $xy = 180$, reducing 'A' in terms of 'x', we get,

$$A = 180 + 5x + 4\left(\frac{180}{x}\right) + 20 = 200 + 5x + \frac{720}{x}$$

$A'(x) = 5 - \frac{720}{x^2}$, Solving $A'(x) = 0$ for critical points,

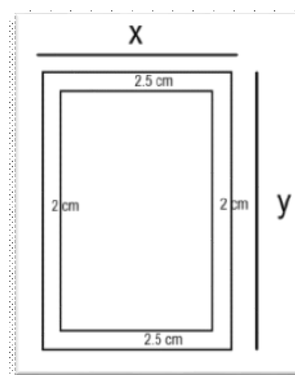


Fig. 26.

we get $x^2 = 144 \Rightarrow x = 12$, as 'x' cannot be negative

$A''(x) = \frac{1440}{x^3} \Rightarrow A''(12) > 0$, i.e. 'A' (the area) is minimum at $x = 12$, and $y = \frac{180}{x} = \frac{180}{12} = 15$

Hence the most economical dimensions of the each printed page is $x + 4 = 12 + 4 = 16$cm and $y + 5 = 15 + 5 = 20$cm.

Example 37

An open tank with a square bottom is to contain 4000 cubic cm of liquid is to be constructed. Find the dimension of the tank so that the surface area of the tank is minimum.

Solution. Let each side of the square base of tank be 'x' cm and its depth be 'y' cm. Then, V (Volume of the tank) $= x^2y = 4000 \Rightarrow y = \frac{4000}{x^2}$

If 'S' is the surface area of the tank, then $S = x^2 + 4xy = x^2 + \frac{16000}{x}$

$S'(x) = 2x - \frac{16000}{x^2}$, For critical points $S'(x) = 0 \Rightarrow x^3 = 8000 \Rightarrow x = 20$

$$\frac{d^2S}{dx^2} = 2 + \frac{32000}{x^3} \quad \Rightarrow \left.\frac{d^2S}{dx^2}\right]_{x=20} = 6 > 0$$

$\therefore$ S (The surface area of the tank) is minimum for x = 20cm, y = 10cm

Example 38

A wire 40 m length is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum?

Solution. Let the length of one piece be 'x' metres which is made into a square. Then the length of the other piece is (40–x) metres which is made into a circle. Let 'r' be the radius of the circle, then

$$2\pi r = 40 - x \Rightarrow r = \frac{40 - x}{2\pi}$$

Let 'A' be the combined area of the square and the circle, then

$$A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{40-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(40-x)^2}{4\pi}$$

$A'(x) = \frac{x}{8} - \frac{40-x}{2\pi}$ For critical points, $A'(x) = 0 \Rightarrow x = \frac{160}{\pi+4}$

$A''(x) = \frac{1}{8} + \frac{1}{2\pi} > 0$, $\therefore$ 'A' is minimum at $x = \frac{160}{\pi+4}$

Length of the first piece = $\frac{160}{\pi+4}$ m and,

The length of the second piece = 40 – x = $\frac{40\pi}{\pi+4}$ m

Example 39

Let the cost function of firm be given by the equation $C(x) = 300x - 10x^2 + \frac{1}{3}x^3$. Find the output at which the marginal cost MC is minimum.

Solution. Let $f(x) = MC = \frac{dC}{dx} = 300 - 20x + x^2$

$f'(x) = -20 + 2x$, for stationary points $f'(x) = 0 \Rightarrow x = 10$

$f''(x) = 2$, $\therefore$ $f''(10) = 2 > 0$, $\therefore$ f(x) or MC is minimum at x = 10.

Example 40

A pen drive manufacturing company charges ₹ 6,000 per unit for an order of 50 pen drives or less. The charge is reduced by ₹ 75 per pen drive for each order in excess of 50. Find the largest size order, the company should allow so as to receive maximum revenue.

Solution. Let 'x' units of pen drives be the total size of order, so that x > 50 to avail the said deduction. Then, revenue function is

$$R(x) = (6000 - 75(x - 50)) \cdot x = 9750x - 75x^2$$

$R'(x) = 9750 - 150x$, and for critical points $R'(x) = 0 \Rightarrow x = 65$

Also, $R''(x) = -150$, $R''(65) = -150 < 0$, $\therefore$ the revenue is maximum, if 65 pen drives are ordered.

Example 41

A manufacturer produces x pants per week at total cost of ₹ ($x^2 + 78x + 2500$). The price per unit is given by $8x = 600 - p$, where 'p' is the price of each set. Find the maximum profit obtained, where the profit function is given by $P(x) = R(x) - C(x)$.

Solution. The revenue function, $R(x) = p \cdot x = (600 - 8x) \cdot x = 600x - 8x^2$

$\therefore$ The profit function is given by,

$$P(x) = (600x - 8x^2) - (x^2 + 78x + 2500) = -9x^2 + 522x - 2500$$

$P'(x) = 522 - 18x$, For critical points $P'(x) = 0 \Rightarrow x = 29$,

$P''(x) = -18 \Rightarrow P''(29) = -18 < 0$

$\therefore$ The profit is maximum when 29 sets are produced per week. And the maximum profit per week is $P(29) =$ ₹ 5069.

Exercise 3.5

1. Find the local maxima, local minima, local minimum value and local maximum value, if any of the following

   i. $f(x) = x^2 - 6x + 16$

   ii. $f(x) = \dfrac{\log x}{x}$, $x > 0$

   iii. $f(x) = (1 - x^2)e^x$

   iv. $f(x) = \dfrac{x^2 - 7x + 6}{x - 10}$

   v. $f(x) = 2x + \dfrac{1}{2x}$

2. The sum of two positive numbers is 16. Find the numbers, if the product of the squares is to be maximum.
3. Show that of all rectangles with a given perimeter, the square has the largest area.
4. Show that the function $f(x) = x^3 - 6x^2 + 12x + 50$ has neither a local maximum nor a local minimum value.
5. The profit function, in rupees, of a firm selling 'x' items ($x \geq 0$) per week is given by $P(x) = (400 - x)x - 3500$. How many items should the firm sell to make the maximum profit? Also find the maximum profit.

6. A tour operator charges ₹ 136 per passenger for 100 passengers with a discount of ₹ 4 for each 10 passengers in excess of 100. Find the number of passengers that will maximise the amount of money the tour operator receives.

7. If price 'p' per unit of an article is p = 75 – 2x and the cost function is $C(x) = 350 + 12x + \frac{x^2}{4}$. Find the number of units and the price at which the total profit is maximum. What is the maximum profit?

8. The cost of fuel in running an engine is proportional to the square of the speed in kms per hour, and is ₹ 48 per hour when the speed is 16 km. Other costs amount to ₹ 300 per hour. Find the most economical speed.

CASE BASED QUESTIONS

Case Study-I.

A farmer has a piece of land. He observed that he got 600 units of fruits per tree by planting upto 25 trees and when 26 trees were grown, he received 15210 units of fruits, for 27 trees he ended up with 15390 fruits, for 28 trees he got 15540 fruits and this sequence of production of fruits continues in the same pattern as more trees, in excess of 25, were grown.

Based on the above information answer the following questions:

1. If 'x' more trees, in excess of 25 are grown, then the number of fruits produced per tree is
   i. 600 – 15x ii. 600 + 15x
   iii. 600x – 15 iv. 600x + 15
2. The production of entire garden if 'x' more trees, in excess of 25, are planted
   i. (25 + x) (600 + 15x) ii. (25 – x) (600 – 15x)
   iii. (25 + x) (600 – 15x) iv. (25 + x) (15x – 600)
3. The marginal production of the garden when 'x' more trees, in excess of 25, are planted
   i. 225 + 30x ii. 225 – 30x
   iii. 225x + 30 iv. 225x –30
4. The critical point of producing 'x' more units of trees is
   i. 7 ii. 8
   iii. 7.5 iv. 8.5
5. The number of trees to be grown to get maximum production is
   i. 30 or 31 trees ii. 32 or 33 trees
   iii. 33 or 34 trees iv. 34 or 35 trees

Case Study- II.

A manufacturing company manufactures toys, the company observed the following costs at different production levels,

Number of toys manufactured	Cost of raw material (₹)	Cost of Production Supply(₹)	Cost of freight (₹)	Property tax (₹)	Salaries (₹)
100	800	2000	1000	5000	20000
150	1200	3000	1500	5000	20000
200	1600	4000	2000	5000	20000
250	2000	5000	2500	5000	20000
300	2400	6000	3000	5000	20000

Based on the above information, answer the following.

1. Which of the following is the fixed cost
   i. Number of toys manufactured   ii. Cost of raw material
   iii. Cost of production supply   iv. Salaries
2. Total cost C(x) of toys for 'x' units of production is
   i. $C(x) = 8x^2 + 30x + 25000$ ii. $C(x) = 8x^2 + 30x + 20000$
   iii. $C(x) = 38x + 25000$ iv. $C(x) = 28x + 25000$
3. If the company observes the price 'p' per unit of item sold $p = 5000 - 10x$, where the 'x' is the number of units sold. Then the revenue function R(x) is given by,
   i. $R(x) = 5000x - 10x^2$ ii. $R(p) = 5000p - 10p^2$
   iii. $R(x) = 5000 - 10x^2$ iv. $R(p) = 5000 - 10p^2$
4. The Marginal revenue (MR) of the company is given by
   i. $5000 - 20x$ ii. $5000 - 20p$
   iii. $-20x$ iv. $-20p$
5. If the profit function $P(x) = R(x) - C(x)$, then it is given by
   i. $-18x^2 + 4970x - 25000$ ii. $-10x^2 + 4962x - 20000$
   iii. $10x^2 + 4962x - 25000$ iv. $-10x^2 + 4962x - 25000$

SUGGESTED PROJECT

Select a fruit seller in your neighbouring market. Select a fruit say Papaya. Collect following data.

1. For cost function, collect cost data from the site https://www.agrifarming.in/papaya-farming-project-report-cost-and-profit
2. Collect transportation cost from your location or any other cost that one may have to spend to bring it to consumer market.
3. For price function, collect price for 30 days in the market near you, collect the number of papaya units sold in 30 days at the given price.
4. Taking 'p' the price and 'x' the number of units sold on daily basis till 30 days. Plot the points (p,x) for 30 days on graph paper and observe the curve around the data, assumimg it to be a negative sloped line, Find the linear relation between 'p' and 'x' by the method of regression line, take help from the site
   https://www.dummies.com/education/math/statistics/how-to-calculate-a-regression-line/

**You may use the following site to calculate the equation of the regression line https://www.socscistatistics.com/tests/regression/default.aspx

5. Make Cost Function from the data collected in step 2, Revenue function from step 4.

Based on the data collected and functions constructed do the following mathematical modelling.

1. Find the intervals in which the cost function, revenue function, profit function increases or decreases.
2. Find MC (Marginal Cost), MR (Marginal Revenue) functions.
3. Find the Profit function, its local maxima, local minima if any.

ANSWERS

EXERCISE 3.1

1. i. $\frac{ay-x^2}{y^2-ax}$ ii. $-\frac{y}{x}$ iii. $\frac{10xy-9x^2-2y^2}{4xy-5x^2+12y^2}$ iv. $-\sqrt[3]{\frac{y}{x}}$

   v. $\frac{x-y}{x(1+\log xy)}$

2. i. $-\frac{1}{t^2}$ ii. $\frac{1-\log t}{t^2(1+\log t)}$ iii. $-\frac{b}{2a}\left(\frac{1-t^2}{t}\right)$

3. i. $\frac{(y-x\log y)y}{(x-y\log x)x}$ ii. $\frac{\log x}{(1+\log x)^2}$ iii. $\frac{2y-x}{y}$ iv. $2\cdot x^{\log x-1}\cdot\log x$

4. i. $\frac{1}{x}$ ii. $e^x(x^2+4x+2)$ iii. $-\frac{\log x+1}{(x\log x)^2}$ iv. $6(2e^{2x}+3e^{3x})$

EXERCISE 3.2

1. 2π 2. 32 cm^2/cm 3. $r=\frac{1}{2\sqrt{\pi}}$ 4. 0.002cm/sec

5. $3,\frac{1}{3}$ 6. $54\pi cm^2$/minute 7. $-\frac{3}{2}$ m/min 8. $-\frac{1}{18\pi}$ cm / min

9. $C(x)=x^2+2x+5000$; ₹ 43 ; ₹ 102

10. $p=11-\frac{x}{200}$; $R=11x-\frac{x^2}{200}$; $MR=11-\frac{x}{100}$

EXERCISE 3.3

1. i. $2, \frac{-1}{2}$ ii. $6;\frac{-1}{6}$ iii. $\frac{-1}{64};64$ iv. $-1;\ 1$

2. i. $9x-y-11=0; x+9y-65=0$ ii. $x-2y+4a=0$; $2x+y-12a=0$

3. $y-6=0$; $y-7=0$

4. $14x-y-20=0; 14x-y+12=0$

5. $x-20y-7=0; 20x+y-140=0$

6. $x+y-3=0$ 7. $(2,14),(-2,2)$

EXERCISE 3.4

1. i. 1, 3 ii. e iii. 2500 iv. 3
2. i. Increasing on $(1,2)$ and $(3,\infty)$; Decreasing on $(-\infty,1)$ and $(2,3)$.

   ii. Decreasing on $\left(-\infty,\frac{1}{2}\right)$ and Increasing on $\left(\frac{1}{2},\infty\right)$

   iii. Increasing on $(-\infty,-2)$ and $(0,\infty)$; Decreasing on $(-2,0)$
5. Increasing if $x < 500$ and decreasing if $x > 500$
6. $R=\frac{1}{3}p^3-2p^2+3$; increasing on $p < 1$ or $p > 3$ and decreasing on $1 < p < 3$.
8. Increasing on $(0,9)$ and decreasing on $(9,\infty)$

EXERCISE 3.5

1. i. $x = 3$ is a point of local minimum, local minimum value $= 7$

   ii. $x=e$ is a point of local maximum, local maximum value $= \frac{1}{e}$

   iii. $x=-1$ is a point of local maximum, local maximum value $= \frac{4}{e}$

   iv. $x = 16$ is a point of local minimum, local minimum value $= 25$; $x = 4$ is a point of local maximum, local maximum value $= 1$

   v. $x = \frac{1}{2}$ is a point of local minimum, local minimum value $= 2$; $x = -\frac{1}{2}$ is a point of local maximum, local maximum value $= -2$
2. 8,85. 200, ₹ 36,500 6. 220 passangers
7. $x = 14$, $p =$ ₹ 47, Maximum Profit = ₹ 83,323. 8. 40 km/hour

3. CASE BASED QUESTIONS

4. Case Study- I

5. 1(i), 2 (iii), 3 (ii), 4 (iii), 5 (ii)

6. Case Study- II

7. 1(iv), 2(iii), 3(i), 4(i), 5(iv)

SUMMARY

1. Implicit functions are those in which the dependent variable is not expressed explicitly in terms of independent variable.

2. If $y = f(t)$ and $x = g(t)$, is a parametric function, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

3. Logarithmic differentiation is applied for the functions of the type $f(x)^{g(x)}$.

4. Second and higher order derivative.

   i. Derivative of $f'(x)$ with respect to 'x' = $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = f''(x)$, is the second order derivative of 'y' or $f(x)$.

   ii. Derivative of $f''(x)$ with respect to 'x' = $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y''' = f'''(x)$, is the third order derivative of 'y' or $f(x)$.

5. Cost function: C(x) = V(x) + k, where V(x) is a variable cost and 'k' is the fixed cost.

6. Revenue Function: R(x) = p·x,, where 'p' is the price per unit and 'x' is the output or sales at price 'p'.

7. $\frac{dy}{dx} =$ Rate (or instantaneous rate) of change of 'y' with respect to 'x'.

   i.e. change in 'y' with respect to very small change in 'x'.

8. $\text{MC (Marginal cost)} = C'(x) = \frac{dC}{dx}$, $\text{MR (Marginal revenue)} = R'(x) = \frac{dR}{dx}$

9. Slope (or gradient) of a tangent at a point $A(x_0, y_0)$ = $\left.\frac{dy}{dx}\right]_{A(x_0,y_0)}$

10. Slope of a normal line to the curve at a point $A(x_0, y_0)$ = $\frac{-1}{\left.\frac{dy}{dx}\right]_{A(x_0,y_0)}}$

11. Equation of tangent to the curve at the point A (x_0,y_0) is $(y - y_0) = \left.\frac{dy}{dx}\right]_{(x_0,y_0)} (x - x_0)$

12. Equation of tangent to the curve at the point A (x_0,y_0) is $(y - y_0) = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0,y_0)}}(x - x_0)$

13. A real function $y = f(x)$ is an increasing function on (a,b) if $f'(x) > 0, \forall x \in (a,b)$

14. A real function $y = f(x)$ is a decreasing function on (a,b) if $f'(x) < 0, \forall x \in (a,b)$

15. A function which is either increasing or decreasing on its domain is termed as a monotonic function.

16. If $y = f(x)$ is a real function then an interior point $x_0 \in D_f$ is called a critical point if either $f'(x_0) = 0$ or f(x) is not differentiable at x_0.

17. If 'f' is a real function defined on the domain D. Then f(c) is the absolute minimum value of $f(x)$ on D if $f(x) \geq f(c), \forall x \in D$.

    f(c) is the absolute maximum value of f(x) on D if $f(x) \leq f(c), \forall x \in D$

    'f' is said to have an absolute extremum value in its domain if there exists a point 'c' in the domain such that f(c) is either the absolute maximum value or the absolute minimum value of the function 'f' in its domain. The number f(c), in this case, is called an absolute extremum value of 'f' in its domain and the point 'c' is called a point of extremum.

18. First Derivative Test: Let 'f' be a function defined on an open interval (a,b). Let 'f' be continuous at a critical point $c \in (a,b)$ then,

    (i) The point 'c' is a point of local minimum if there exists h > 0 such that

    $$f'(x) < 0, \forall x \in (c-h, c) \text{ and } f'(x) > 0, \forall x \in (c, c+h)$$

    i.e. f(x) is decreasing in the left neighbourhood of 'c' and increasing in the right neighbourhood of 'c' (or f '(x) changes its sign from negative to positive as x increases through 'c'), also f(c) is a local minimum value of f(x).

    (ii) The point 'c' is a point of local maximum if there exists h > 0 such that

    $f'(x) > 0, \forall x \in (c-h, c)$ and $f'(x) < 0, \forall x \in (c, c+h)$

    i.e. f(x) is increasing in the left neighbourhood of 'c' and decreasing in the right neighbourhood of 'c' (or f '(x) changes its sign from positive to negative as x increases through 'c'), also f(c) is a local maximum value of f(x) .

    (iii) If f '(x) does not change sign as x increases through 'c', then c is neither a point of local maximum nor a point of local minimum. Such a point is a point of inflexion.

19. Second derivative Test: Let 'f' be a function defined on an open interval (a,b). Let 'f' be continuous at a critical point $c \in (a,b)$ then,

    i. The point 'c' is a point of local minimum if f '(c) = 0 and f " (c) > 0 and we say f(c) is a local minimum value of f(x)

    ii. The point 'c' is a point of local maximum if f '(c) = 0 and f " (c) < 0 and we say f(c) is a local maximum value of f(x)

    iii. If f " (c) = 0, we say second derivative test fails and then we apply first derivative test to check whether 'c' is a point of local maximum, local minimum or a point of inflexion.

20. Absolute Maximum and Minimum on closed interval.

    Step 1: Find all critical points of the function by solving f '(x) = 0 in the closed interval [a,b].

    Step 2: If $x_1, x_2, ..., x_n \in [a,b]$ are the 'n' critical points of the function, then we find n + 2 values of the function at the points $a, x_1, x_2, ..., x_n, b$, i.e., including the end points of the interval.

    Step 3: The highest among $f(a), f(x_1), f(x_2), ..., f(x_n), f(b)$ is the absolute maximum value of the function and the least an absolute minimum value.

❑❑❑