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Continuity and Differentiability

Short Answer Type Questions

Q. 1 Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

Thinking Process

We know that, function f will be continuous at $x = a$, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

Sol. We have, $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

$$\therefore \quad \lim_{x \to 1^+} f(x) = \lim_{h \to 0} (1 + h)^3 + 2(1 + h)^2 - 1 = 2$$

and
$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1 - h)^3 + 2(1 - h)^2 - 1 = 2$$

$$\therefore \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$$

and
$$f(1) = 1 + 2 - 1 = 2$$

So, $f(x)$ is continuous at $x = 1$.

Note *Every polynomial function is continuous at any real point.*

Q. 2 $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$.

Sol. We have, $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$.

At $x = 2$,
$$\text{LHL} = \lim_{x \to 2^-} (x)^2$$
$$= \lim_{h \to 0} (2 - h)^2 = \lim_{h \to 0} (4 + h^2 - 4h) = 4$$

and
$$\text{RHL} = \lim_{x \to 2^+} (3x + 5)$$
$$= \lim_{h \to 0} [3(2 + h) + 5] = 11$$

Since, $\text{LHL} \neq \text{RHL}$ at $x = 2$

So, $f(x)$ is discontinuous at $x = 2$.

Q. 3 $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have,
$$f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

At $x = 0$,
$$\text{LHL} = \lim_{x \to 0^-} \frac{1-\cos 2x}{x^2}$$
$$= \lim_{h \to 0} \frac{1-\cos 2(0-h)}{(0-h)^2}$$
$$= \lim_{h \to 0} \frac{1-\cos 2h}{h^2} \quad [\because \cos(-\theta) = \cos\theta]$$
$$= \lim_{h \to 0} \frac{1-1+2\sin^2 h}{h^2} \quad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$
$$= \lim_{h \to 0} \frac{2(\sin h)^2}{(h)^2} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$
$$= 2$$
$$\text{RHL} = \lim_{x \to 0^+} \frac{1-\cos 2x}{x^2}$$
$$= \lim_{h \to 0} \frac{1-\cos 2(0+h)}{(0+h)^2}$$
$$= \lim_{h \to 0} \frac{2\sin^2 h}{h^2} = 2 \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

and $f(0) = 5$

Since, $\text{LHL} = \text{RHL} \neq f(0)$

Hence, $f(x)$ is not continuous at $x = 0$.

Q. 4 $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ at $x = 2$.

Sol. We have, $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ at $x = 2$.

At $x = 2$,
$$\text{LHL} = \lim_{x \to 2^-} \frac{2x^2-3x-2}{x-2}$$
$$= \lim_{h \to 0} \frac{2(2-h)^2 - 3(2-h) - 2}{(2-h)-2}$$
$$= \lim_{h \to 0} \frac{8 + 2h^2 - 8h - 6 + 3h - 2}{-h}$$
$$= \lim_{h \to 0} \frac{2h^2 - 5h}{-h} = \lim_{h \to 0} \frac{h(2h-5)}{-h} = 5$$
$$\text{RHL} = \lim_{x \to 2^+} \frac{2x^2-3x-2}{x-2}$$

$$= \lim_{h \to 0} \frac{2(2+h)^2 - 3(2+h) - 2}{(2+h) - 2}$$

$$= \lim_{h \to 0} \frac{8 + 2h^2 + 8h - 6 - 3h - 2}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 5h}{h} = \lim_{h \to 0} \frac{h(2h+5)}{h} = 5$$

and $\quad f(2) = 5$

$\therefore \quad$ LHL = RHL = $f(2)$

So, $f(x)$ is continuous at $x = 2$.

Q. 5 $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ at $x = 4$.

Sol. We have, $\quad f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ at $x = 4$.

At $x = 4$, $\quad$ LHL $= \lim_{x \to 4^-} \dfrac{|x-4|}{2(x-4)}$

$$= \lim_{h \to 0} \frac{|4-h-4|}{2[(4-h)-4]} = \lim_{h \to 0} \frac{|0-h|}{(8-2h-8)}$$

$$= \lim_{h \to 0} \frac{h}{-2h} = \frac{-1}{2} \quad \text{and} \quad f(4) = 0 \neq \text{LHL}$$

So, $f(x)$ is discontinuous at $x = 4$.

Q. 6 $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have, $\quad f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$, $\quad$ LHL $= \lim_{x \to 0^-} |x| \cos \dfrac{1}{x} = \lim_{h \to 0} |0-h| \cos \dfrac{1}{0-h}$

$$= \lim_{h \to 0} h \cos\left(\frac{-1}{h}\right)$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

RHL $= \lim_{x \to 0^+} |x| \cos \dfrac{1}{x}$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)}$$

$$= \lim_{h \to 0} h \cos \frac{1}{h}$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

and $\quad f(0) = 0$

Since, $\quad$ LHL = RHL = $f(0)$

So, $f(x)$ is continuous at $x = 0$.

Q. 7 $f(x)=\begin{cases} |x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq 0 \\ 0, & \text{if } x=a \end{cases}$ at $x=a$.

Sol. We have, $f(x)=\begin{cases} |x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq 0 \\ 0, & \text{if } x=a \end{cases}$ at $x=a$

At $x=a$,

$$\text{LHL}=\lim_{x\to a^-}|x-a|\sin\frac{1}{x-a}$$

$$=\lim_{h\to 0}|a-h-a|\sin\left(\frac{1}{a-h-a}\right)$$

$$=\lim_{h\to 0}-h\sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta)=-\sin\theta]$$

$= 0\times$[an oscillating number between -1 and 1] $= 0$

$$\text{RHL}=\lim_{x\to a^+}|x-a|\sin\left(\frac{1}{x-a}\right)$$

$$=\lim_{h\to 0}|a+h-a|\sin\left(\frac{1}{a+h-a}\right)=\lim_{h\to 0}h\sin\frac{1}{h}$$

$= 0\times$[an oscillating number between -1 and 1] $= 0$

and $f(a)=0$

$\therefore$ LHL = RHL = $f(a)$

So, $f(x)$ is continuous at $x=a$.

Q. 8 $f(x)=\begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$ at $x=0$

At $x=0$,

$$\text{LHL}=\lim_{x\to 0^-}\frac{e^{1/x}}{1+e^{1/x}}=\lim_{h\to 0}\frac{e^{1/0-h}}{1+e^{1/0-h}}$$

$$=\lim_{h\to 0}\frac{e^{-1/h}}{1+e^{-1/h}}=\lim_{h\to 0}\frac{1}{e^{1/h}\,(1+e^{-1/h})}$$

$$=\lim_{h\to 0}\frac{1}{e^{1/h}+1}=\frac{1}{e^{\infty}+1}=\frac{1}{\infty+1} \qquad [\because e^{\infty}=\infty]$$

$$=\frac{1}{\frac{1}{0}}=0$$

$$\text{RHL}=\lim_{x\to 0^+}\frac{e^{1/x}}{1+e^{1/x}}$$

$$=\lim_{h\to 0}\frac{e^{1/0+h}}{1+e^{1/0+h}}=\lim_{h\to 0}\frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h} + 1} = \frac{1}{e^{-\infty} + 1}$$

$$= \frac{1}{0+1} = 1 \qquad [\because e^{-\infty} = 0]$$

Hence, LHL $\neq$ RHL at $x = 0$.

So, $f(x)$ is discontinuous at $x = 0$.

Q. 9 $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$.

Sol. We have, $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$

At $x = 1$,

$$\text{HL} = \lim_{x \to 1^-} \frac{x^2}{2} = \lim_{h \to 0} \frac{(1-h)^2}{2}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right)$$

$$= \lim_{h \to 0} \left[2(1+h)^2 - 3(1+h) + \frac{3}{2}\right]$$

$$= \lim_{h \to 0} \left(2 + 2h^2 + 4h - 3 - 3h + \frac{3}{2}\right) = -1 + \frac{3}{2} = \frac{1}{2}$$

and $f(1) = \dfrac{1^2}{2} = \dfrac{1}{2}$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Q. 10 $f(x) = |x| + |x - 1|$ at $x = 1$.

Sol. We have, $f(x) = |x| + |x-1|$ at $x = 1$

At $x = 1$, $\text{LHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$

$$= \lim_{h \to 0} [|1 - h| + |1 - h - 1|] = 1 + 0 = 1$$

and $\text{RHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$

$$= \lim_{h \to 0} [|1 + h| + |1 + h - 1|] = 1 + 0 = 1$$

and $f(1) = |1| + |0| = 1$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Note *Every modulus function is a continuous function at any real point.*

Q. 11 $f(x) = \begin{cases} 3x - 8, \text{ if } x \le 5 \\ 2k, \quad \text{if } x > 5 \end{cases}$ at $x = 5$.

Sol. We have, $f(x) = \begin{cases} 3x - 8, \text{ if } x \le 5 \\ 2k, \quad \text{if } x > 5 \end{cases}$ at $x = 5$

Since, $f(x)$ is continuous at $x = 5$.

$\therefore$ $\text{LHL} = \text{RHL} = f(5)$

Now, $\text{LHL} = \lim_{x \to 5^-} (3x - 8) = \lim_{h \to 0} [3(5 - h) - 8]$

$= \lim_{h \to 0} [15 - 3h - 8] = 7$

$\text{RHL} = \lim_{x \to 5^+} 2k = \lim_{h \to 0} 2k = 2k = 7$ $[\because \text{LHL} = \text{RHL}]$

and $f(5) = 3 \times 5 - 8 = 7$

$\therefore$ $2k = 7 \Rightarrow k = \frac{7}{2}$

Q. 12 $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$ at $x = 2$.

Sol. We have, $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$ at $x = 2$

Since, $f(x)$ is continuous at $x = 2$.

$\therefore$ $\text{LHL} = \text{RHL} = f(2)$

At $x = 2$, $\lim_{x \to 2} \dfrac{2^x \cdot 2^2 - 2^4}{4^x - 4^2} = \lim_{x \to 2} \dfrac{4 \cdot (2^x - 4)}{(2^x)^2 - (4)^2}$

$= \lim_{x \to 2} \dfrac{4 \cdot (2^x - 4)}{(2^x - 4)(2^x + 4)}$ $[\because a^2 - b^2 = (a + b)(a - b)]$

$= \lim_{x \to 2} \dfrac{4}{2^x + 4} = \dfrac{4}{8} = \dfrac{1}{2}$

But $f(2) = k$

$\therefore$ $k = \frac{1}{2}$

Q.13 $f(x) = \begin{cases} \dfrac{\sqrt{1 + kx} - \sqrt{1 - kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x + 1}{x - 1}, & \text{if } 0 \le x \le 1 \end{cases}$ at $x = 0$.

Sol. We have, $f(x) = \begin{cases} \dfrac{\sqrt{1 + kx} - \sqrt{1 - kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x + 1}{x - 1}, & \text{if } 0 \le x \le 1 \end{cases}$ at $x = 0$.

$\therefore$ $$\text{LHL} = \lim_{x \to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

$$= \lim_{x \to 0^-} \left(\frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} \right) \cdot \left(\frac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}} \right)$$

$$= \lim_{x \to 0^-} \frac{1+kx-1+kx}{x[\sqrt{1+kx} + \sqrt{1-kx}]}$$

$$= \lim_{x \to 0^-} \frac{2kx}{x\sqrt{1+kx} + \sqrt{1-kx}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1+k(0-h)} + \sqrt{1-k(0-h)}}$$

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} \quad = \frac{2k}{2} = k$$

and $$f(0) = \frac{2 \times 0 + 1}{0 - 1} = -1$$

$\Rightarrow$ $k = -1$ $\quad [\because \text{LHL} = \text{RHL} = f(0)]$

Q. 14 $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have, $$f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases} \text{ at } x = 0$$

At $x = 0$, $$\text{LHL} = \lim_{x \to 0^-} \frac{1-\cos kx}{x \sin x} = \lim_{h \to 0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)}$$

$$= \lim_{h \to 0} \frac{1-\cos(-kh)}{-h\sin(-h)}$$

$$= \lim_{h \to 0} \frac{1-\cos kh}{h \sin h} \quad [\because \cos(-\theta) = \cos\theta, \ \sin(-\theta) = -\sin\theta]$$

$$= \lim_{h \to 0} \frac{1-1+2\sin^2 \frac{kh}{2}}{h \sin h} \quad \left[\because \cos\theta = 1 - 2\sin^2 \frac{\theta}{2}\right]$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{kh}{2}}{h \sin h}$$

$$= \lim_{h \to 0} \frac{2\sin \frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{\sin \frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{1}{\frac{\sin h}{h}} \cdot \frac{k^2 h/4}{h}$$

$$= \frac{2k^2}{4} = \frac{k^2}{2} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Also, $$f(0) = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1 \quad \text{p}$$

Q. 15 Prove that the function f defined by $f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$

remains discontinuous at $x=0$, regardless the choice of k.

Sol. We have, $$f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

At $x=0$, $$\text{LHL}=\lim_{x\to 0^-}\frac{x}{|x|+2x^2}=\lim_{h\to 0}\frac{(0-h)}{|0-h|+2(0-h)^2}$$

$$=\lim_{h\to 0}\frac{-h}{h+2h^2}=\lim_{h\to 0}\frac{-h}{h(1+2h)}=-1$$

$$\text{RHL}=\lim_{x\to 0^+}\frac{x}{|x|+2x^2}=\lim_{h\to 0}\frac{0+h}{|0+h|+2(0+h)^2}$$

$$=\lim_{h\to 0}\frac{h}{h+2h^2}=\lim_{h\to 0}\frac{h}{h(1+2h)}=1$$

and $f(0)=k$

Since, LHL $\neq$ RHL for any value of k.

Hence, $f(x)$ is discontinuous at $x=0$ regardless the choice of k.

Q. 16 Find the values of a and b such that the function f defined by

$$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

is a continuous function at $x=4$.

Sol. We have, $$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

At $x=4$, $$\text{LHL}=\lim_{x\to 4^-}\frac{x-4}{|x-4|}+a$$

$$=\lim_{h\to 0}\frac{4-h-4}{|4-h-4|}+a=\lim_{h\to 0}\frac{-h}{h}+a$$

$$=-1+a$$

$$\text{RHL}=\lim_{x\to 4^+}\frac{x-4}{|x-4|}+b$$

$$=\lim_{h\to 0}\frac{4+h-4}{|4+h-4|}+b=\lim_{h\to 0}\frac{h}{h}+b=1+b$$

$$f(4)=a+b\Rightarrow -1+a=1+b=a+b$$

$\Rightarrow$ $-1+a=a+b$ and $1+b=a+b$

$\therefore$ $b=-1$ and $a=1$

Q. 17 If the function $f(x) = \frac{1}{x+2}$, then find the points of discontinuity of the composite function $y = f\{f(x)\}$.

Sol. We have, $$f(x) = \frac{1}{x+2}$$

$\therefore$ $$y = f\{f(x)\}$$

$$= f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2} + 2}$$

$$= \frac{1}{1 + 2x + 4} \cdot (x+2) = \frac{(x+2)}{(2x+5)}$$

So, the function y will not be continuous at those points, where it is not defined as it is a rational function.

Therefore, $y = \frac{x+2}{(2x+5)}$ is not defined, when $2x + 5 = 0$

$\therefore$ $$x = \frac{-5}{2}$$

Hence, y is discontinuous at $x = \frac{-5}{2}$.

Q. 18 Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

Sol. We have, $$f(t) = \frac{1}{t^2 + t - 2} \text{ and } t = \frac{1}{x-1}$$

$\therefore$ $$f(t) = \frac{1}{\left(\frac{1}{x^2 + 1 - 2x}\right) + \left(\frac{1}{x-1}\right) - \frac{2}{1}}$$

$$= \frac{1}{\left(\frac{1 + x - 1 + [-2(x-1)^2]}{(x^2 + 1 - 2x)}\right)}$$

$$= \frac{x^2 + 1 - 2x}{x - 2x^2 - 2 + 4x}$$

$$= \frac{x^2 + 1 - 2x}{-2x^2 + 5x - 2}$$

$$= \frac{(x-1)^2}{-(2x^2 - 5x + 2)}$$

$$= \frac{(x-1)^2}{(2x-1)(2-x)}$$

So, $f(t)$ is discontinuous at $2x - 1 = 0 \Rightarrow x = 1/2$

and $2 - x = 0 \Rightarrow x = 2$.

Q. 19 Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

Sol. We have, $f(x) = |\sin x + \cos x|$ at $x = \pi$

Let $g(x) = \sin x + \cos x$

and $h(x) = |x|$

$\therefore$ $hog(x) = h[g(x)]$

$= h(\sin x + \cos x)$

$= |\sin x + \cos x|$

Since, $g(x) = \sin x + \cos x$ is a continuous function as it is forming with addition of two continuous functions sin x and cos x.

Also, $h(x) = |x|$ is also a continuous function. Since, we know that composite functions of two continuous functions is also a continuous function.

Hence, $f(x) = |\sin x + \cos x|$ is a continuous function everywhere.

So, $f(x)$ is continuous at $x = \pi$.

Q. 20 Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \text{ at } x = 2.$$

Thinking Process

We know that, a function f is differentiable at a point a in its domain, if both $Lf'(a)$ and $Rf'(a)$ are finite and equal, where $Lf'(c) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ and $Rf'(c) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

Sol. We have, $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x & \text{if } 2 \le x < 3 \end{cases}$ at $x = 2$.

At $x = 2$, $Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$

$= \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)2}{-h}$

$\{\because [a-h] = [a-1]$, where a is any positive number$\}$

$= \lim_{h \to 0} \frac{(2-h)(1) - 2}{-h}$

$= \lim_{h \to 0} \frac{2-h-2}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$

$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \frac{(2+h-1)(2+h) - (2-1)\cdot 2}{h}$

$= \lim_{h \to 0} \frac{(1+h)(2+h) - 2}{h}$

$= \lim_{h \to 0} \frac{2+h+2h+h^2 - 2}{h}$

$= \lim_{h \to 0} \frac{h^2 + 3h}{h} = \lim_{h \to 0} \frac{h(h+3)}{h} = 3$

$\therefore$ $Lf'(2) \ne Rf'(2)$

So, $f(x)$ is not differentiable at $x = 2$.

Q. 21 $f(x)=\begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

For differentiability at $x = 0$,

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \sin\left(\frac{1}{0-h}\right)}{0-h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{-1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} + h \sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= 0 \times [\text{an oscillating number between } -1 \text{and} 1] = 0$$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right)}{0+h} = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h}$$

$$= \lim_{h \to 0} h \sin(1/h)$$

$$= 0 \times [\text{an oscillating number between } -1 \text{ and } 1] = 0$$

$\because$ $Lf'(0) = Rf'(0)$

So, $f(x)$ is differentiable at $x = 0$.

Q. 22 $f(x)=\begin{cases} 1+x, & \text{if } x \leq 2 \\ 5-x, & \text{if } x > 2 \end{cases}$ at $x=2$.

Sol. We have, $f(x)=\begin{cases} 1+x, \text{ if } x \leq 2 \\ 5-x, \text{ if } x > 2 \end{cases}$ at $x = 2$.

For differentiability at $x = 2$,

$$Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(1+x) - (1+2)}{x - 2}$$

$$= \lim_{h \to 0} \frac{(1+2-h) - 3}{2-h-2} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(5-x) - 3}{x - 2}$$

$$= \lim_{h \to 0} \frac{5-(2+h)-3}{2+h-2}$$

$$= \lim_{h \to 0} \frac{5-2-h-3}{h} = \lim_{h \to 0} \frac{-h}{+h}$$

$$= -1$$

$\because$ $Lf'(2) \neq Rf'(2)$

So, $f(x)$ is not differentiable at $x = 2$.

Q. 23 Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

Sol. We have, $f(x) = |x - 5|$

$\therefore$ $f(x) = \begin{cases} -(x-5), & \text{if } x < 5 \\ x - 5, & \text{if } x \geq 5 \end{cases}$

For continuity at $x = 5$,

$$\text{LHL} = \lim_{x \to 5^-} (-x + 5)$$
$$= \lim_{h \to 0} [-(5-h) + 5] = \lim_{h \to 0} h = 0$$
$$\text{RHL} = \lim_{x \to 5^+} (x - 5)$$
$$= \lim_{h \to 0} (5 + h - 5) = \lim_{h \to 0} h = 0$$

$\therefore$ $f(5) = 5 - 5 = 0$

$\Rightarrow$ LHL = RHL = $f(5)$

Hence, $f(x)$ is continuous at $x = 5$.

Now, $$Lf'(5) = \lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5}$$
$$= \lim_{x \to 5^-} \frac{-x + 5 - 0}{x - 5} = -1$$
$$Rf'(5) = \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5}$$
$$= \lim_{x \to 5^+} \frac{x - 5 - 0}{x - 5} = 1$$

$\therefore$ $Lf'(5) \neq Rf'(5)$

So, $f(x) = |x - 5|$ is not differentiable at $x = 5$.

Q. 24 A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$, then prove that $f'(x) = 2f(x)$.

Sol. Let $f : R \to R$ satisfies the equation $f(x + y) = f(x) \cdot f(y)$, $\forall\, x, y \in R$, $f(x) \neq 0$.

Let $f(x)$ is differentiable at $x = 0$ and $f'(0) = 2$.

$\Rightarrow$ $$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$\Rightarrow$ $$2 = \lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

$\Rightarrow$ $$2 = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{0 + h}$$

$\Rightarrow$ $$2 = \lim_{h \to 0} \frac{f(0) \cdot f(h) - f(0)}{h}$$

$\Rightarrow$ $$2 = \lim_{h \to 0} \frac{f(0)\,[f(h) - 1]}{h} \quad [\because f(0) = f(h)] \ldots \text{(i)}$$

Also, $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h} \quad [\because f(x + y) = f(x) \cdot f(y)]$$
$$= \lim_{h \to 0} \frac{f(x)\,[f(h) - 1]}{h} = 2f(x) \quad \text{[using Eq. (i)]}$$

$\therefore$ $f'(x) = 2f(x)$

Q. 25 $2^{\cos^2 x}$

Sol. Let $$y = 2^{\cos^2 x}$$

$\therefore$ $$\log y = \log 2^{\cos^2 x} = \cos^2 x \cdot \log 2$$

On differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y . \frac{dy}{dx} = \frac{d}{dx}\log 2 \cdot \cos^2 x$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2 \frac{d}{dx}(\cos x)^2$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2 \cdot [2\cos x]\cdot\frac{d}{dx}\cos x$$

$$= \log 2 \cdot 2\cos x \cdot (-\sin x)$$

$$= \log 2 \cdot [-(\sin 2x)]$$

$\therefore$ $$\frac{dy}{dx} = -y \cdot \log 2 (\sin 2x)$$

$$= -2^{\cos^2 x}\cdot \log 2 (\sin 2x)$$

Q. 26 $\dfrac{8^x}{x^8}$

Sol. Let $$y = \frac{8^x}{x^8} \Rightarrow \log y = \log \frac{8^x}{x^8}$$

$\Rightarrow$ $$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[\log 8^x - \log x^8]$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[x \cdot \log 8 - 8 \cdot \log x]$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8 \cdot 1 - 8 \cdot \frac{1}{x}$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$\therefore$ $$\frac{dy}{dx} = y\left(\log 8 - \frac{8}{x}\right) = \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right)$$

Q. 27 $\log (x + \sqrt{x^2 + a})$

Sol. Let $$y = \log (x + \sqrt{x^2 + a})$$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx}\log (x + \sqrt{x^2 + a})$$

$$= \frac{1}{(x + \sqrt{x^2 + a})} . \frac{d}{dx}[x + \sqrt{x^2 + a}]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\left[1 + \frac{1}{2}(x^2 + a)^{-1/2}\cdot 2x\right]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\cdot\left(1 + \frac{x}{\sqrt{x^2 + a}}\right)$$

$$= \frac{(\sqrt{x^2 + a} + x)}{(x + \sqrt{x^2 + a})(\sqrt{x^2 + a})} = \frac{1}{(\sqrt{x^2 + a})}$$

Q. 28 $\log[\log\ (\log x^5)]$

Sol. Let $\quad y = \log[\log(\log x^5)]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log\ (\log\ \log x^5)]$$

$$= \frac{1}{\log\log x^5} \cdot \frac{d}{dx}(\log \cdot \log x^5)$$

$$= \frac{1}{\log\log x^5} \cdot \left(\frac{1}{\log x^5}\right) \cdot \frac{d}{dx}\log x^5$$

$$= \frac{1}{\log\log x^5} \cdot \frac{1}{\log x^5} \cdot \frac{d}{dx}(5\log x) = \frac{5}{x \cdot \log(\log x^5) \cdot \log(x^5)}$$

Q. 29 $\sin\sqrt{x} + \cos^2\sqrt{x}$

Sol. Let $\quad y = \sin\sqrt{x} + (\cos\sqrt{x})^2$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^{1/2}) + \frac{d}{dx}[\cos(x^{1/2})]^2$$

$$= \cos x^{1/2} \cdot \frac{d}{dx}x^{1/2} + 2\cos(x^{1/2})\ \frac{d}{dx}[\cos(x^{1/2})]$$

$$= \cos(x^{1/2})\ \frac{1}{2}x^{-1/2} + 2 \cdot \cos(x^{1/2}) \cdot \left[-\sin(x^{1/2}) \cdot \frac{d}{dx}x^{1/2}\right]$$

$$= \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}[-2\cos(x^{1/2})] \cdot \sin x^{1/2} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}[\cos(\sqrt{x}) - \ \sin(2\sqrt{x})]$$

Q. 30 $\sin^n(ax^2 + bx + c)$

Sol. Let $\quad y = \sin^n(ax^2 + bx + c)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\sin(ax^2 + bx + c)]^n$$

$$= n \cdot [\sin(ax^2 + bx + c)]^{n-1} \cdot \frac{d}{dx}\sin(ax^2 + bx + c)$$

$$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot \frac{d}{dx}(ax^2 + bx + c)$$

$$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot (2ax + b)$$

$$= n \cdot (2ax + b) \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c)$$

Q. 31 $\cos(\tan\sqrt{x+1})$

Sol. Let $\quad y = \cos(\tan\sqrt{x+1})$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\cos(\tan\sqrt{x+1}) = -\sin(\tan\sqrt{x+1}) \cdot \frac{d}{dx}(\tan\sqrt{x+1})$$

$$= -\sin(\tan\sqrt{x+1}) \cdot \sec^2\sqrt{x+1} \cdot \frac{d}{dx}(x+1)^{1/2} \qquad \left[\because \frac{d}{dx}(\tan x) = \sec^2 x\right]$$

$$= -\sin(\tan\sqrt{x+1}) \cdot (\sec\sqrt{x+1})^2 \cdot \frac{1}{2}(x+1)^{-1/2} \cdot \frac{d}{dx}(x+1)$$

$$= \frac{-1}{2\sqrt{x+1}} \cdot \sin(\tan\sqrt{x+1}) \cdot \sec^2(\sqrt{x+1})$$

Q. 32 $\sin x^2 + \sin^2 x + \sin^2 (x^2)$

Sol. Let $y = \sin x^2 + \sin^2 x + \sin^2 (x^2)$

$\therefore$ $\frac{dy}{dx} = \frac{d}{dx}\sin(x^2) + \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\sin x^2)^2$

$= \cos(x^2)\frac{d}{dx}(x^2) + 2\sin x \cdot \frac{d}{dx}\sin x + 2\sin x^2 \cdot \frac{d}{dx}\sin x^2$

$= \cos x^2 2x + 2 \cdot \sin x \cdot \cos x + 2\sin x^2 \cos x^2 \cdot \frac{d}{dx} x^2$

$= 2x\cos(x)^2 + 2 \cdot \sin x \cdot \cos x + 2\sin x^2 \cdot \cos x^2 \cdot 2x$

$= 2x\cos(x)^2 + \sin 2x + \sin 2(x)^2 \cdot 2x$

$= 2x\cos(x^2) + 2x \cdot \sin 2(x^2) + \sin 2x$

Q. 33 $\sin^{-1}\frac{1}{\sqrt{x+1}}$

Sol. Let $y = \sin^{-1}\frac{1}{\sqrt{x+1}}$

$\therefore$ $\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\frac{1}{\sqrt{x+1}}$

$= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}} \cdot \frac{d}{dx}\frac{1}{(x+1)^{1/2}}$ $\left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$

$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}} \cdot \frac{d}{dx} \cdot (x+1)^{-1/2}$

$= \sqrt{\frac{x+1}{x}} \cdot \frac{-1}{2}(x+1)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(x+1)$

$= \frac{(x+1)^{1/2}}{x^{1/2}} \cdot \left(-\frac{1}{2}\right)(x+1)^{-3/2} = \frac{-1}{2\sqrt{x}} \cdot \left(\frac{1}{x+1}\right)$

Q. 34 $(\sin x)^{\cos x}$

Sol. Let $y = (\sin x)^{\cos x}$

$\Rightarrow$ $\log y = \log(\sin x)^{\cos x} = \cos x \log \sin x$

$\therefore$ $\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x \cdot \log \sin x)$

$\Rightarrow$ $\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \frac{d}{dx}\log \sin x + \log \sin x \cdot \frac{d}{dx}\cos x$

$= \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}\sin x + \log \sin x \cdot (-\sin x)$

$= \cot x \cdot \cos x - \log(\sin x) \cdot \sin x$ $\left[\because \cot x = \frac{\cos x}{\sin x}\right]$

$\therefore$ $\frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x)\right]$

$= \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x)\right]$

Q. 35 $\sin^m x \cdot \cos^n x$

Sol. Let $y = \sin^m x \cdot \cos^n x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(\sin x)^m \cdot (\cos x)^n]$$

$$= (\sin x)^m \cdot \frac{d}{dx}(\cos x)^n + (\cos x)^n \cdot \frac{d}{dx}(\sin x)^m$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1} \cdot \frac{d}{dx}\cos x + (\cos x)^n \, m(\sin x)^{m-1} \cdot \frac{d}{dx}\sin x$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1} \, (-\sin x) + (\cos x)^n \cdot m(\sin x)^{m-1}\cos x$$

$$= -n\sin^m x \cdot \cos^{n-1} x \cdot (\sin x) + m\cos^n x \cdot \sin^{m-1} x \cdot \cos x$$

$$= -n \cdot \sin^m x \cdot \sin x \cdot \cos^n x \cdot \frac{1}{\cos x} + m \cdot \sin^m x . \frac{1}{\sin x} \cdot \cos^n x \cdot \cos x$$

$$= -n \cdot \sin^m x \cdot \cos^n x \cdot \tan x + m\sin^m x \cdot \cos^n x \cdot \cot x$$

$$= \sin^m x \cdot \cos^n x \, [-n\tan x + m\cot x]$$

Q. 36 $(x+1)^2(x+2)^3(x+3)^4$

Sol. Let $y = (x+1)^2 (x+2)^3 (x+3)^4$

$$\therefore \quad \log y = \log\{(x+1)^2 \cdot (x+2)^3 (x+3)^4\}$$

$$= \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

and
$$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[2\log(x+1)] + \frac{d}{dx}[3\log(x+2)] + \frac{d}{dx}[4\log(x+3)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{(x+1)} \cdot \frac{d}{dx}(x+1) + 3 \cdot \frac{1}{(x+2)} \cdot \frac{d}{dx}(x+2)$$

$$+ 4 \cdot \frac{1}{(x+3)} \cdot \frac{d}{dx}(x+3) \qquad \left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4 \left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4 \left[\frac{2(x+2)(x+3) + 3(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= \frac{(x+1)^2 (x+2)^3 (x+3)^4}{(x+1)(x+2)(x+3)} [2(x^2+5x+6) + 3(x^2+4x+3) + 4(x^2+3x+2)]$$

$$= (x+1)(x+2)^2(x+3)^3 [2x^2 + 10x + 12 + 3x^2 + 12x + 9 + 4x^2 + 12x + 8]$$

$$= (x+1)(x+2)^2(x+3)^3 [9x^2 + 34x + 29]$$

Q. 37 $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$

Sol. Let $y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\frac{-1}{\sqrt{1-\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)^2}}\cdot\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$\left[\because \frac{d}{dx}(\cos x)=-\frac{1}{\sqrt{1-x^2}}\right]$$

$$=\frac{-1}{\sqrt{4-\frac{(\sin^2 x+\cos^2 x+2\sin x\cdot\cos x)}{2}}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$=\frac{-1\cdot\sqrt{2}}{\sqrt{1-\sin 2x}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$[\because 1-\sin 2x=(\cos x-\sin x)^2=\cos^2 x+\sin^2 x-2\sin x\cos x]$$

$$=\frac{-1(\cos x-\sin x)}{(\cos x-\sin x)}=-1$$

Q. 38 $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}, -\frac{\pi}{4}<x<\frac{\pi}{4}$

Sol. Let $y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

$$\therefore \quad \frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$=\frac{1}{1+\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)^2}}\cdot\frac{d}{dx}\left[\frac{1-\cos x}{1+\cos x}\right]^{1/2} \qquad \left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$=\frac{1}{1+\frac{1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{1-\cos x}{1+\cos x}\right]^{-1/2}\cdot\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$=\frac{1}{\frac{1+\cos x+1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{(1-\cos x)}{(1+\cos x)}\cdot\frac{(1-\cos x)}{(1-\cos x)}\right]^{-1/2}$$

$$\cdot\frac{(1+\cos x)\cdot\sin x+(1-\cos x)\cdot\sin x}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2} \left[\frac{(1-\cos x)^2}{\sin x} \right]^{-1/2} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2} \cdot \frac{\sin x}{(1-\cos x)} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin^2 x}{4(1+\cos x)(1-\cos x)} = \frac{1}{2} \cdot \frac{\sin^2 x}{(1-\cos^2 x)}$$

$$= \frac{1}{2} \cdot \frac{\sin^2 x}{\sin^2 x} = \frac{1}{2}$$

Alternate Method

Let $$y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{1-1+2\sin^2 \frac{x}{2}}{1+2\cos^2 \frac{x}{2} - 1}}\right) \qquad \left[\because \cos x = 1 - 2\sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2} - 1\right]$$

$$= \tan^{-1}\left(\tan \frac{x}{2}\right) = \frac{x}{2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

Q. 39 $\tan^{-1}(\sec x + \tan x), \frac{-\pi}{2} < x < \frac{\pi}{2}$

Sol. Let $$y = \tan^{-1}(\sec x + \tan x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx} \tan^{-1}(\sec x + \tan x)$$

$$= \frac{1}{1+(\sec x + \tan x)^2} \cdot \frac{d}{dx}(\sec x + \tan x) \qquad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \cdot \tan x} \cdot [\sec x \cdot \tan x + \sec^2 x]$$

$$= \frac{1}{(\sec^2 x + \sec^2 x + 2\sec x \cdot \tan x)} \cdot \sec x \cdot (\sec x + \tan x)$$

$$= \frac{1}{2\sec x(\tan x + \sec x)} \cdot \sec x(\sec x + \tan x) = \frac{1}{2}$$

Q. 40 $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), \frac{-\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$.

Sol. Let $$y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$$

$$= \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

$$= \tan^{-1}\frac{a}{b} - \tan^{-1}\tan x \qquad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\frac{a}{b} - x$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x)$$

$$= 0 - 1 \quad \left[\because \frac{d}{dx}\left(\frac{a}{b}\right) = 0\right]$$

$$= -1$$

Q. 41 $\sec^{-1}\left(\frac{1}{4x^3 - 3x}\right), 0 < x < \frac{1}{\sqrt{2}}$

Sol. Let $y = \sec^{-1}\left(\frac{1}{4x^3 - 3x}\right)$ ...(i)

On putting $x = \cos\theta$ in Eq. (i), we get

$$y = \sec^{-1}\frac{1}{4\cos^3\theta - 3\cos\theta}$$

$$= \sec^{-1}\frac{1}{\cos 3\theta}$$

$$= \sec^{-1}(\sec 3\theta) = 3\theta$$

$$= 3\cos^{-1}x \quad [\because \theta = \cos^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(3\cos^{-1}x)$$

$$= 3\cdot\frac{-1}{\sqrt{1-x^2}}$$

Q. 42 $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), \frac{-1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$

Sol. Let $y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

Put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$$\therefore \quad y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right] \quad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$= 3\tan^{-1}\frac{x}{a} \quad \left[\because \theta = \tan^{-1}\frac{x}{a}\right]$$

$$\therefore \quad \frac{dy}{dx} = 3\cdot\frac{d}{dx}\tan^{-1}\frac{x}{a} = 3\cdot\left[\frac{1}{1+\frac{x^2}{a^2}}\right]\cdot\frac{d}{dx}\cdot\left(\frac{x}{a}\right)$$

$$= 3\cdot\frac{a^2}{a^2+x^2}\cdot\frac{1}{a} = \frac{3a}{a^2+x^2}$$

Q. 43 $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right], -1 < x < 1, x \neq 0$

Sol. Let $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Put $x^2 = \cos 2\theta$

$\therefore$ $y = \tan^{-1}\left(\dfrac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right) = \tan^{-1}\left[\dfrac{\sqrt{2}(\cos\theta+\sin\theta)}{\sqrt{2}(\cos\theta-\sin\theta)}\right]$

$= \tan^{-1}\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\dfrac{\dfrac{\cos\theta+\sin\theta}{\cos\theta}}{\dfrac{\cos\theta-\sin\theta}{\cos\theta}}\right)$

$= \tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$= \tan^{-1}\tan\left(\dfrac{\pi}{4}+\theta\right)$ $\quad\left[\because \tan(a+b) = \dfrac{\tan a + \tan b}{1-\tan a \cdot \tan b}\right]$

$= \dfrac{\pi}{4}+\theta = \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2$ $\quad\left[\because 2\theta = \cos^{-1}x^2 \Rightarrow \theta = \dfrac{1}{2}\cos^{-1}x^2\right]$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{4}\right)+\dfrac{d}{dx}\left(\dfrac{1}{2}\cos^{-1}x^2\right)$

$= 0+\dfrac{1}{2}\cdot\dfrac{-1}{\sqrt{1-x^4}}\cdot\dfrac{d}{dx}x^2 = \dfrac{1}{2}\cdot\dfrac{-2x}{\sqrt{1-x^4}} = \dfrac{-x}{\sqrt{1-x^4}}$

Find $\dfrac{dy}{dx}$ of each of the functions expressed in parametric form.

Q. 44 $x = t+\dfrac{1}{t},\ y = t-\dfrac{1}{t}$

Sol. $\because$ $x = t+\dfrac{1}{t}$ and $y = t-\dfrac{1}{t}$

$\therefore$ $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(t+\dfrac{1}{t}\right)$ and $\dfrac{dy}{dt} = \dfrac{d}{dt}\left(t-\dfrac{1}{t}\right)$

$\Rightarrow$ $\dfrac{dx}{dt} = 1+(-1)t^{-2}$ and $\dfrac{dy}{dt} = 1-(-1)t^{-2}$

$\Rightarrow$ $\dfrac{dx}{dt} = 1-\dfrac{1}{t^2}$ and $\dfrac{dy}{dt} = 1+\dfrac{1}{t^2}$

$\Rightarrow$ $\dfrac{dx}{dt} = \dfrac{t^2-1}{t^2}$ and $\dfrac{dy}{dt} = \dfrac{t^2+1}{t^2}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t^2+1/t^2}{t^2-1/t^2} = \dfrac{t^2+1}{t^2-1}$

Q. 45 $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Sol. $\because$ $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

$\therefore$ $$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[e^{\theta}\cdot\left(\theta + \frac{1}{\theta}\right)\right]$$

$$= e^{\theta}\cdot\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\cdot\frac{d}{d\theta}e^{\theta}$$

$$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right)e^{\theta}$$

$$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right)$$

$$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right) \quad \text{...(i)}$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[e^{-\theta}\cdot\left(\theta - \frac{1}{\theta}\right)\right]$$

$$= e^{-\theta}\cdot\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \frac{d}{d\theta}e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$

$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}\cdot\frac{d}{d\theta}(-\theta)$$

$$= e^{-\theta}\left[\frac{\theta^2 + 1}{\theta^2} - \frac{\theta^2 - 1}{\theta}\right] = e^{-\theta}\left[\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right] \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

Q. 46 $x = 3\cos\theta - 2\cos^3\theta, \; y = 3\sin\theta - 2\sin^3\theta$

Sol. $\because$ $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$

$\therefore$ $$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos\theta) - \frac{d}{d\theta}(2\cos^3\theta)$$

$$= 3\cdot(-\sin\theta) - 2\cdot 3\cos^2\theta\cdot\frac{d}{d\theta}\cdot\cos\theta$$

$$= -3\sin\theta + 6\cos^2\theta\sin\theta$$

and $$\frac{dy}{d\theta} = 3\cos A - 2\cdot 3\sin^2\theta\cdot\frac{d}{d\theta}\cdot\sin\theta$$

$$= 3\cos\theta - 6\sin^2\theta\cdot\cos\theta$$

Now, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$$

$$= \frac{3\cos\theta(1 - 2\sin^2\theta)}{3\sin\theta(-1 + 2\cos^2\theta)} = \cot\theta\cdot\frac{\cos 2\theta}{\cos 2\theta} = \cot\theta$$

Q. 47 $\sin x = \dfrac{2t}{1+t^2}$, $\tan y = \dfrac{2t}{1-t^2}$

Sol. $\because$

$$\sin x = \frac{2t}{1+t^2} \qquad \ldots(i)$$

and

$$\tan y = \frac{2t}{1-t^2} \qquad \ldots(ii)$$

$$\therefore \quad \frac{d}{dx}\sin x \cdot \frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\Rightarrow \quad \cos x \frac{dx}{dt} = \frac{(1+t^2)\cdot \frac{d}{dt}(2t) - (2t)\cdot \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2(1+t^2) - 2t\cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \qquad \ldots(iii)$$

Also,

$$\frac{d}{dy}\tan y\cdot\frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y\,\frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t\cdot\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}\cdot\frac{1}{\sec^2 y}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \qquad \ldots(iv)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/1+t^2}{2/1+t^2} = 1$$

[from Eqs. (iii) and (iv)]

Q. 48 $x = \dfrac{1+\log t}{t^2}$, $y = \dfrac{3+2\log t}{t}$

Sol. $\because$

$$x = \frac{1+\log t}{t^2} \text{ and } y = \frac{3+2\log t}{t}$$

$$\therefore \quad \frac{dx}{dt} = \frac{t^2.\frac{d}{dt}(1+\log t) - (1+\log t).\frac{d}{dt}t^2}{(t^2)^2}$$

$$= \frac{t^2 \cdot \frac{1}{t} - (1 + \log t) \cdot 2t}{t^4} = \frac{t - (1 + \log t) \cdot 2t}{t^4}$$

$$= \frac{t}{t^4} [1 - 2 (1 + \log t) = \frac{-1 - 2 \log t}{t^3} \quad \text{... (i)}$$

and $$\frac{dy}{dt} = \frac{t \cdot \frac{d}{dt} (3 + 2 \log t) - (3 + 2 \log t) \cdot \frac{d}{dt} t}{t^2}$$

$$= \frac{t \cdot 2 \cdot \frac{1}{t} - (3 + 2 \log t) \cdot 1}{t^2}$$

$$= \frac{2 - 3 - 2 \log t}{t^2} = \frac{-1 - 2 \log t}{t^2} \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1 - 2 \log t / t^2}{-1 - 2 \log t / t^3} = t$$

Q. 49 If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

Sol. $\because$ $$x = e^{\cos 2t} \text{ and } y = e^{\sin 2t}$$

$\therefore$ $$\frac{dx}{dt} = \frac{d}{dt} e^{\cos 2t} = e^{\cos 2t} \cdot \frac{d}{dt} \cos 2t$$

$$= e^{\cos 2t} \cdot (-\sin 2t) \cdot \frac{d}{dt} (2t)$$

$$\frac{dx}{dt} = -2 e^{\cos 2t} \cdot \sin 2t \quad \text{...(i)}$$

and $$\frac{dy}{dt} = \frac{d}{dt} e^{\sin 2t} = e^{\sin 2t} \cdot \frac{d}{dt} \sin 2t$$

$$= e^{\sin 2t} \cos 2t \cdot \frac{d}{dt} 2t$$

$$= 2e^{\sin 2t} \cdot \cos 2t \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t} \cdot \cos 2t}{-2e^{\cos 2t} \cdot \sin 2t}$$

$$= \frac{e^{\sin 2t} \cdot \cos 2t}{e^{\cos 2t} \cdot \sin 2t} \quad \text{...(iii)}$$

We know that, $$\log x = \cos 2t \cdot \log e = \cos 2t \quad \text{...(iv)}$$

and $$\log y = \sin 2t \cdot \log e = \sin 2t \quad \text{...(v)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{-y \log x}{x \log y}$$

[using Eqs. (iv) and (v) in Eq. (iii) and $x = e^{\cos 2t}$, $y = e^{\sin 2t}$]

Hence proved.

Q. 50 If $x = a \sin 2t (1 + \cos 2t)$ and $y = b \cos 2t (1 - \cos 2t)$, then show that $\left(\frac{dy}{dx}\right)_{t = \pi/4} = \frac{b}{a}$.

Sol. $\because$ $$x = a \sin 2t (1 + \cos 2t) \text{ and } y = b \cos 2t (1 - \cos 2t)$$

$\therefore$ $$\frac{dx}{dt} = a \left[\sin 2t \cdot \frac{d}{dt} (1 + \cos 2t) + (1 + \cos 2t) \cdot \frac{d}{dt} \sin 2t\right]$$

$$= a\left[\sin 2t\cdot(-\sin 2t)\cdot\frac{d}{dt}2t + (1+\cos 2t)\cdot\cos 2t\cdot\frac{d}{dt}2t\right]$$

$$= -2a\sin^2 2t + 2a\cos 2t\,(1+\cos 2t)$$

$\Rightarrow$ $$\frac{dx}{dt} = -2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)] \quad \text{...(i)}$$

and $$\frac{dy}{dt} = b\left[\cos 2t\cdot\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\cdot\frac{d}{dt}\cos 2t\right]$$

$$= b\left[\cos 2t\cdot(\sin 2t)\frac{d}{dt}2t + (1-\cos 2t)(-\sin 2t)\cdot\frac{d}{dt}2t\right]$$

$$= b\,[2\sin 2t\cdot\cos 2t + 2(1-\cos 2t)(-\sin 2t)]$$

$$= 2b\,[\sin 2t\cdot\cos 2t - (1-\cos 2t)\sin 2t] \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2b\,[-\sin 2t\cdot\cos 2t + (1-\cos 2t)\sin 2t]}{-2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)]}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{t=\pi/4} = \frac{b}{a}\,\frac{\left[-\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \left(1-\cos\frac{\pi}{2}\right)\sin\frac{\pi}{2}\right]}{\left[\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\right)\right]}$$

$$= \frac{b}{a}\cdot\frac{(0+1)}{(1-0)} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{b}{a}$$

Hence proved.

Q. 51 If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, then find $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{3}$.

Sol. $\because$ $$x = 3\sin t - \sin 3t \text{ and } y = 3\cos t - \cos 3t$$

$\therefore$ $$\frac{dx}{dt} = 3\cdot\frac{d}{dt}\sin t - \frac{d}{dt}\sin 3t$$

$$= 3\cos t - \cos 3t\cdot\frac{d}{dt}3t = 3\cos t - 3\cos 3t \quad \text{...(i)}$$

and $$\frac{dy}{dt} = 3\cdot\frac{d}{dt}\cos t - \frac{d}{dt}\cos 3t$$

$$= -3\sin t + \sin 3t\cdot\frac{d}{dt}3t$$

$$\frac{dy}{dt} = 3\sin 3t - 3t\sin t \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3(\sin 3t - \sin t)}{3(\cos t - \cos 3t)}$$

Now, $$\left(\frac{dy}{dx}\right)_{t=\pi/3} = \frac{\sin\frac{3\pi}{3} - \sin\frac{\pi}{3}}{\left(\cos\frac{\pi}{3} - \cos 3\frac{\pi}{3}\right)} = \frac{0-\sqrt{3}/2}{\frac{1}{2}-(-1)}$$

$$= \frac{-\sqrt{3}/2}{3/2} = \frac{-\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

Q. 52 Differentiate $\dfrac{x}{\sin x}$ w.r.t. $\sin x$.

Sol. Let $u = \dfrac{x}{\sin x}$ and $v = \sin x$

$$\therefore \quad \frac{du}{dx} = \frac{\sin x \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx} \sin x}{(\sin x)^2}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x} \quad \text{...(i)}$$

and $$\frac{dv}{dx} = \frac{d}{dx} \sin x = \cos x \quad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\sin x - x \cos x / \sin^2 x}{\cos x}$$

$$= \frac{\sin x - x \cos x}{\sin^2 x \cos x} = \frac{\frac{\sin x - x \cos x}{\cos x}}{\frac{\sin^2 x \cos x}{\cos x}}$$

[dividing by $\cos x$ in both numerator and denominator]

$$= \frac{\tan x - x}{\sin^2 x}$$

Q. 53 Differentiate $\tan^{-1} \dfrac{\sqrt{1+x^2} - 1}{x}$ w.r.t. $\tan^{-1} x$, when $x \neq 0$.

Sol. Let $u = \tan^{-1} \dfrac{\sqrt{1+x^2} - 1}{x}$ and $v = \tan^{-1} x$

$\therefore \quad x = \tan \theta$

$$\Rightarrow \quad u = \tan^{-1} \frac{\sqrt{1 + \tan^2 \theta} - 1}{\tan \theta}$$

$$= \tan^{-1} \frac{(\sec \theta - 1) \cos \theta}{\sin \theta}$$

$$= \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right)$$

$$= \tan^{-1} \left[\frac{1 - 1 + 2\sin^2 \theta/2}{2\sin \theta/2 \cdot \cos \theta/2} \right] \quad [\because \cos \theta = 1 - 2\sin^2 \theta]$$

$$= \tan^{-1} \left[\tan \frac{\theta}{2} \right]$$

$$= \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2} \frac{d}{dx} \tan^{-1} x = \frac{1}{2} \cdot \frac{1}{1 + x^2} \quad \text{...(i)}$$

and $$\frac{dv}{dx} = \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \quad \text{...(ii)}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{1/2\,(1 + x^2)}{1/(1 + x^2)} = \frac{(1 + x^2)}{2(1 + x^2)} = \frac{1}{2}$$

Find $\frac{dy}{dx}$ when x and y are connected by the relation given.

Q. 54 $\sin(xy) + \frac{x}{y} = x^2 - y$

Sol. We have, $$\sin(xy) + \frac{x}{y} = x^2 - y$$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}x^2 - \frac{d}{dx}y$$

$$\Rightarrow \quad \cos xy \cdot \frac{d}{dx}(xy) + \frac{y\frac{d}{dx}x - x\cdot\frac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy \cdot \left[x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}\cdot x\right] + \frac{y - x\frac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad x\cos xy \cdot \frac{dy}{dx} + y\cos xy + \frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[x\cos xy - \frac{x}{y^2} + 1\right] = 2x - y\cos xy - \frac{y}{y^2}$$

$$\therefore \quad \frac{dy}{dx} = \left[\frac{2xy - y^2\cos xy - 1}{y}\right]\left[\frac{y^2}{xy^2\cos xy - x + y^2}\right]$$

$$= \frac{(2xy - y^2\cos xy - 1)\,y}{(xy^2\cos xy - x + y^2)}$$

Q. 55 $\sec(x + y) = xy$

Sol. We have, $\sec(x + y) = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\sec(x + y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad \sec(x + y)\cdot\tan(x + y)\cdot\frac{d}{dx}(x + y) = x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad \sec(x + y)\cdot\tan(x + y)\cdot\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \sec(x + y)\tan(x + y) + \sec(x + y)\cdot\tan(x + y)\,.\,\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \frac{dy}{dx}[\sec(x + y)\cdot\tan(x + y) - x] = y - \sec(x + y)\cdot\tan(x + y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y - \sec(x + y)\cdot\tan(x + y)}{\sec(x + y)\cdot\tan(x + y) - x}$$

Q. 56 $\tan^{-1}(x^2 + y^2) = a$

Sol. We have, $\tan^{-1}(x^2 + y^2) = a$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\tan^{-1}(x^2 + y^2) = \frac{d}{dx}(a)$$

$$\Rightarrow \quad \frac{1}{1 + (x^2 + y^2)^2} \cdot \frac{d}{dx}(x^2 + y^2) = 0$$

$$\Rightarrow \quad 2x + \frac{d}{dy} y^2 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y \cdot \frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Q. 57 $(x^2 + y^2)^2 = xy$

Sol. We have, $(x^2 + y^2)^2 = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x^2 + y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad 2(x^2 + y^2) \cdot \frac{d}{dx}(x^2 + y^2) = x \cdot \frac{d}{dx} y + y \cdot \frac{d}{dx} x$$

$$\Rightarrow \quad 2(x^2 + y^2) \cdot \left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad 2x^2 \cdot 2x + 2x^2 \cdot 2y\frac{dy}{dx} + 2y^2 \cdot 2x + 2y^2 \cdot 2y\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \frac{dy}{dx}[4x^2y + 4y^3 - x] = y - 4x^3 - 4xy^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{(y - 4x^3 - 4xy^2)}{(4x^2y + 4y^3 - x)}$$

Q. 58 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$.

Sol. We have, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$\Rightarrow \quad 2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + b \cdot 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow \quad \frac{dy}{dx}[2hx + 2by + 2f] = -2ax - 2hy - 2g$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2(ax + hy + g)}{2(hx + by + f)}$$

$$= \frac{-(ax + hy + g)}{(hx + by + f)} \quad \text{...(ii)}$$

Now, differentiating Eq. (i) w.r.t. y, we get

$$\frac{d}{dy}(ax^2)+\frac{d}{dy}(2hxy)+\frac{d}{dy}(by^2)+\frac{d}{dy}(2gx)+\frac{d}{dy}(2fy)+\frac{d}{dy}(c)=0$$

$$\Rightarrow \quad a\cdot 2x\cdot\frac{dx}{dy}+2h\cdot\left(x\cdot\frac{d}{dy}y+y\cdot\frac{d}{dy}x\right)+b\cdot 2y+2g\cdot\frac{dx}{dy}+2f+0=0$$

$$\Rightarrow \quad \frac{dx}{dy}[2ax+2hy+2g]=-2hx-2by-2f$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2(hx+by+f)}{2(ax+hy+g)}=\frac{-(hx+by+f)}{(ax+hy+g)} \quad \text{... (iii)}$$

$$\therefore \quad \frac{dy}{dx}\cdot\frac{dx}{dy}=\frac{-(ax+hy+g)}{(hx+by+f)}\cdot\frac{-(hx+by+f)}{(ax+hy+g)} \quad \text{[using Eqs. (ii) and (iii)]}$$

$$=1=\text{RHS}$$

Hence proved.

Q. 59 If $x=e^{x/y}$, then prove that $\dfrac{dy}{dx}=\dfrac{x-y}{x\log x}$.

Sol. We have, $x=e^{x/y}$

$$\therefore \quad \frac{d}{dx}x=\frac{d}{dx}e^{x/y}$$

$$\Rightarrow \quad 1=e^{x/y}\cdot\frac{d}{dx}(x/y)$$

$$\Rightarrow \quad 1=e^{x/y}\cdot\left[\frac{y\cdot 1-x\cdot dy/dx}{y^2}\right]$$

$$\Rightarrow \quad y^2=y\cdot e^{x/y}-x\cdot\frac{dy}{dx}\cdot e^{x/y}$$

$$\Rightarrow \quad x\cdot\frac{dy}{dx}\cdot e^{x/y}=ye^{x/y}-y^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{y(e^{x/y}-y)}{x\cdot e^{x/y}}$$

$$=\frac{(e^{x/y}-y)}{e^{x/y}\cdot\frac{x}{y}} \quad \left[\because x=e^{x/y}\Rightarrow \log x=\frac{x}{y}\right]$$

$$=\frac{x-y}{x\cdot\log x}$$

Hence proved.

Q. 60 If $y^x=e^{y-x}$, then prove that $\dfrac{dy}{dx}=\dfrac{(1+\log y)^2}{\log y}$.

Sol. We have, $y^x=e^{y-x}$

$$\Rightarrow \quad \log y^x=\log e^{y-x}$$

$$\Rightarrow \quad x\log y=y-x\cdot\log_e=(y-x) \quad [\because \log_e=1]$$

$$\Rightarrow \quad \log y=\frac{(y-x)}{x} \quad \text{...(i)}$$

Now, differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y\cdot\frac{dy}{dx}=\frac{d}{dx}\frac{(y-x)}{x}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{x\cdot\frac{d}{dx}(y-x)-(y-x)\cdot\frac{d}{dx}\cdot x}{x^2}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{x\left(\frac{dy}{dx}-1\right)-(y-x)}{x^2}$$

$$\Rightarrow \quad \frac{x^2}{y}\cdot\frac{dy}{dx} = x\frac{dy}{dx}-x-y+x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x^2}{y}-x\right) = -y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2}{x^2-xy} = \frac{-y^2}{x(x-y)}$$

$$= \frac{y^2}{x(y-x)}\cdot\frac{x}{x} = \frac{y^2}{x^2}\cdot\frac{1}{\frac{(y-x)}{x}}$$

$$= \frac{(1+\log y)^2}{\log y} \quad \left[\because \log y = \frac{y-x}{x}\ \log y = \frac{y}{x}-1 \Rightarrow 1+\log y = \frac{y}{x}\right]$$

Hence proved.

Q. 61 If $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$, then show that $\frac{dy}{dx} = \frac{y^2\tan x}{y\log\cos x - 1}$.

Sol. We have,

$$y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$$

$$\Rightarrow \quad y = (\cos x)^y$$

$$\therefore \quad \log y = \log(\cos x)^y$$

$$\Rightarrow \quad \log y = y\log\cos x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{d}{dx}\log\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{\cos x}\cdot\frac{d}{dx}\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y}-\log\cos x\right] = \frac{-y\sin x}{\cos x} = -y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2\tan x}{(1-y\log\cos x)}$$

$$= \frac{y^2\tan x}{y\log\cos x - 1}$$

Hence proved.

Q. 62 If $x\sin(a+y)+\sin a\cdot\cos(a+y)=0$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Sol. We have,

$$x\sin(a+y)+\sin a\cdot\cos(a+y) = 0$$

$$\Rightarrow \quad x\sin(a+y) = -\sin a\cdot\cos(a+y)$$

$$\Rightarrow \quad x = \frac{-\sin a\cdot\cos(a+y)}{\sin(a+y)}$$

$$\Rightarrow \quad x = -\sin a \cdot \cot(a+y)$$

$$\therefore \quad \frac{dx}{dy} = -\sin a \cdot [-\text{cosec}^2(a+y)] \cdot \frac{d}{dy}(a+y)$$

$$= \sin a \cdot \frac{1}{\sin^2(a+y)} \cdot 1$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

Hence proved.

Q. 63 If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Sol. We have,

$$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

On putting $x = \sin\alpha$ and $y = \sin\beta$, we get

$$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \quad \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \quad 2\cos\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2} = a\left(2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = a\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \quad \cot\frac{\alpha-\beta}{2} = a$$

$$\Rightarrow \quad \frac{\alpha-\beta}{2} = \cot^{-1}a$$

$$\Rightarrow \quad \alpha - \beta = 2\cot^{-1}a$$

$$\Rightarrow \quad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a \qquad [\because x = \sin\alpha \text{ and } y = \sin\beta]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Hence proved.

Q. 64 If $y = \tan^{-1}x$, then find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol. We have, $y = \tan^{-1}x$ [on differentiating w.r.t. x]

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2} \qquad \text{[again differentiating w.r.t. } x]$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1}$$

$$= -1(1+x^2)^{-2}.\frac{d}{dx}(1+x^2)$$

$$= -\frac{1}{(1+x^2)^2}\cdot 2x$$

$$= \frac{-2\tan y}{(1+\tan^2 y)^2} \qquad [\because y = \tan^{-1}x \Rightarrow \tan y = x]$$

$$= \frac{-2\tan y}{(\sec^2 y)^2}$$

$$= -2\frac{\sin y}{\cos y}\cdot\cos^2 y\cdot\cos^2 y$$

$$= -\sin 2y\cdot\cos^2 y \qquad [\because \sin 2x = 2\sin x\cos x]$$

Verify the Rolle's theorem for each of the functions in following questions.

Q. 65 $f(x) = x(x-1)^2$ in $[0,1]$

Thinking Process

We know that, Rolle's theorem states that, if f be a real valued function, defined in the closed interval [a, b], such that (i) f is continuous on [a, b]. (ii) f is differentiable on]a, b[.(iii) f(a) = f(b).

Then, there exists a real number c in the open interval] a, b [, such that f′(c) = 0′. Here, we shall verify the Rolle's theorem for the given function.

Sol. We have, $f(x) = x(x-1)^2$ in [0, 1].

(i) Since, $f(x) = x(x-1)^2$ is a polynomial function.

So, it is continuous in [0, 1].

(ii) Now,

$$f'(x) = x\cdot\frac{d}{dx}(x-1)^2 + (x-1)^2\frac{d}{dx}x$$

$$= x\cdot 2(x-1)\cdot 1 + (x-1)^2$$

$$= 2x^2 - 2x + x^2 + 1 - 2x$$

$$= 3x^2 - 4x + 1, \text{ which exists in } (0, 1).$$

So, $f(x)$ is differentiable in (0, 1).

(iii) Now, $f(0) = 0$ and $f(1) = 0 \Rightarrow f(0) = f(1)$

f satisfies the above conditions of Rolle's theorem.

Hence, by Rolle's theorem $\exists\, c \in (0, 1)$ such that

$$f'(c) = 0$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = \frac{1}{3}, 1 \Rightarrow \frac{1}{3} \in (0, 1)$$

Thus, we see that there exists a real number c in the open interval (0, 1).

Hence, Rolle's theorem has been verified.

Q. 66 $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

Sol. We have, $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$ ...(i)

(i) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

[since, $\sin^4 x$ and $\cos^4 x$ are continuous functions and we know that, if g and h be continuous functions, then $(g + h)$ is a continuous function.]

(ii)
$$f'(x) = 4(\sin x)^3 \cdot \cos x + 4(\cos x)^3 \cdot (-\sin x)$$
$$= 4\sin^3 x \cdot \cos x - 4 \sin x \cdot \cos^3 x$$
$$= 4\sin x \cos x\,(\sin^2 x - \cos^2 x) \text{ which exists in } \left(0, \frac{\pi}{2}\right) \quad ...(ii)$$

Hence, $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(iii) Also, $f(0) = 0 + 1 = 1$ and $f\left(\frac{\pi}{2}\right) = 1 + 0 = 1$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

Conditions of Rolle's theorem are satisfied.

Hence, there exists atleast one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

$$\therefore \quad 4 \sin c \cos c\,(\sin^2 c - \cos^2 c) = 0$$
$$\Rightarrow \quad 4 \sin c \cos c\,(-\cos 2c) = 0$$
$$\Rightarrow \quad -2 \sin 2c \cdot \cos 2c = 0$$
$$\Rightarrow \quad -\sin 4c = 0$$
$$\Rightarrow \quad \sin 4c = 0$$
$$\Rightarrow \quad 4c = \pi$$
$$\Rightarrow \quad c = \frac{\pi}{4}$$
$$\text{and} \quad \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem has been verified.

Q. 67 $f(x) = \log(x^2 + 2) - \log 3$ in $[-1, 1]$

Sol. We have, $f(x) = \log(x^2 + 2) - \log 3.$

(i) Logarithmic functions are continuous in their domain.
Hence, $f(x) = \log(x^2 + 2) - \log 3$ is continuous in $[-1, 1]$.

(ii)
$$f'(x) = \frac{1}{x^2 + 2} \cdot 2x - 0$$
$$= \frac{2x}{x^2 + 2}, \text{ which exists in } (-1, 1).$$

Hence, $f(x)$ is differentiable in $(-1, 1)$.

(iii) $f(-1) = \log[(-1)^2 + 2] - \log 3 = \log 3 - \log 3 = 0$ and

$f(1) = \log(1^2 + 2) - \log 3 = \log 3 - \log 3 = 0$

$\Rightarrow \quad f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$$f'(c) = 0.$$

$$\Rightarrow \quad \frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1)$$

Hence, Rolle's theorem has been verified.

Q. 68 $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

Sol. We have, $\quad f(x) = x(x+3)e^{-x/2}$

(i) $f(x)$ is a continuous function. [since, it is a combination of polynomial functions $x(x+3)$ and an exponential function $e^{-x/2}$ which are continuous functions]

So, $f(x) = x(x+3)e^{-x/2}$ is continuous in $[-3, 0]$.

(ii) $\therefore$

$$f'(x) = (x^2 + 3x)\cdot\frac{d}{dx}e^{-x/2} + e^{-x/2}\cdot\frac{d}{dx}(x^2 + 3x)$$

$$= (x^2 + 3x)\cdot e^{-x/2}\cdot\left(-\frac{1}{2}\right) + e^{-x/2}\cdot(2x + 3)$$

$$= e^{-x/2}\left[2x + 3 - \frac{1}{2}\cdot(x^2 + 3x)\right]$$

$$= e^{-x/2}\left[\frac{4x + 6 - x^2 - 3x}{2}\right]$$

$$= e^{-x/2}\cdot\frac{1}{2}[-x^2 + x + 6]$$

$$= \frac{-1}{2}e^{-x/2}[x^2 - x - 6]$$

$$= \frac{-1}{2}e^{-x/2}[x^2 - 3x + 2x - 6]$$

$$= \frac{-1}{2}e^{-x/2}[(x+2)(x-3)] \text{ which exists in } (-3, 0).$$

Hence, $f(x)$ is differentiable in $(-3, 0)$.

(iii) $\therefore \quad f(-3) = -3(-3+3)e^{-3/2} = 0$

and $\quad f(0) = 0(0+3)e^{-0/2} = 0$

$\Rightarrow \quad f(-3) = f(0)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c) = 0$

$$\Rightarrow \quad -\frac{1}{2}e^{-c/2}(c+2)(c-3) = 0$$

$$\Rightarrow \quad c = -2, 3, \text{ where } -2 \in (-3, 0)$$

Therefore, Rolle's theorem has been verified.

Q. 69 $f(x)=\sqrt{4-x^2}$ in $[-2, 2]$

Sol. We have, $f(x)=\sqrt{4-x^2}=(4-x^2)^{1/2}$

(i) $f(x)=\sqrt{4-x^2}$ is a continuous function.

[since every polynomial function is a continuous function]

Hence, $f(x)$ is continuous in $[-2, 2]$.

(ii) $f'(x)=\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x)$

$= -x.\frac{1}{\sqrt{4-x^2}}$, which exists everywhere except at $x=\pm 2$.

Hence, $f(x)$ is differentiable in $(-2, 2)$.

(iii) $f(-2)=\sqrt{(4-4)}=0$ and $f(2)=\sqrt{(4-4)}=0$

$\Rightarrow \quad f(-2)=f(2)$

conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c)=0$.

$$\Rightarrow \quad -c\,\frac{1}{\sqrt{4-c^2}}=0$$

$$\Rightarrow \quad c=0\in(-2, 2)$$

Hence, Rolle's theorem has been verified.

Q. 70 Discuss the applicability of Rolle's theorem on the function given by

$$f(x)=\begin{cases}x^2+1, & \text{if } 0\le x\le 1\\ 3-x, & \text{if } 1\le x\le 2\end{cases}$$

Sol. We have,
$$f(x)=\begin{cases}x^2+1, \text{ if } 0\le x\le 1\\ 3-x, \ \text{ if } 1\le x\le 2\end{cases}$$

We know that, polynomial function is everywhere continuous and differentiability.
So, $f(x)$ is continuous and differentiable at all points except possibly at $x=1$.
Now, check the differentiability at $x=1$,
At $x=1$,

$$\text{LDH}=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1}\frac{(x^2+1)-(1+1)}{x-1} \qquad [\because f(x)=x^2+1,\ \forall\ 0\le x\le 1]$$

$$=\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}$$

$$=2$$

and
$$\text{RDH}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(3-x)\,f(1+1)}{(x-1)}$$

$$=\lim_{x\to 1}\frac{3-x-2}{x-1}=\lim_{x\to 1}\frac{-(x-1)}{x-1}=-1$$

$\therefore \quad$ LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x=1$.
Hence, polle's theorem is not applicable on the interval [0, 2].

Q. 71 **Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to X-axis.**

Thinking Process

We know that, if f be a real valued function defined in the closed interval $[a, b]$ such that it follows all the three conditions of Rolle's theorem, then $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has a slope 0, i.e., it is parallel to the X-axis. So, by getting the value of c' we can get the required point.

Sol. The equation of the curve is $y = \cos x - 1$.

Now, we have to find a point on the curve in $[0, 2\pi]$,

where the tangent is parallel to X-axis *i.e.,* the tangent to the curve at $x = c$ has a slope o, where $c \in]0, 2\pi[$.

Let us apply Rolle's theorem to get the point.

(i) $y = \cos x - 1$ is a continuous function in $[0, 2\pi]$.

[since it is a combination of cosine function and a constant function]

(ii) $y' = -\sin x$, which exists in $(0, 2\pi)$.

Hence, y is differentiable in $(0, 2\pi)$.

(iii) $y(0) = \cos 0 - 1 = 0$ and $y(2\pi) = \cos 2\pi - 1 = 0$,

$\therefore \quad y(0) = y(2\pi)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$$f'(c) = 0$$

$\Rightarrow \quad -\sin c = 0$

$\Rightarrow \quad c = \pi$ or 0, where $\pi \in (0, 2\pi)$

$\Rightarrow \quad x = \pi$

$\therefore \quad y = \cos \pi - 1 = -2$

Hence, the required point on the curve, where the tangent drawn is parallel to the X-axis is $(\pi, -2)$.

Q. 72 **Using Rolle's theorem, find the point on the curve $y = x(x - 4)$, $x \in [0, 4]$, where the tangent is parallel to X-axis.**

Sol. We have, $y = x(x - 4)$, $x \in [0, 4]$

(i) y is a continuous function since $x(x - 4)$ is a polynomial function.

Hence, $y = x(x - 4)$ is continuous in $[0, 4]$.

(ii) $y' = (x - 4) \cdot 1 + x \cdot 1 = 2x - 4$ which exists in $(0, 4)$.

Hence, y is differentiable in $(0, 4)$.

(iii) $y(0) = 0(0 - 4) = 0$

and $\quad y(4) = 4(4 - 4) = 0$

$\Rightarrow \quad y(0) = y(4)$

Sicne, conditions of Rolle's theorem are satisfied.

Hence, there exists a point c such that

$f'(c) = 0$ in $(0, 4)$ $\quad [\because f'(x) = y']$

$\Rightarrow \quad 2c - 4 = 0$

$\Rightarrow \quad c = 2$

$\Rightarrow \quad x = 2;\ y = 2(2 - 4) = -4$

Thus, $(2, -4)$ is the point on the curve at which the tangent drawn is parallel to X-axis.

Verify mean value theorem for each of the functions.

Q. 73 $f(x) = \dfrac{1}{4x-1}$ in [1, 4]

Thinking Process

We know that, mean value theorem states that, if f be a real function such that
(i) $f(x)$ *is continuous on* [a,b]
(ii) $f(x)$ *is differentiable on*]a,b[

Then, there exists a real number $c \in]a, b[$ *such that* $f'(c) = \dfrac{f(b) - f(a)}{b-a}$*, thus we can verify it for given function.*

Sol. We have, $f(x) = \dfrac{1}{4x-1}$ in [1, 4]

(i) $f(x)$ is continuous in [1, 4].

Also, at $x = \dfrac{1}{4}$, $f(x)$ is discontinuous.

Hence, $f(x)$ is continuous in [1, 4].

(ii) $f'(x) = -\dfrac{4}{(4x-1)^2}$, which exists in (1, 4).

Since, conditions of mean value theorem are satisfied.
Hence, there exists a real number $c \in]1, 4[$ such that

$$f'(c) = \frac{f(4) - f(1)}{4-1}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2} = \frac{\frac{1}{16-1} - \frac{1}{4-1}}{4-1} = \frac{\frac{1}{15} - \frac{1}{3}}{3}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2} = \frac{1-5}{45} = \frac{-4}{45}$$

$$\Rightarrow \quad (4c-1)^2 = 45$$

$$\Rightarrow \quad 4c - 1 = \pm 3\sqrt{5}$$

$$\Rightarrow \quad c = \frac{3\sqrt{5}+1}{4} \in (1, 4) \qquad \text{[neglecting (– ve) value]}$$

Hence, mean value theorem has been verified.

Q. 74 $f(x) = x^3 - 2x^2 - x + 3$ in [0, 1]

Sol. We have, $f(x) = x^3 - 2x^2 - x + 3$ in [0, 1]

(i) Since, $f(x)$ is a polynomial function.
Hence, $f(x)$ is continuous in [0, 1].

(ii) $f'(x) = 3x^2 - 4x - 1$, which exists in (0,1).

Hence, $f(x)$ is differentiable in (0,1).
Since, conditions of mean value theorem are satisfied.
Therefore, by mean value theorem $\exists\, c \in (0,1)$, such that

$$f'(c) = \frac{f(1) - f(0)}{1-0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{[1-2-1+3]-[0+3]}{1-0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{-2}{1}$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = 1/3, 1, \text{ where } \frac{1}{3} \in (0, 1)$$

Hence, the mean value theorem has been verified.

Q. 75 $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

Sol. We have, $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

(i) Since, we know that sine functions are continuous functions hence $f(x) = \sin x - \sin 2x$ is a continuous function in $[0, \pi]$.

(ii) $f'(x) = \cos x - \cos 2x \cdot 2 = \cos x - 2\cos 2x$, which exists in $(0, \pi)$.

So, $f(x)$ is differentiable in $(0, \pi)$. Conditions of mean value theorem are satisfied.

Hence, $\exists c \in (0, \pi)$ such that, $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$$\Rightarrow \quad \cos c - 2\cos 2c = \frac{\sin \pi - \sin 2\pi - \sin 0 + \sin 2 \cdot 0}{\pi - 0}$$

$$\Rightarrow \quad 2\cos 2c - \cos c = \frac{0}{\pi}$$

$$\Rightarrow \quad 2 \cdot (2\cos^2 c - 1) - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - 2 - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\therefore \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

Also, $\quad \cos^{-1}\left(\dfrac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$

Hence, mean value theorem has been verified.

Q. 76 $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

Sol. We have, $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

(i) Since, $f(x) = (25 - x^2)^{1/2}$, where $25 - x^2 \geq 0$

$$\Rightarrow \quad x^2 \leq \pm 5 \Rightarrow -5 \leq x \leq 5$$

Hence, $f(x)$ is continuous in $[1, 5]$.

(ii) $f'(x) = \dfrac{1}{2}(25 - x^2)^{-1/2} \cdot -2x = \dfrac{-x}{\sqrt{25 - x^2}}$, which exists in $(1, 5)$.

Hence, $f'(x)$ is differentiable in $(1, 5)$.

Since, conditions of mean value theorem are satisfied.

By mean value theorem $\exists\, c \in (1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} \Rightarrow \frac{-c}{\sqrt{25 - c^2}} = \frac{0 - \sqrt{24}}{4}$$

$$\Rightarrow \quad \frac{c^2}{25 - c^2} = \frac{24}{16}$$

$$\Rightarrow \quad 16c^2 = 600 - 24c^2$$

$$\Rightarrow \quad c^2 = \frac{600}{40} = 15$$

$$\therefore \quad c = \pm\sqrt{15}$$

Also, $\quad c = \sqrt{15} \in (1, 5)$

Hence, the mean value theorem has been verified.

Q. 77 Find a point on the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Thinking Process

We know that, if $y = f(x)$ be a function defined on $[a, b]$ which follows mean value theorem, then there exists atleast one point c in (a, b) such that the tangent at the point $[c, f(c)]$ is parallel to the secant joining the points $[a, f(a)]$ and $[b, f(b)]$. So, we shall use this concept.

Sol. We have, $y = (x - 3)^2$, which is continuous in $x_1 = 3$ and $x_2 = 4$ *i.e.*, [3, 4].

Also, $y' = 2(x - 3) \cdot 1 = 2(x - 3)$ which exists in (3, 4).

Hence, by mean value theorem there exists a point on the curve at which tangent drawn is parallel to the chord joining the points (3,0) and (4,1).

Thus, $\quad f'(c) = \dfrac{f(4) - f(3)}{4 - 3}$

$$\Rightarrow \quad 2(c - 3) = \frac{(4 - 3)^2 - (3 - 3)^2}{4 - 3}$$

$$\Rightarrow \quad 2c - 6 = \frac{1 - 0}{1} \Rightarrow c = \frac{7}{2}$$

For $x = \dfrac{7}{2}$, $\quad y = \left(\dfrac{7}{2} - 3\right)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

So, $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$ is the point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

Q. 78 Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points $A(1, 0)$ and $B(2, 1)$, where tangent is parallel to the chord AB. Also, find that point.

Sol. We have, $y = 2x^2 - 5x + 3$, which is continuous in [1, 2] as it is a polynomial function.

Also, $y' = 4x - 5$, which exists in (1, 2).

By mean value theorem, $\exists\, c \in (1, 2)$ at which drawn tangent is parallel to the chord AB, where A and B are (1, 0) and (2,1), respectively.

$$\therefore \quad f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow \quad 4c - 5 = \frac{(8 - 10 + 3) - (2 - 5 + 3)}{1}$$

$$\Rightarrow \quad 4c - 5 = 1$$

$$\therefore \quad c = \frac{6}{4} = \frac{3}{2} \in (1, 2)$$

For $x = \frac{3}{2}$, $\quad y = 2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3$

$$= 2 \times \frac{9}{4} - \frac{15}{2} + 3 = \frac{9 - 15 + 6}{2} = 0$$

Hence, $\left(\frac{3}{2}, 0\right)$ is the point on the curve $y = 2x^2 - 5x + 3$ between the points A (1, 0) and B (2, 1), where tangent is parallel to the chord AB.

Long Answer Type Questions

Q. 79 Find the values of p and q, so that $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Sol. We have, $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

$$\therefore \quad Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x^2 + 3x + p) - (1 + 3 + p)}{x - 1}$$

$$= \lim_{h \to 0} \frac{[(1 - h)^2 + 3(1 - h) + p] - [1 + 3 + p]}{(1 - h) - 1}$$

$$= \lim_{h \to 0} \frac{[1 + h^2 - 2h + 3 - 3h + p] - [4 + p]}{-h}$$

$$= \lim_{h \to 0} \frac{[h^2 - 5h + p + 4 - 4 - p]}{-h} = \lim_{h \to 0} \frac{h[h - 5]}{-h}$$

$$= \lim_{h \to 0} -[h - 5] = 5$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(qx + 2) - (1 + 3 + p)}{x - 1}$$

$$= \lim_{h \to 0} \frac{[q(1 + h) + 2] - (4 + p)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{[q + qh + 2 - 4 - p]}{h} = \lim_{h \to 0} \frac{qh + (q - 2 - p)}{h}$$

$$\Rightarrow \quad q - 2 - p = 0 \Rightarrow p - q = -2 \qquad \text{...(i)}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{qh + 0}{h} = q \qquad \text{[for existing the limit]}$$

If $Lf'(1) = Rf'(1)$, then $5 = q$

$$\Rightarrow \quad p - 5 = -2 \Rightarrow p = 3$$

$$\therefore \quad p = 3 \text{ and } q = 5$$

Q. 80 If $x^m \cdot y^n = (x+y)^{m+n}$, prove that

(i) $\dfrac{dy}{dx} = \dfrac{y}{x}$ and (ii) $\dfrac{d^2y}{dx^2} = 0$

Sol. We have, $x^m \cdot y^n = (x+y)^{m+n}$ ...(i)

(i) Differentiating Eq. (i) w.r.t. x, we get

$$\frac{d}{dx}(x^m \cdot y^n) = \frac{d}{dx}(x+y)^{m+n}$$

$$\Rightarrow \quad x^m \cdot \frac{d}{dy} y^n \cdot \frac{dy}{dx} + y^n \cdot \frac{d}{dx} x^m = (m+n)(x+y)^{m+n-1} \frac{d}{dx}(x+y)$$

$$\Rightarrow \quad x^m \cdot ny^{n-1} \frac{dy}{dx} + y^n \cdot mx^{m-1} = (m+n)(x+y)^{m+n-1}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}[x^m \cdot ny^{n-1} - (m+n)\cdot(x+y)^{m+n-1}] = (m+n)(x+y)^{m+n-1} - y^n mx^{m-1}$$

$$\Rightarrow \quad \frac{dy}{dx}[nx^m y^{n-1} - (m+n)(x+y)^{m+n-1}] = (m+n)\cdot(x+y)^{m+n-1} - \frac{y^{n-1}\cdot y \cdot mx^m}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{(m+n)(x+y)^{m+n}}{(x+y)} - \dfrac{y^{n-1}\cdot y \cdot mx^m}{x}}{\dfrac{n\,x^m y^n}{y} - (m+n)(x+y)^{m+n}\dfrac{1}{(x+y)}}$$

$$= \frac{\dfrac{x(m+n)(x+y)^{m+n} - (x+y)\cdot y\cdot^{n-1} y \cdot mx^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,n\,x^m y^n - y(m+n)(x+y)^{m+n}}{(x+y)\cdot y}}$$

$$= \frac{\dfrac{x(m+n)\cdot x^m \cdot y^n - m(x+y)\,y^n\,x^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,nx^m \cdot y^n - y(m+n)\cdot x^m \cdot y^n}{(x+y)\cdot y}} \qquad [\because (x+y)^{m+n} = x^m \cdot y^n]$$

$$= \frac{x^m\,y^n[mx + nx - mx - my]\cdot(x+y)\,y}{x^m\,y^n[nx + ny - my - ny]\cdot(x+y)\cdot x}$$

$$= \frac{y}{x} \qquad \text{...(ii)}$$

Hence proved.

(ii) Further, differentiating Eq. (ii) *i.e.*, $\dfrac{dy}{dx} = \dfrac{y}{x}$ on both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x \cdot \dfrac{dy}{dx} - y \cdot 1}{x^2}$$

$$= \frac{x \cdot \dfrac{y}{x} - y}{x^2} \qquad \left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= 0$$

Hence proved.

Q. 81 If $x = \sin t$ and $y = \sin pt$, then prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0.$$

Sol. We have, $x = \sin t$ and $y = \sin pt$

$$\therefore \quad \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \cos pt \cdot p$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p \cdot \cos pt}{\cos t} \qquad \ldots(i)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos t \cdot \frac{d}{dt}(p \cdot \cos pt)\frac{dt}{dx} - p\cos pt \cdot \frac{d}{dt}\cos t \cdot \frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t \cdot p \cdot (-\sin pt) \cdot p - p\cos pt \cdot (-\sin t)]\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[-p^2 \sin pt \cdot \cos t + p \sin t \cdot \cos pt] \cdot \frac{1}{\cos t}}{\cos^2 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t}{\cos^3 t} \qquad \ldots(ii)$$

Since, we have to prove

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

$$\therefore \quad \text{LHS} = (1 - \sin^2 t)\frac{[-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t]}{\cos^3 t} - \sin t \cdot \frac{p\cos pt}{\cos t} + p^2 \sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}(1 - \sin^2 t)(-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t) \\ - p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t\end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}-p^2 \sin pt \cdot \cos^3 t + p\cos pt \cdot \sin t \cdot \cos^2 t \\ - p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t\end{bmatrix} \quad [\because 1 - \sin^2 t = \cos^2 t]$$

$$= \frac{1}{\cos^3 t} \cdot 0$$

$$= 0$$

Hence proved.

Q. 82 Find the value of $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}}$.

Sol. We have,

$$y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}} \qquad \ldots(i)$$

Taking $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2 + 1}{2}}$,

$$\log u = \tan x \log x \qquad \ldots(ii)$$

and

$$v^2 = \frac{x^2 + 1}{2} \qquad \ldots(iii)$$

On, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \tan x \cdot \frac{1}{x} + \log x \cdot \sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x \cdot \sec^2 x\right]$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x \cdot \sec^2 x\right] \quad \ldots\text{(iv)}$$

Also, differentiating Eq. (iii) w.r.t. x, we get

$$2v \cdot \frac{dv}{dx} = \frac{1}{2}(2x) \Rightarrow \frac{dv}{dx} = \frac{1}{4v}\cdot(2x)$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{4\cdot\sqrt{\frac{x^2+1}{2}}}\cdot 2x = \frac{x\cdot\sqrt{2}}{2\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{x}{\sqrt{2(x^2+1)}} \quad \ldots\text{(v)}$$

Now, $y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x \cdot \sec^2 x\right] + \frac{x}{\sqrt{2(x^2+1)}}$$

Objective Type Questions

Q. 83 If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function?

(a) $f(x) + g(x)$

(b) $f(x) - g(x)$

(c) $f(x)\cdot g(x)$

(d) $\frac{g(x)}{f(x)}$

Sol. *(d)* We know that, if f and g be continuous functions, then

(a) $f + g$ is continuous

(b) $f - g$ is continuous.

(c) fg is continuous

(d) $\frac{f}{g}$ is continuous at these points, where $g(x) \neq 0$.

Here, $$\frac{g(x)}{f(x)} = \frac{\frac{x^2}{2}+1}{2x} = \frac{x^2+2}{4x}$$

which is discontinuous at $x = 0$.

Q. 84 The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is

(a) discontinuous at only one point
(b) discontinuous at exactly two points
(c) discontinuous at exactly three points
(d) None of the above

Sol. *(c)* We have,

$$f(x) = \frac{4 - x^2}{4x - x^3} = \frac{(4 - x^2)}{x\,(4 - x^2)}$$

$$= \frac{(4 - x^2)}{x\,(2^2 - x^2)} = \frac{4 - x^2}{x\,(2 + x)\,(2 - x)}$$

Clearly, $f(x)$ is discontinuous at exactly three points $x = 0$, $x = -2$ and $x = 2$.

Q. 85 The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) R
(b) $R - \left(\dfrac{1}{2}\right)$
(c) $(0, \infty)$
(d) None of these

Sol. *(b)* We have, $f(x) = |2x - 1| \sin x$

At $x = \dfrac{1}{2}$, $f(x)$ is not differentiable.

Hence, $f(x)$ is differentiable in $R - \left(\dfrac{1}{2}\right)$.

$\because$

$$Rf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} + h\right) - 1\right| \sin\left(\frac{1}{2} + h\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{|2h| \cdot \sin\left(\frac{1 + 2h}{2}\right)}{h} = 2 \cdot \sin\frac{1}{2}$$

and

$$Lf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} - h\right)^{-1}\right| - \sin\left(\frac{1}{2} - h\right) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{|0 - 2h| - \sin\left(\frac{1}{2} - h\right)}{-h} = -2\sin\left(\frac{1}{2}\right)$$

$\because$

$$Rf'\left(\frac{1}{2}\right) \neq Lf'\left(\frac{1}{2}\right)$$

So, $f(x)$ is not differentiable at $x = \dfrac{1}{2}$.

Q. 86 The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$
(b) $\{x = 2n\pi : n \in Z\}$
(c) $\left\{ x = (2n+1)\frac{\pi}{2}; n \in Z \right\}$
(d) $\left\{ x = \frac{n\pi}{2}; n \in Z \right\}$

Sol. *(a)* We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \frac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $n\pi$, $n \in Z$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

Q. 87 The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$
(b) continuous and differentiable everywhere
(c) not continuous at $x = 0$
(d) None of the above

Sol. *(a)* Let $u(x) = |x|$ and $v(x) = e^x$

$\therefore$ $f(x) = vou(x) = v[u(x)]$
$= v|x| = e^{|x|}$

Since, $u(x)$ and $v(x)$ are both continuous functions.

So, $f(x)$ is also continuous function but $u(x) = |x|$ is not differentiable at $x = 0$, whereas $v(x) = e^x$ is differentiable at everywhere.

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = 0$.

Q. 88 If $f(x) = x^2 \sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0
(b) -1
(c) 1
(d) None of these

Sol. *(a)* $\because$ $f(x) = x^2 \sin\left(\frac{1}{x}\right)$, where $x \neq 0$

Hence, value of the function f at $x = 0$, so that it is continuous at $x = 0$ is 0.

Q. 89 If $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(a) $m = 1, n = 0$
(b) $m = \frac{n\pi}{2} + 1$
(c) $n = \frac{m\pi}{2}$
(d) $m = n = \frac{\pi}{2}$

Sol. *(c)* We have, $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ (\sin x + n), & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ $$\text{LHL} = \lim_{x \to \frac{\pi^-}{2}} (mx + 1) = \lim_{h \to 0} \left[m\left(\frac{\pi}{2} - h\right) + 1 \right] = \frac{m\pi}{2} + 1$$

and $$\text{RHL} = \lim_{x \to \frac{\pi^+}{2}} (\sin x + n) = \lim_{h \to 0} \left[\sin\left(\frac{\pi}{2} + h\right) + n \right]$$

$$= \lim_{h \to 0} \cos h + n = 1 + n$$

$\therefore$ $\text{LHL} = \text{RHL}$ $\left[\text{to be continuous at } x = \frac{\pi}{2}\right]$

$\Rightarrow$ $m \cdot \frac{\pi}{2} + 1 = n + 1$

$\therefore$ $n = m \cdot \frac{\pi}{2}$

Q. 90 If $f(x) = |\sin x|$, then

(a) f is everywhere differentiable

(b) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$

(c) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$

(d) None of the above

Sol. ***(b)*** We have, $f(x) = |\sin x|$

Let $f(x) = vou\ (x) = v\ [u(x)]$ [where, $u\ (x) = \sin x$ and $v\ (x) = |x|$]

$= v\ (\sin x) = |\sin x|$

where, $u(x)$ and $v\ (x)$ are both continuous.

Hence, $f(x) = vo\ u(x)$ is also a continuous function but $v(x)$ is not differentiable at $x = 0$.

So, $f(x)$ is not differentiable where $\sin x = 0 \Rightarrow x = n\pi$, $n \in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = n\pi$, $n \in Z$.

Q. 91 If $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{4x^3}{1 - x^4}$ (b) $\frac{-4x}{1 - x^4}$ (c) $\frac{1}{4 - x^4}$ (d) $\frac{-4x^3}{1 - x^4}$

Sol. ***(b)*** We have, $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\frac{1 - x^2}{1 + x^2}} \cdot \frac{d}{dx}\left(\frac{1 - x^2}{1 + x^2}\right)$$

$$= \frac{(1 + x^2)}{(1 - x^2)} \cdot \frac{(1 + x^2)\cdot(-2x) - (1 - x^2)\cdot 2x}{(1 + x^2)^2}$$

$$= \frac{-2x[1 + x^2 + 1 - x^2]}{(1 - x^2)\cdot(1 + x^2)} = \frac{-4x}{1 - x^4}$$

Q. 92 If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{\cos x}{2y-1}$ (b) $\dfrac{\cos x}{1-2y}$ (c) $\dfrac{\sin x}{1-2y}$ (d) $\dfrac{\sin x}{2y-1}$

Sol. *(a)* $\because \quad y = (\sin x + y)^{1/2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2} \cdot \frac{d}{dx}(\sin x + y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{(\sin x + y)^{1/2}} \cdot \left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \qquad [\because (\sin x + y)^{1/2} = y]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y} \cdot \frac{2y}{2y-1} = \frac{\cos x}{2y-1}$$

Q. 93 The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1} x$ is

(a) 2 (b) $\dfrac{-1}{2\sqrt{1-x^2}}$

(b) $\dfrac{2}{x}$ (d) $1 - x^2$

Sol. *(a)* Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1} x$

$$\therefore \quad \frac{dv}{dx} = \frac{+-1}{\sqrt{1-(2x^2-1)^2}} \cdot 4x = \frac{-4x}{\sqrt{1-(4x^4+1-4x^2)}}$$

$$= \frac{-4x}{\sqrt{-4x^4+4x^2}} = \frac{-4x}{\sqrt{4x^2(1-x^2)}}$$

$$= \frac{-2}{\sqrt{1-x^2}}$$

and $$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dx}{dv} = \frac{du/dx}{dv/dx} = \frac{-2/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}} = 2$$

Q. 94 If $x = t^2$ and $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is equal to

(a) $\dfrac{3}{2}$ (b) $\dfrac{3}{4t}$ (c) $\dfrac{3}{2t}$ (d) $\dfrac{3}{2t}$

Sol. *(b)* We have, $x = t^2$ and $y = t^3$

$$\therefore \quad \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$$

On further differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{3}{2} \cdot \frac{d}{dt}t \cdot \frac{dt}{dx}$$

$$= \frac{3}{2} \cdot \frac{1}{2t} \qquad \left[\because \frac{dt}{dx} = \frac{1}{2t}\right]$$

$$= \frac{3}{4t}$$

Q. 95 The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1 (b) -1 (c) $\frac{3}{2}$ (d) $\frac{1}{3}$

Sol. *(a)* $\because \quad f'(c) = 0 \qquad [\because f'(x) = 3x^2 - 3]$

$\Rightarrow \quad 3c^2 - 3 = 0$

$\Rightarrow \quad c^2 = \frac{3}{3} = 1$

$\Rightarrow \quad c = \pm 1$, where $1 \in (0, \sqrt{3})$

$\therefore \quad c = 1$

Q. 96 For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of c for mean value theorem is

(a) 1 (b) $\sqrt{3}$

(c) 2 (d) None of these

Sol. *(b)* $\because \quad f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\left[3 + \frac{1}{3}\right] - \left[1 + \frac{1}{1}\right]}{3 - 1} \qquad \left[\because f'(x) = 1 - \frac{1}{x^2} \text{ and } b = 3, a = 1\right]$

$\Rightarrow \quad \frac{c^2 - 1}{c^2} = \frac{\frac{10}{3} - 2}{2}$

$\Rightarrow \quad \frac{c^2 - 1}{c^2} = \frac{4}{3 \times 2} = \frac{2}{3}$

$\Rightarrow \quad 3(c^2 - 1) = 2c^2$

$\Rightarrow \quad 3c^2 - 2c^2 = 3$

$\Rightarrow \quad c^2 = 3 \Rightarrow c = \pm\sqrt{3}$

$\because \quad c = \sqrt{3} \in (1, 3)$

Fillers

Q. 97 An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is ....

Sol. $|x| + |x-1|$ is continuous everywhere but fails to be differentiable exactly at two points $x = 0$ and $x = 1$.

So, there can be more such examples of functions.

Q. 98 Derivative of x^2 w.r.t. x^3 is ....

Sol. Derivative of x^2 w.r.t. x^3 is $\frac{2}{3x}$.

Let $u = x^2$ and $v = x^3$

$\therefore \quad \frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3x^2$

$\Rightarrow \quad \frac{du}{dv} = \frac{2x}{3x^2} = \frac{2}{3x}$

Q. 99 If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to ....

Sol. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$

$\because \quad 0 < x < \frac{\pi}{2}, \cos x > 0.$

$f(x) = +\cos x$

$\therefore \quad f'(x) = (-\sin x)$

$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}} \quad \left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

Q. 100 If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right)$ is equal to ....

Sol. $\because \quad f(x) = |\cos x - \sin x|,$

$\therefore \quad f'\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2}$

We know that, $\frac{\pi}{4} < x < \frac{\pi}{2}, \sin x > \cos x$

$\therefore \cos x - \sin x \le 0$ *i.e.*, $\quad f(x) = -(\cos x - \sin x)$

$f'(x) = -[-\sin x - \cos x]$

$\therefore \quad f'\left(\frac{\pi}{3}\right) = -\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}\right) = \left(\frac{\sqrt{3}+1}{2}\right)$

Q. 101 For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is ....

Sol. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is -1.

We have, $\sqrt{x} + \sqrt{y} = 1$

$\Rightarrow$ $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$

$\Rightarrow$ $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{\left(\frac{1}{4}, \frac{1}{4}\right)} = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$

True/False

Q. 102 Rolle's theorem is applicable for the function $f(x) = |x - 1|$ in [0, 2].

Sol. *False*

Hence, $f(x) = |x - 1|$ in $[0, 2]$ is not differentiable at $x = 1 \in (0, 2)$.

Q. 103 If f is continuous on its domain D, then $|f|$ is also continuous on D.

Sol. *True*

Q. 104 The composition of two continuous function is a continuous function.

Sol. *True*

Q. 105 Trigonometric and inverse trigonometric functions are differentiable in their respective domain.

Sol. *True*

Q. 106 If $f \cdot g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

Sol. *False*

Let $f(x) = \sin x$ and $g(x) = \cot x$

$\therefore$ $f(x) \cdot g(x) = \sin x \cdot \frac{\cos x}{\sin x} = \cos x$

which is continuous at $x = 0$ but $\cot x$ is not continuous at $x = 0$.