Chapter-VIII

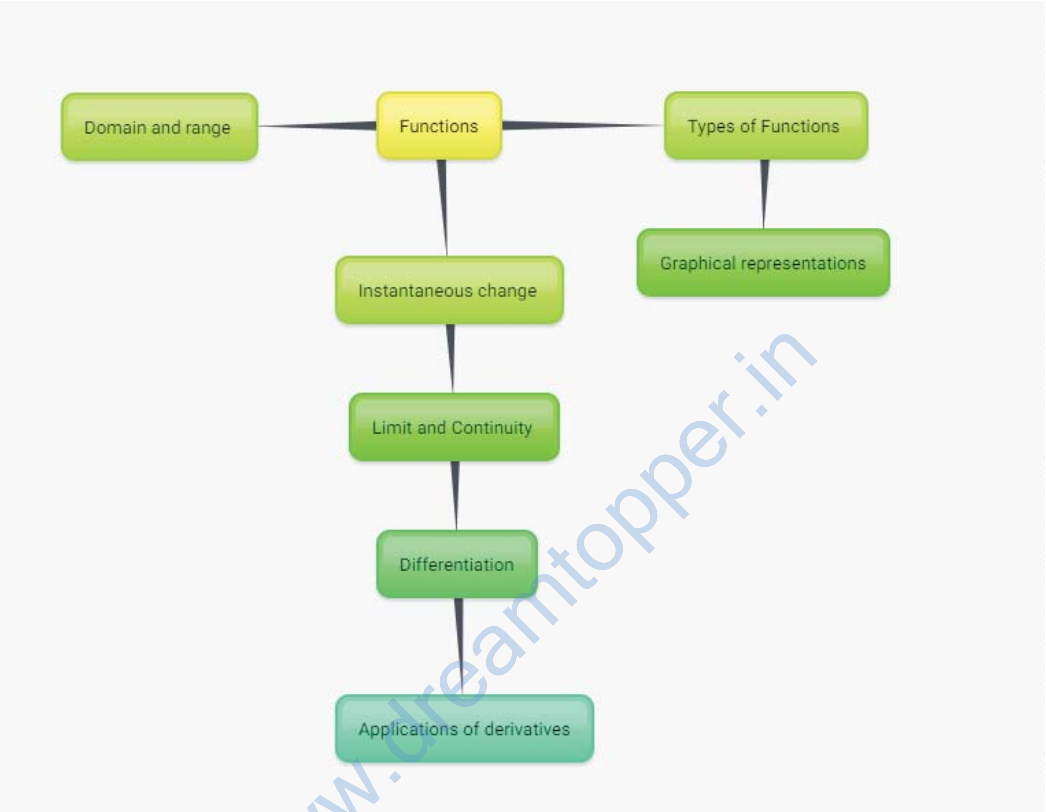
# Calculus

**Learning Objectives:**

After completion of this unit, student will be able to :

- Identify a function and find out domain and range of a function
- Classify functions into different types like Polynomial function, Composite function, Logarithmic function, Exponential function, Modulus function, Greatest integer function, Signum function
- Represent a function graphically and identify the function corresponding to a graph
- Explain the concepts of limit and continuity
- Correlate instantaneous change with differentiation
- Find out derivatives of algebraic functions using first principle, standard formulae and Chain Rule
- find out equations of tangents to the curves, using derivatives

## Concept Map






**Before you start:**

You should know about numbers, polynomials, algebraic expressions and plotting graphs.

## 8.1 Introduction

Calculus helps us in predictions in the instance of change. For example velocity denotes a change in position with respect to time. We can study this change using Calculus. It provides a way to construct simple quantitative models of change, and to deduce their consequences. By studying Calculus, we will come to know how to control the system to do make it do what we want it to do. The development of Calculus and its applications to Physics and Engineering is probably the most significant factor in the development of modern Science.

Study of changes tend to be simpler if we study it by considering changes over tiny interval of time ( instantaneous) rather than studying changes over a period of time.

When we talk about change, we talk about variation of one quantity with respect to another. This can be represented by a function. In this unit, we are going to learn about functions, its types and graphical representations. While studying the instantaneous change, we will develop the idea of instantaneous changing, the concept of limits and continuity of a function. The basic process of differentiation is dealt thereafter with derivatives of algebraic functions and Chain Rule. Tangent lines and equations of tangent are also studied using the concept of differentiation.



## 8.2 Functions

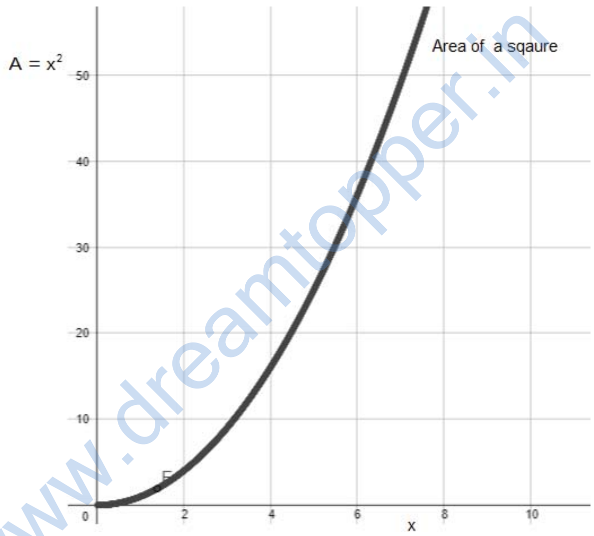
Quantities that change are called **variables.** For example, temperature at a place is a variable as it changes over a period of time. In case of a moving vehicle, speed is a variable. Calculus can be applied to any quantity that varies. In nature if we see that one quantity is connected to another. It means that variation in one quantity affect another. If two variables are connected to each other, one can be seen as a function of another.

**Function, in mathematics is an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable).**

Let us look into the area A of a square. It depends on the length of the side a of square(x). We write $A = x^2$, Here, area (A) is called a dependent variable and it is a function of side of the square (x), which is an independent variable.

There are many ways to express the same function. A function can be represented by an equation, a graph or a table. The function 'area of a square' can be expressed with a table. This can also be plotted on a graph with area on the y-axis and side on the x-axis as shown below:

| x ( cm) | $A = x^2$ ($cm^2$) |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |
| 5 | 25 |






Let us see how to express a function. In the above graph we have seen that A depends on x. Let us write A = f(x) means A is a function of (x)

What will happen if the value of x is negative? Can we have a negative value of length? So the equation of this function will be complete only with some additional information $x \geq 0$.

Instead of using A let us take y as the "dependent variable" and x as the "independent variable". The equation can be written as: $y=x^2$, for $x \geq 0$.

The value of independent variable can be decided by us. The variable y depends on the value of x that we take. This is only a convention. We can also take x as independent and y as dependent variable. In this case, we say that y is a function of x. Hence we can write: y= f(x), for $x \geq 0$.

It can also be written as: $f(x) = x^2$ , for $x \geq 0$.

We read that "y equals to $x^2$ for x greater than or equal to zero. In general, $f(x)=x^2$ is defined for any real number x.

Remember that in a **function**, the **input** value must **have one** and only **one** value for the **output**. Vertical line test is used to determine whether a curve represents a function or not. If any curve cuts a vertical line at more than one points then the curve does not represent a function.

Let us see another function $f(x) = x^2+1$

The output value or value of f(x) when x=0 can be written as $f(0) = 0^2 + 1 = 0 + 1 = 1$.

Similarly we can find out the value of function when the input is x=-1 as below:

$f(-1) = (-1)^2 + 1 = 1+1 = 2$

If you know how to do arithmetic operations of the polynomials you can very well do the same with functions.

| | | |
|---|---|---|
| Addition | $f(x) + g(x)$ | $(f + g)(x)$ |
| Subtraction | $f(x) - g(x)$ | $(f - g)(x)$ |
| Multiplication | $f(x) \bullet g(x)$ | $(f \bullet g)(x)$ |
| Division | $\frac{f(x)}{g(x)}$ If $g(x) \neq 0$ | $\frac{f}{g}(x)$ |

**Illustration:**

If $f(x) = 2x+3$ and $g(x) = x^2+3$

Then $f(x)+g(x) = (f+g)(x) = x^2+2x+6$

Where $(f+g)$ is another function which operates on x to get $x^2+2x+6$

Now try to find out the function (f.g) which product of these two functions.

**Check your Progress 1 :**

a) State whether y is a function of x in the following two cases. Justify your answer.

(i)

| *x* | *y* |
|---|---|
| –3 | –6 |
| –2 | –1 |
| 1 | 0 |
| 1 | 5 |
| 2 | 0 |

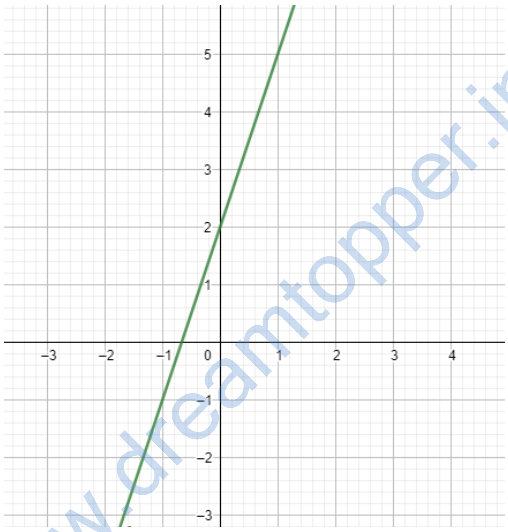
(ii)

| *x* | *y* |
|---|---|
| –3 | 4 |
| –2 | 4 |
| –1 | 4 |
| 2 | 4 |
| 3 | 4 |

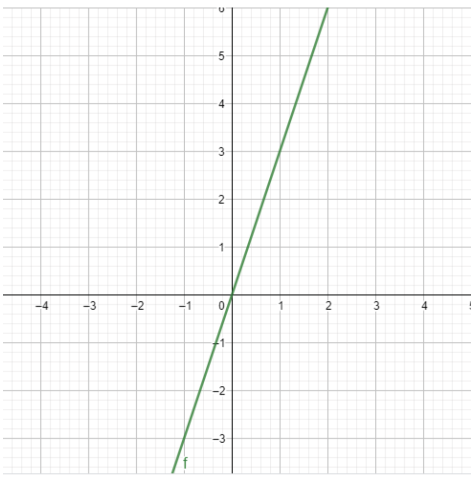
b) If $f(x) = x+1$ and $g(x) = x^2-2x+5$ Find out $(f+g)(x)$. Also plot graphs of $f(x)$, $g(x)$ and $(f+g)(x)$

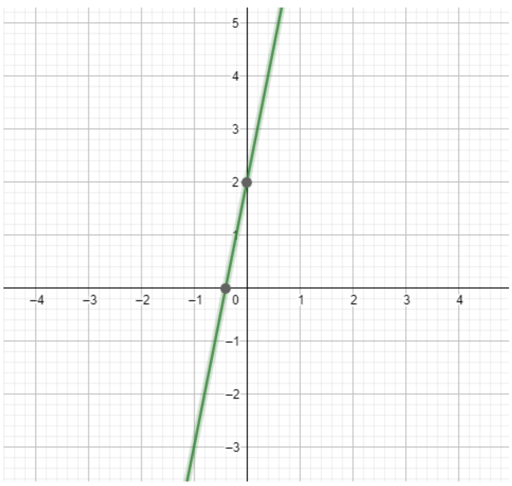
## 8.3 Graphical representation of functions:

Graphical representation of the function finds many applications. In Finance, Science and Technology graphs are the tools used for many purposes. In the simplest case one variable is plotted as a function of another, typically using rectangular axes. It is therefore important that we understand the nature of graph representing different functions. Let us plot the graph of function $f(x) = 3x+2$



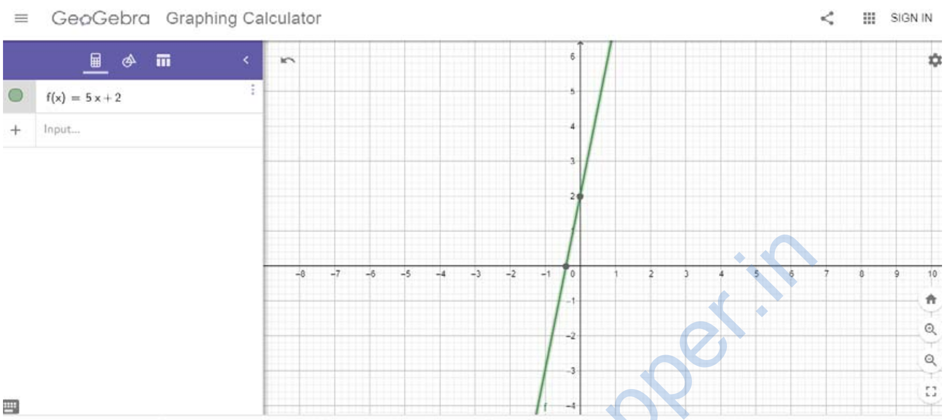
Graph of $f(x) = 3x$ will be different in its y intercept and if we plot $y = 5x+2$ will be a straight line with a greater slope.





**Check your Progress 2 :**

Above graphs are plotted using GeoGebra Graphing calculator which is available at : https://www.geogebra.org/graphing?lang=en.
Explore this useful tool in plotting different functions.




Plot the graph of following functions using GeoGebra Graphing calculator:

a) $f(x) = x^2$

b) $f(x) = x^3$

c) $f(x) = 1/x$



## 8.4 Domain and range of a function:

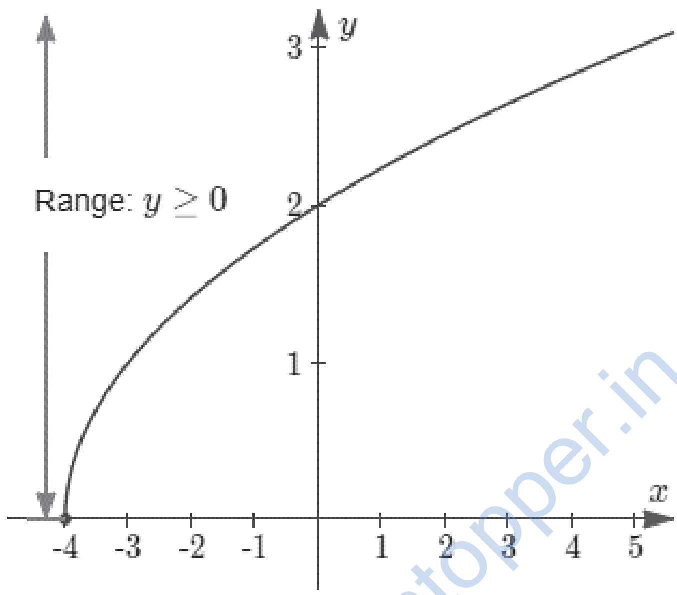
Functions assign outputs to inputs. The domain of a function is the set of all possible inputs for the function. For example, the domain of $f(x)=x^2$ is all real numbers, and the domain of $g(x)=1/x$ is all real numbers except for $x=0$. We can also define special functions whose domains are more limited.

- **Thus, the domain of a function is the set of all values the independent variable can take.**
- **The domain can be specified explicitly or implicitly. When it is implicit, the domain is the set of all real numbers for which the function makes sense.**

Let us take an example:

$$f(x) = \sqrt{x+4}$$

If we plot a graph it will look like the one given below:



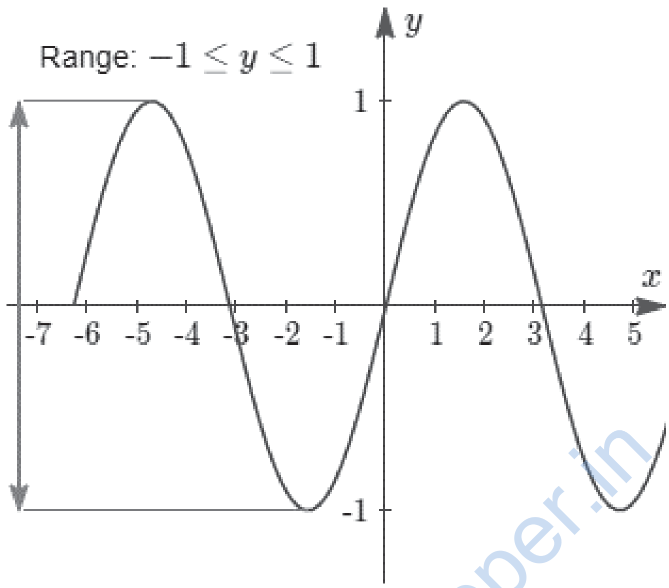

The domain of this function is $x \geq -4$, since $x$ cannot be less than $-4$. If we try to put value less than -4 we cannot calculate the square root.

**The range of a function is the complete set of all possible resulting values of the dependent variable after we have substituted the domain.**

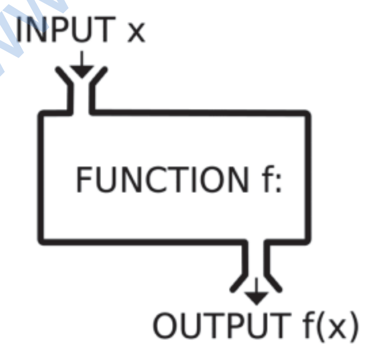
We notice that the curve is either on or above the horizontal axis. For $x \geq -4$, we will always get a zero or positive value of $y$. We say the **range** in this case is

$y \geq 0$.

Similarly for $y = \sin x$, we can see that range to be between $-1$ and 1.




**A function can be considered as a machine that processes elements of its domain to produce elements of its range.** When an element x of domain is fed into input of the machine, the element f(x) of the range comes out in the output as illustrated in the following diagram:






**Check your Progress 3 :**

a) Click on the following link of a GeoGebra applet to understand the concept of range and domain of a function :
https://www.geogebra.org/m/VGCbyDfr

b) Plot a graph using a spreadsheet and find out the range of the following functions :

$$f(x) = \cos x \text{ and } f(x) = \tan x.$$

## 8.5 Types of functions:

There are different types of functions. We will try to explore some functions

### a) Polynomial Functions

A function $f : R \to R$ is said to be a polynomial function, if for each $x \in R$.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

where $a_0, a_1, a_2, a_3, \quad , a_n$ are real numbers and n is a non-negative integer. The degree of the polynomial is n if $a_0 \neq 0$.

**Example:** $f(x) = x^3 - x^2 + \sqrt{2}x + 3$ is a polynomial of degree 3.

$f(x) = x^{3/2} + 2x + 1$ is not a polynomial.

$f(x) = 3x + 1$ is a linear polynomial.

$f(x) = -7x^0$ is polynomial of degree 0.

### b) Rational Function:

A function defined by the quotient of two polynomials is called a rational function.

Thus $f(x) = \dfrac{f(x)}{g(x)}, g(x) \neq 0$ is a rational function.

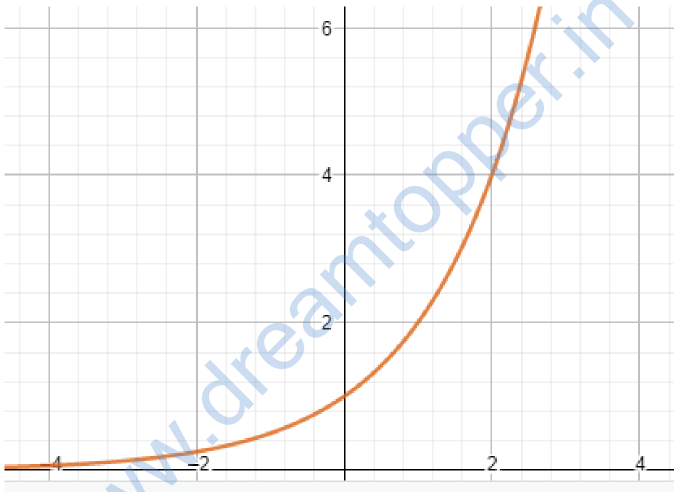
$f(x) = \dfrac{2x^3 + x + 5}{x^2 - 3x}$ is a rational function.

**Note:** This function is not defined at $x = 0$ and $x = 3$. ( Why?)

**c)** **Exponential Function:**

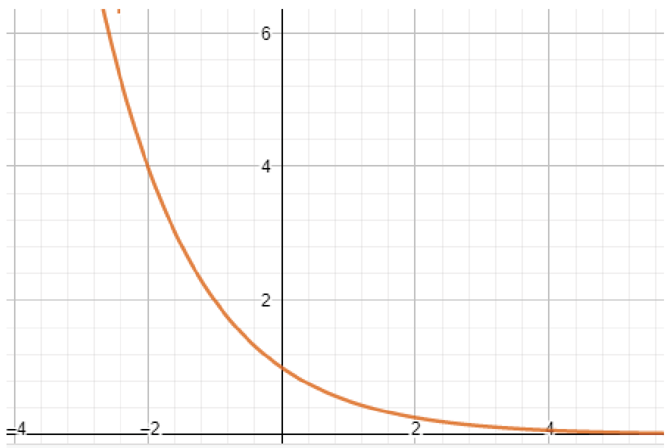
Let $x$ be any real number and $a > 0$ but $a \neq 1$, $f(x) = a^x$ is called exponential function with base $a$. For different values to the base $a$, the function $f(x) = a^x$ have different characteristics.

If $a > 1$, $f(x) = a^x$ is strictly increasing.

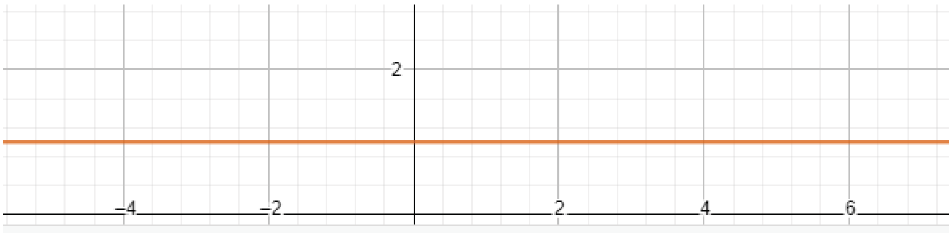





If $0 < a < 1$, $f(x) = a^x$ is strictly decreasing.




If $a = 1$, the graph of $f(x) = a$ is a horizontal line.



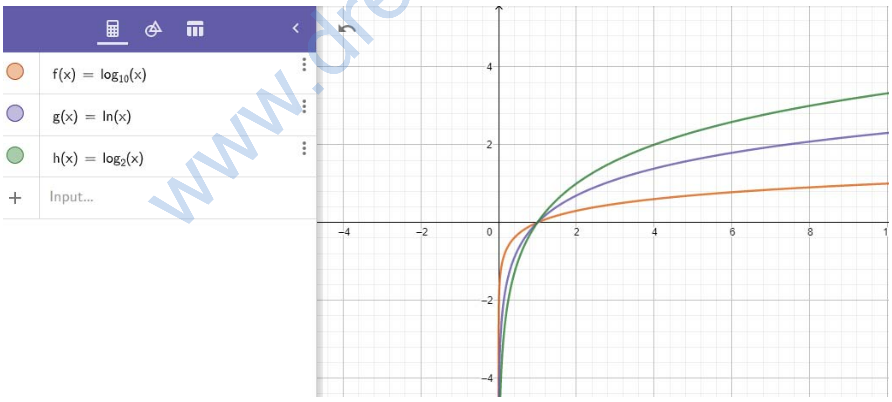

The domain of $a^x$ (with given restrictions on a) is R.

The range of $a^x$ is R i.e., $a^x$ is always positive.

**d) Logarithmic functions :**

These functions are the inverses of exponential functions. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential equation $x = a^y$ only under the following conditions: $x = a^y$, $a > 0$, and $a \neq 1$. It is called the logarithmic function with base a. The graphs of logarithmic functions with bases 2, e and 10 are given below.



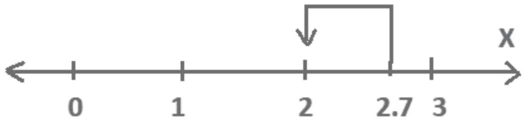



**e) Greatest Integer Function**

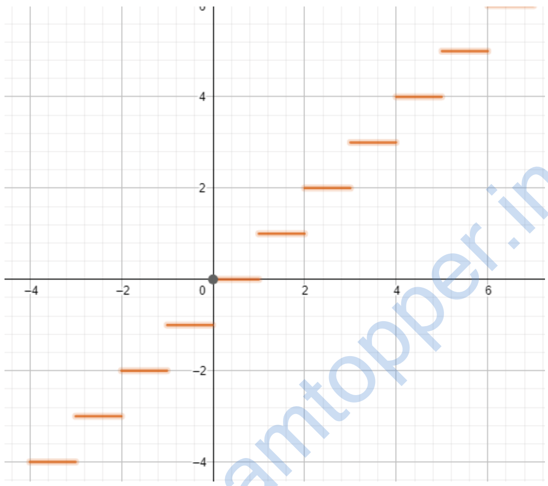
Let x be a real number. The function $f(x) = \lfloor x \rfloor$ where $f(x) = \lfloor x \rfloor$ is the greatest integer $\leq x$ is

called greatest integer function. This is also called step function.

If we examine a number line with the integers and 2.7 plotted on it, we see $\lfloor x \rfloor$




If we plot f(x) = ⌊ x ⌋ we get the following graph

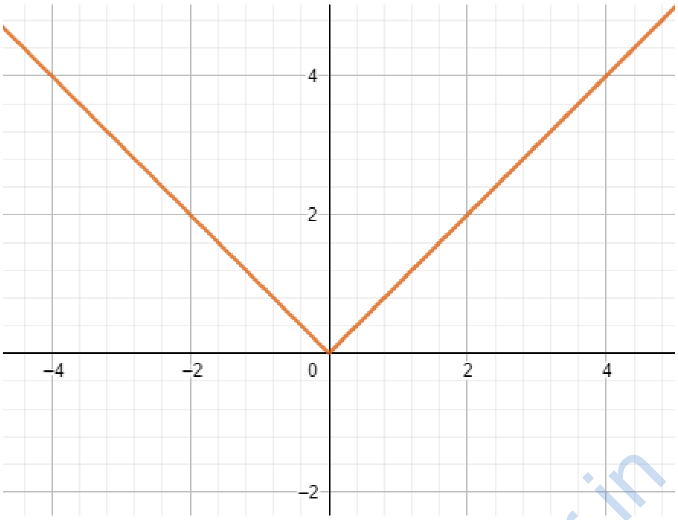


f) **Modulus Function**

For each non-negative value of x, f(x) is equal to x. But for negative values of x, the value of f(x) is the negative of the value of x.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It is also called the **absolute value function.** Graph of | x | is given below:



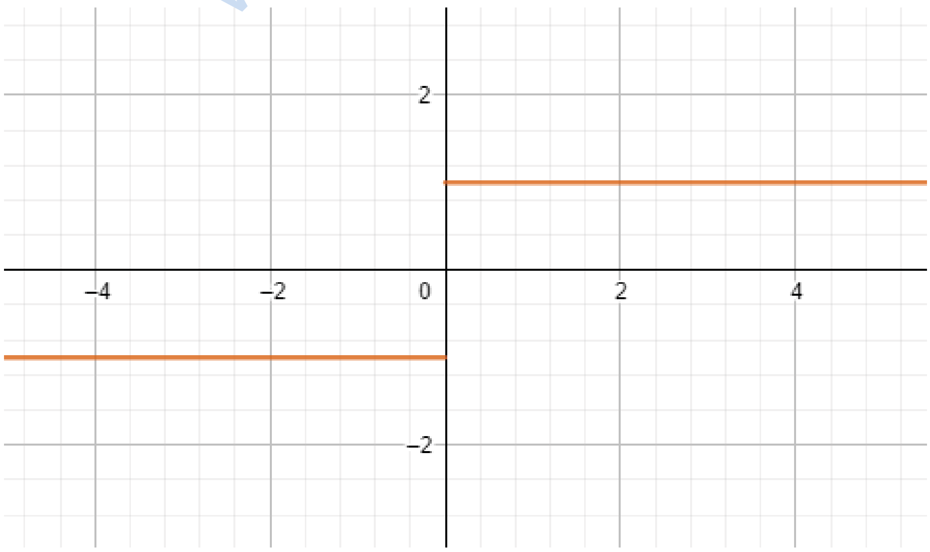

## g) Signum Function

The signum of a real number, also called sgn or signum, is -1 for a negative number (i.e., one with a minus sign "), 0 for the number zero, or +1 for a positive number (i.e., one with a plus sign"). In other words, for any real x

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

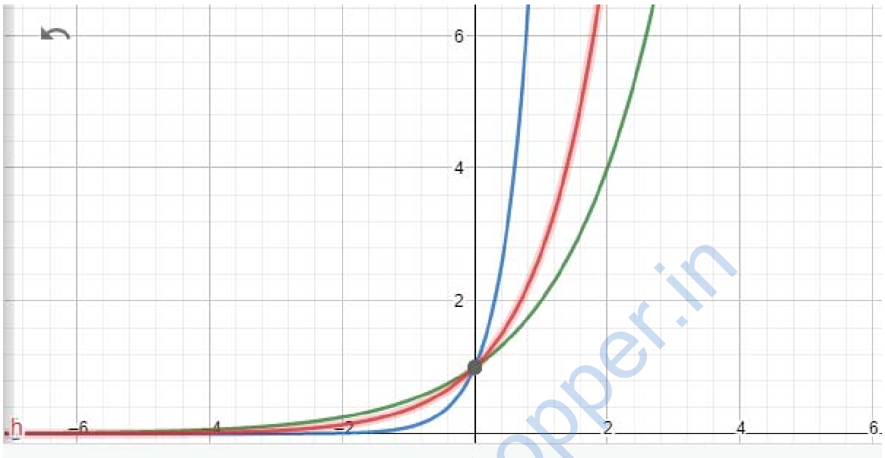


Plot of signum function is given below:



**Check your Progress 4:**

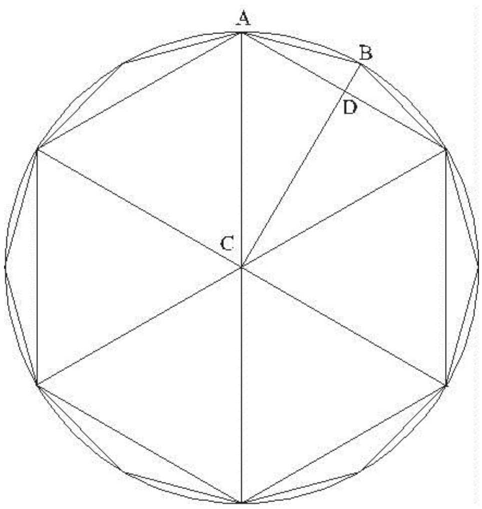
Following are the graphs of h (x) = $e^x$, g(x) = $10^x$ and f(x) = $2^x$ plotted using Geo Gebra graphing calculator. Identify the colour of the graph corresponding to each function.




## 8.6 Limit of a function:

For understanding Calculus, we need to understand the concept of limit or limiting process. The concept of limit was used to understand the area of a circle in earlier days. The area enclosed by a circle of radius r is given by $\pi r^2$.

We can view the circle as a limit of a sequence of regular polygons as shown above:

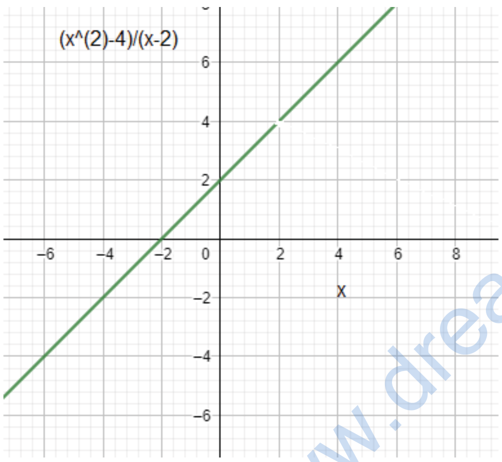


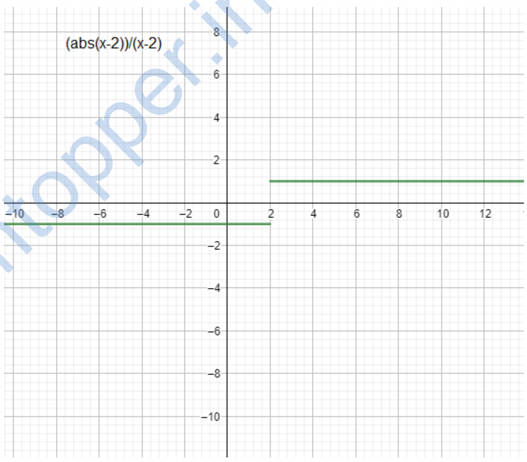
Area of a regular polygon can be obtained by multiplication of half of its perimeter with the distance from the centre to its sides.

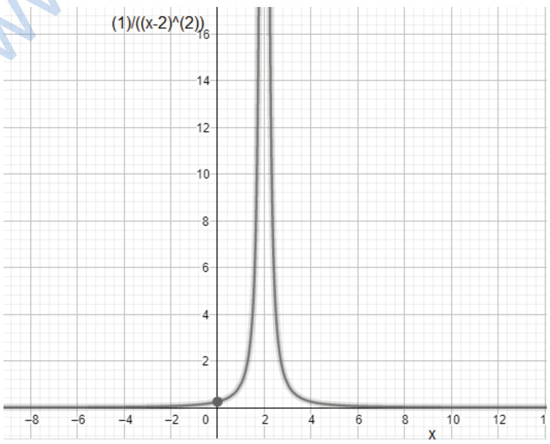
Thus area of circle can be calculated as below if the side of the polygon tends to zero (limiting case).

$A = 1/\boxed{2} \times 2\pi r \times r = \pi r^2$.

Let us understand the concept of limit in case of a function in intuitive way before the formal definition. Note the behavior of each graph at x=2



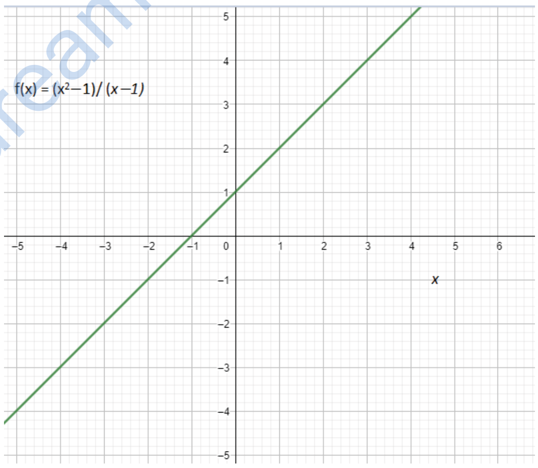
$$f(x) = \frac{x^2 - 4}{x - 2} \qquad g(x) = \frac{|x-2|}{x-2} \qquad h(x) = \frac{1}{(x-2)^2}$$

(a) (b) (c)

We can easily find that these functions are undefined at x=2. Such a statement does not give us a complete picture of the function at this value. To explain the behaviour of graph near x=2 we need to introduce the concept of limit.

Let us consider another function : $f(x) = (x^2 - 1)/(x - 1)$. When we put the value of x= 1 we find that the function is undefined. Now let us put some values other than x = 1 and find out the value of the function as given below.

| X | $(x^2 - 1)/(x - 1)$ |
|---|---|
| 0.5 | 1.50000 |
| 0.9 | 1.90000 |
| 0.99 | 1.99000 |
| 0.999 | 1.99900 |
| 0.9999 | 1.99990 |
| 0.99999 | 1.99999 |
| ... | ... |
| 1.0001 | 2.00010 |
| 1.001 | 2.00100 |
| 1.01 | 2.01000 |
| 1.1 | 2.10000 |
| 1.5 | 2.50000 |



It can be observed that as we get close to 1 we get the value of function close to 2. It means that though the function at x=1 is not defined as we get the value close to 1 we get the value of the function approaches 2. In Mathematics we use a concept of limit to express this situation. This result can be stated as :

The **limit** of $(x^2-1)/$ *(x– 1)* as x approaches 1 is **2**

Symbolically same can be written as : $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}=2$

**Illustrations:**

Let us find out the limit of some functions:

1. Find the limit of $f(x)=4x$, as x approaches 3.

   Substitute the value of x and we get f(3) = 4x3 =12

2. Let us now find out limit of another function as x tends to zero

   $$\lim_{x\to 0}\frac{6x^2-7x}{x}$$

   When we substitute x =0 we do not get a definite result.

   Let us simplify and see what happens.

   $$\frac{6x^2-7x}{x}=\frac{x(6x-7)}{x}=6x-7$$

   Now when we put the value of x=0 we get

   $$\lim_{x\to 0}(6x^2-7)=-7$$

**Algebra of Limits:**

| | |
|---|---|
| The limit of a sum is the sum of the limits. | $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$<br>if both limits on the right-hand side exist and are finite. |
| The limit of a product is the product of the limits. | $\lim_{x \to a} (f.g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$<br>if both limits on the right-hand side exist and are finite. |
| The limit of a quotient is the quotient of the limits. | $\lim_{x \to a} (f/g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$<br>if both limits on the right-hand side exist and are finite and the limit in the denominator is not zero |



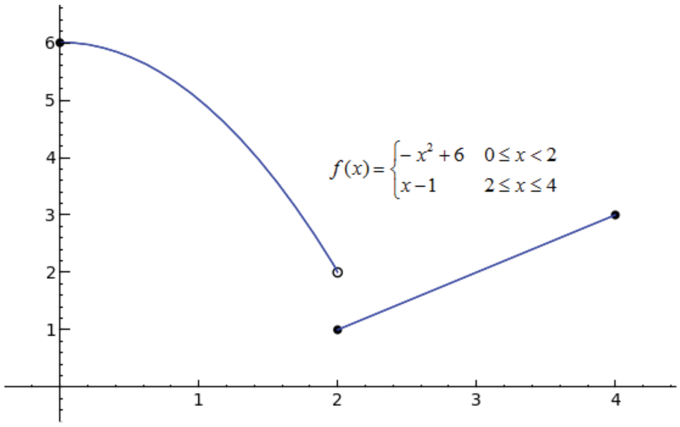
**Check your progress: 5**

a) Find : $\lim_{x \to 2} (8 - 3x + 12x^2)$

b) Find: $\lim_{x \to 8} \dfrac{2z^2 - 17z + 8}{8 - z}$

## 8.7 Continuity of a function:

In the case of f(x) = $(x^2 - 1)/(x - 1)$ discussed earlier, irrespective of value of x approaches to 1 from a value less than one or greater than one, the value of the limit of the function is 2. We may not always get a unique value in these two cases. See the above example.




We can say that the limit of f(x) as x approaches 2 from the left is 2, and the limit of f(x) as x approaches 2 from the right is 1. This can be written as:

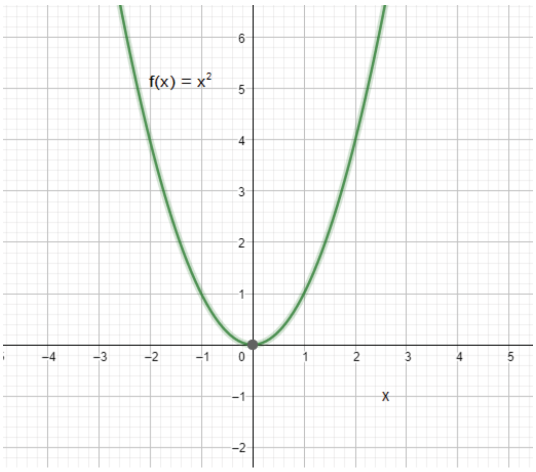
$$\lim_{x\to 2^-} f(x)=2$$

$$\lim_{x\to 2^+} f(x)=1$$

The (-) sign indicates "from the left", and (+) sign indicates "from the right".

The function is said to be **discontinuous.** In other words we can say that limit of this function does not exist at x=2.

On the other hand, for the function f(x) = $x^2$, it can be easily verified that at any point, x=a, lim f(x) = f(a) and the graph of the function is smooth curve having no break anywhere.

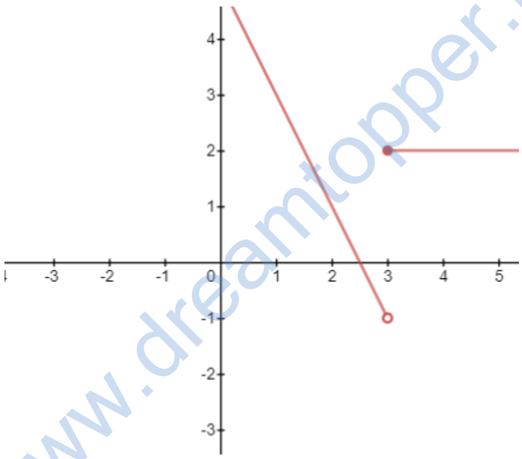


The function f(x) is said to be continuous at the point x = a if its finite value at x = a is equal to the limiting value of f(x) at the point x = a. Thus for continuity at x = a

Left hand limit = right hand limit = value of the function

**Check your progress 6:**

a) Study the graph given below:



(i) What y-value is the function approaching as x approaches 3 from the left?

(ii) What y-value is the function approaching as x approaches 3 from the right?

(iii) What (if any) is the actual y-value at x=3? What can you conclude about the function?

b) Following are the examples of Some Continuous Functions. Reflect anc discuss with your friends.

(i) A constant function f(x) = c is continuous everywhere.

(ii) Function f(x) nxn; n ∊ N is continuous on R.

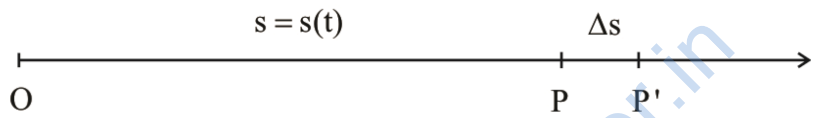
(iii) sin x, cos x are continuous functions on R.

(iv) $f(x) = |x|$ is continuous function on R.

(v) Polynomial functions are always continuous.

## 8.8 Instantaneous rate of change and derivative:

One of the important applications of the Calculus is the motion of a particle along a straight line.



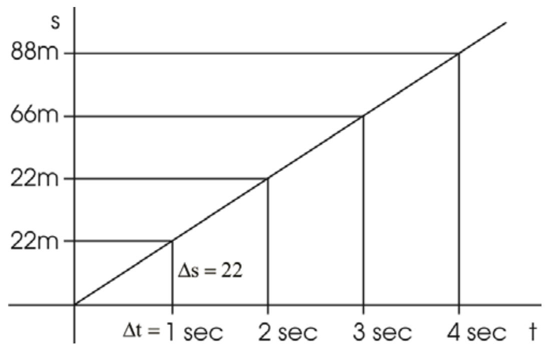

Consider a particle moving in a straight line from a fixed point O to a given point P, and let t be the time elapsed. Then to each value of t there will correspond a distance s, which will be a function of t:

$s = s(t)$.

If the particle moves with a constant velocity of 22 m/s, we can write the distance covered

$s = 22t$

If we plot the function s(t) with respect of t we get the following graph:

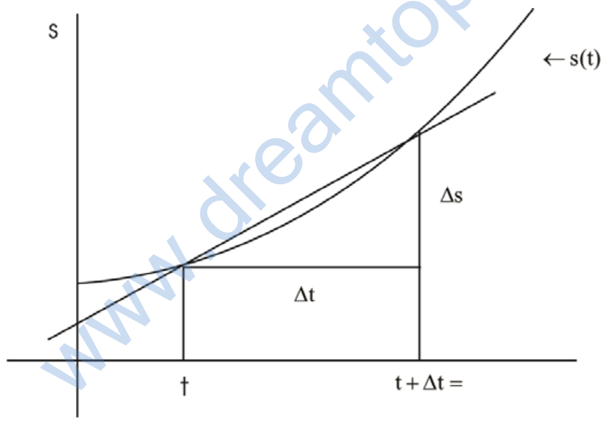


Velocity, which is rate of change of s with respect of t can be calculated by finding out the slope of the graph

$$\frac{\Delta s}{\Delta t} = 22 \text{ meters per second}$$

We can easily see that the value of the slope is constant as velocity is constant in this case.

But in the following case where the velocity of the object is not constant. One can find out average value of velocity over a period of time. But how can we find out the velocity of an object at a particular instant t or when $\Delta t = 0$




You may wonder how can we find the value of velocity when $\Delta t = 0$. Here the concept of limit will help us in finding out a definite value of this instantaneous velocity. We can define instantaneous velocity using slope as the limiting value of $\Delta s / \Delta t$ when $\Delta t$ tends to zero. This can be written as :

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

It can also be said to be the value of slope of the tangent line at t.

Thus instantaneous velocity is said to be derivative of s with respect to t .
**Differentiation** is the action of computing a derivative.

The derivative of a function y = f(x) of a variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x. If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of the tangent to this graph at each point.

For example derivative of the function f at a can be written as :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

When the limit exists, f is said to be differentiable at a. Here f′(a) is one of several common notations for the derivative If f(x) is a function of x derivative can also be written as: $\frac{df}{dx}$

If we use this definition to obtain the derivative then we are using first principle to find out derivative.

Illustration : Let us find out derivative of $y = 2x^2 + 3x$. using first principle:

$$f(x) = 2x^2 + 3x \text{ so}$$
$$f(x+h) = 2(x+h)^2 + 3(x+h)$$
$$= 2(x^2 + 2xh + h^2) + (3x + 3h)$$
$$= 2x^2 + 4xh + 2h^2 + 3x + 3h$$

We now need to find

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\left[2x^2+4xh+2h^2+3x+3h\right]-\left[2x^2+3x\right]}{h} \\
&= \lim_{h\to 0}\frac{4xh+2h2+3h}{h} \\
&= \lim_{h\to 0}(4x+2h+3) \\
&= 4x+3
\end{aligned}
$$

It will be worthwhile to know derivatives of certain functions.

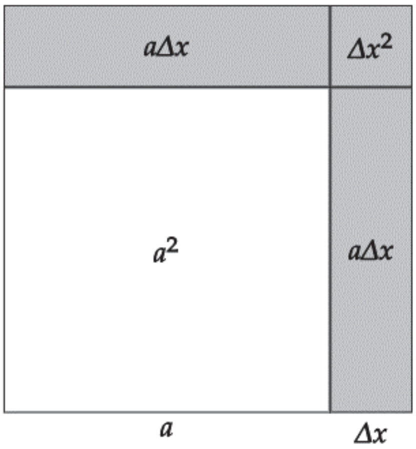


| Common Functions | Function | Derivative |
|---|---|---|
| Constant | c | 0 |
| Line | x | 1 |
| | ax | a |
| Square | $x^2$ | 2x |
| Square Root | √x | $(½)x^{-½}$ |
| Exponential | $e^x$ | $e^x$ |
| | $a^x$ | ln(a) $a^x$ |
| Logarithms | ln(x) | 1/x |
| | $\log_a(x)$ | 1 / (x ln(a)) |

**Check your Progress 7:**

a) Show that the derivative of a constant is zero and derivate of ax with respect to x is a.

b) Let function $y=x^2$ that measures the area of a metallic square of side x. If at any given time the side of the square is a, and we heat the square uniformly increasing the side, what is the tendency of change of the area in that moment?




Some rules which will be of help in finding derivatives are:

| **Rules** | **Function** | **Derivative** |
|---|---|---|
| Multiplication by constant | cf | cf′ |
| Power Rule | $x^n$ | $nx^{n-1}$ |
| Sum Rule | f + g | f′ + g′ |
| Difference Rule | f - g | f′ – g′ |
| Product Rule | fg | f g′ + f′ g |
| Quotient Rule | f/g | $(f' g - g' f)/g^2$ |
| Reciprocal Rule | 1/f | $-f'/f^2$ |
| Trigonometry (x is radians) | sin(x) | cos(x) |
| | cos(x) | –sin x |
| | tan(x) | $\sec^2 x$ |

## 8.9 Derivatives of algebraic functions using chain rule :

A function is *composite* if you can write it as *f*(*g*(*x*). In other words, it is a function within a function, or a function of a function. For example $(5x-2)^3$ is composite function. Here 5x -2 can also be considered as a function. Chain rule helps us in finding the derivative of a composite function.

If f is a composite function then derivative of f with respect x can be written using chain rule as :

$$\frac{d}{x}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$$

Where g is the inner function and f is the outer function.

Let us find out the derivative $(5x\text{-}2)^3$ using chain rule

$(5x\text{-}2)^3$ is made up of $g^3$ and 5x-2:

- $f(g) = g^3$
- $g(x) = 5x–2$

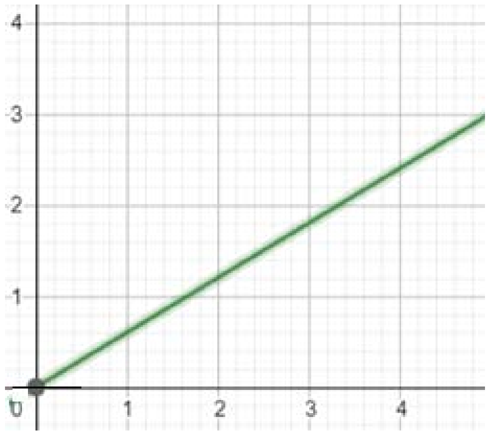
The individual derivatives are:

$f'(g) = 3g^2$ (by the Power Rule)

$g'(x) = 5$

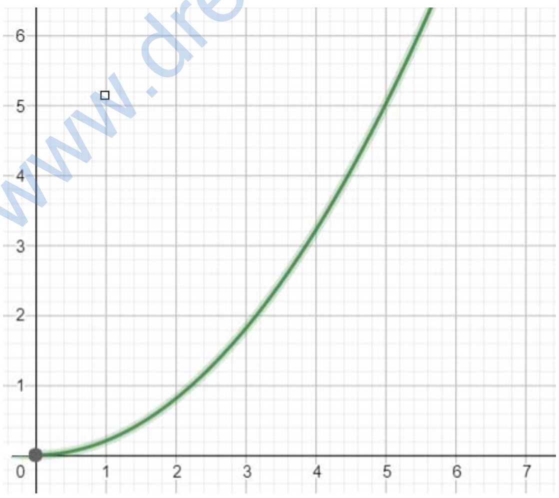
Therefore derivative $(5x\text{-}2)^3 = 3\ (5x\text{-}2)^2 x5\ = 15\ (5x\text{-}2)^2$

One way to interpret the derivative f' is to say that f'(k) at k is the rate of change of f at x =k Let us say that a water tank is filled at constant rate and the volume of water in the tank is a function of time given by V (t) =3/5 t . If we plot a graph of V(t) with respect to t we get the following :




Here 3/5 (which is a constant) is the slope of the graph and can be determined by finding the derivative of V(t) with respect to t. That means the water tank is being filled at the rate of 3/5 liters per second. But the situation may not be that simple. If the volume of water being filled in the tank is non linear and is given by the $V(t) = 0.2\ t^2$ . Here we will not have constant rate of change of volume. Let us plot the graph and see.






We can find instantaneous rate of change of volume of water in the tank by finding the derivative of V(t), which comes out to be 0.2x2t = 0.4t. This value changes with change in time (t). If we want to find the rate of change of volume with respect to time at any instant we just substitute the value. For example, the rate of change will with 2 L/s at t =5 second

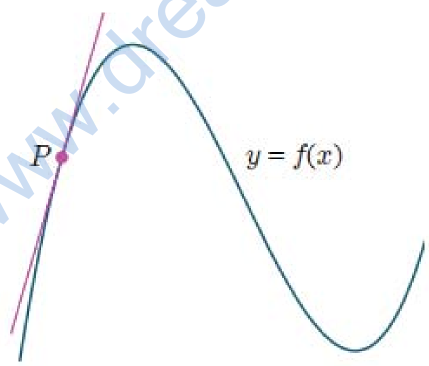
**Check your Progress 8 :**

a) Find the rate of change of the area of a circle with respect to its radius r when r = 5 cm.

b) On heating, the volume of a metal cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of an edge is 10 centimeters?

## 8.10 Tangent line and equation of tangent :

A tangent line (or simply tangent) to a plane curve at a given point is a straight line that just touches the curve at that point. More precisely a straight line is said to be a tangent of a curve y = f(x) at point x = c if the line passes through the point ( c, f ( c) ) on the curve and has a slope f'(c ) where f' is the derivative of f. In the following diagram a tangent is drawn at the point P.

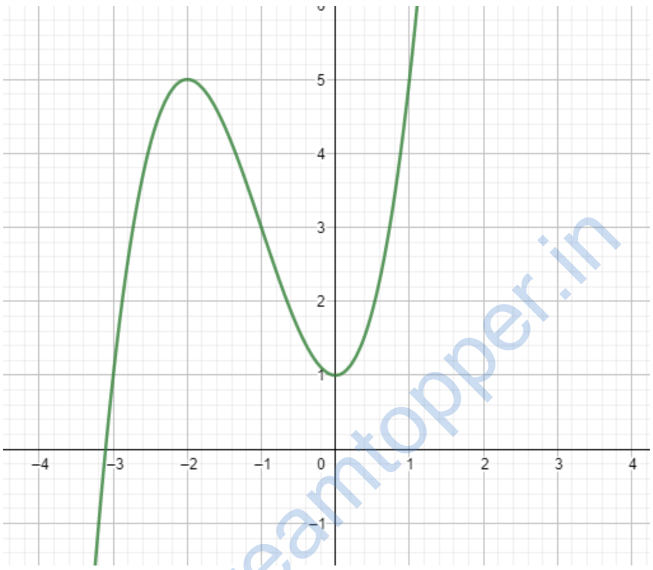





When we say tangent line for a function can be found by computing the derivative, it only means that " the derivative measures the slope of the tangent line". Following step can be used to find the slop of the a tangent line for a function f(x) at a given point x = a.

1. Find the derivative f'(x)
2. Put the value of x =a to get the slope. i.e slope m = f'(a)
3. Find out the y coordinate of the point which is b = f(a)

Use the formula y = m(x-a) +b  and put values of m,a and b to get the equation of tangent line at point ( a,b) Let us take a concrete example to understand this procedure. For a function $f(x) = x^3 + 3x^2 + 1$. We'll find the tangent lines at a few different points. Plot of the graph is as under:



Let us find out the equation of tangent lines at the points x = -3,

Case : 1    x = a = -3

The derivative of $x^3 + 3x^2 + 1$. Is $3x^2 + 6x$

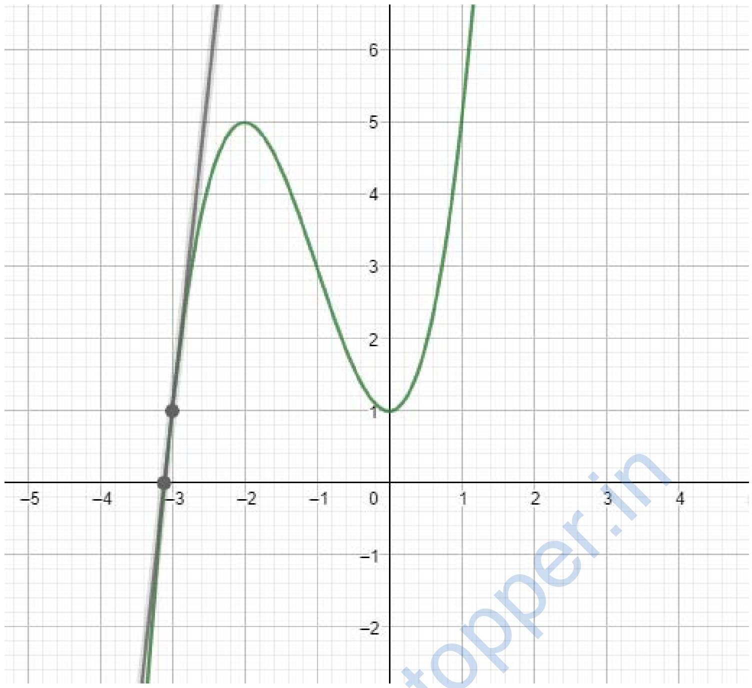
Slope m = f'(a) = 9

If the value of x = a= -3 value of y = b = 1

Thus equation of the tangent line is y =m (x-a)+b

Thus equation of tangent line at point ( -3,1) to the function $f(x) = x^3 + 3x^2 + 1$ is y = 9(x+3)+1

If we draw this tangent line it will look like the one given below:






**Check your Progress 9:**

a. For the function $f(x) = x^3 + 3x^2 + 1$ given above, tangent line at point -3 is drawn using GeoGebra graphing calculator. Draw the tangent line using this application at x= -2 and x=1.

b. Find the equations of a line tangent to $y = x^3-2x^2+x-3$ at the point x=1.

## Summary:

- Function, in mathematics is an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable).
- In a function, the input value must have one and only one value for the output.
- The domain of a function is the set of all values, the independent variable can take.
- The range of a function is the complete set of all possible

resulting values of the dependent variable after we have substituted the domain.

- Different types of functions are Polynomial function, rational function, Exponential function, Logarithmic function, Greatest integer function, Signum functions etc. All these functions can be plotted on a graph.
- Concept of limit can be used to find out the value of function when it is not defined at a particular value.
- The function f(x) is said to be continuous at the point x = a if its finite value at x = a is equal to the limiting value of f(x) at point x = a. Thus for continuity at x = a, Left hand limit = right hand limit = value of the function
- The derivative of a function y = f(x) of variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x.
- If x and y are real numbers, and if the graph of f is plotted against x, the derivative is the slope of the tangent to this graph at each point.
- Derivative of the function f at a can be written as :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- If f is a composite function then derivative of f with respect x can be written using chain rule as :

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) g'(x)$$

Where g is the inner function and f is the outer function.

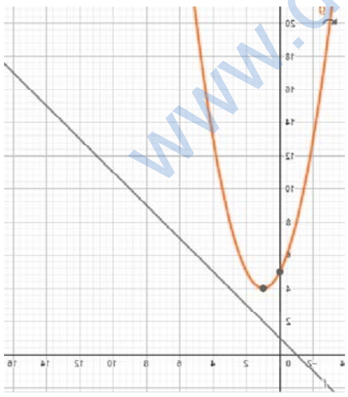
- The concept of derivative can be used to find the slope of the tangent line for a function f(x) at a given point x = a. Steps are:
  - Find the derivative $f'(x)$
  - Put the value of x =a to get the slope. i.e slope m = $f'(a)$
  - Find out the y coordinate of the point which is $b = f(a)$
  - Use the formula $y = m(x-a) + b$ and put values of m,a and b to get the equation of tangent line at point (a, b)

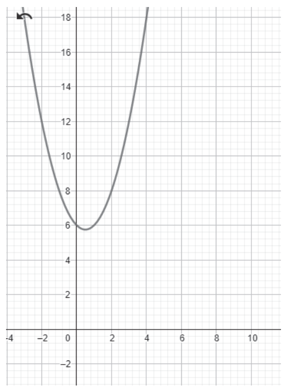
**Solutions to Check your Progress :**

1 a (i) is not a function as input 1 has two outputs: 0 and 5.

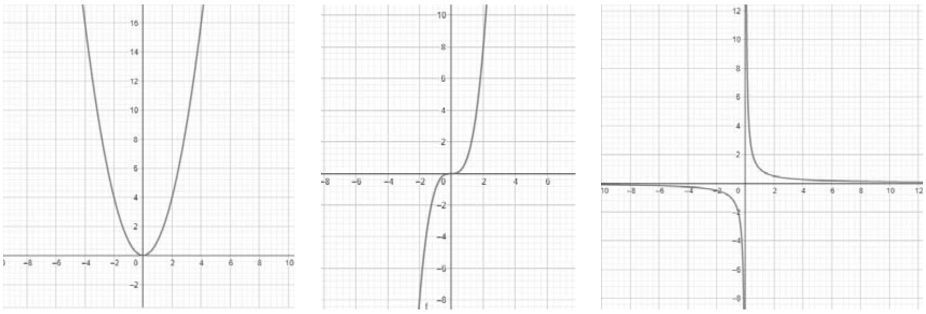
(ii) is a function. Each input has only one output. Same output (4) for same point does not matter.

b) $(f+g)(x) = x^2-x+6$ and the graphs are as under:





2. Graphs are as under :



a) b) c)

3.

- $f(x) = 2^x$
- $g(x) = 10^x$
- $h(x) = e^x$



4. Answers:

a) $\lim\limits_{x \to 2} (8 - 3x + 12x^2) = 8 - 3(2) + 12(4) = \boxed{50}$

b) $\lim\limits_{z \to 8} \dfrac{(2z^2 - 17z + 8)}{8 - z} = \lim\limits_{z \to 8} \dfrac{(2z-1)(z-8)}{-(z-8)} = \lim\limits_{z \to 8} \dfrac{2z-1}{-1} = \boxed{-15}$

5 a)

(i) y = -1
(ii) y=2
(iii) y = 2 , the function is not continuous at x=2

6 b) Rate of change of area with respect to side can be calculated by finding the derivative of $x^2$ with respect to x and if we find its value of at 'a' we get the tendency of change of the area in that moment. By using first principle we get this value = 2a

7 a) Putting value of r in 2 π r we get $10\pi\ cm^2 / cm$ .

b) $3.6\ cm^2 /s$

8. b) $y' = 3x^2 - 4x + 1$

At x=1 we get the value of slope = $3(1)^2 -4(1) +1 = 3-4+1= 0$.

Value of y at x =1 is $b = (1)^3-2(1)^2+1-3 = 1-2+1-3 = -3$.

Using the formula : $y = m(x-a) +b = -3$

$y=-3$ is the equation of tangent at $x=1$