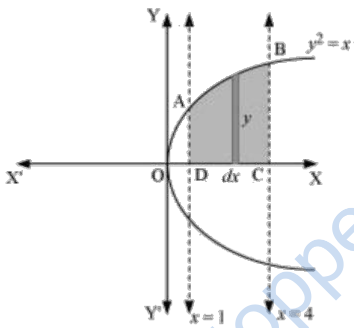
Chapter 8 Applications of Integrals

EXERCISE 8.1

Question 1:

Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1, x = 4$ and the x-axis in the first quadrant.

Solution:

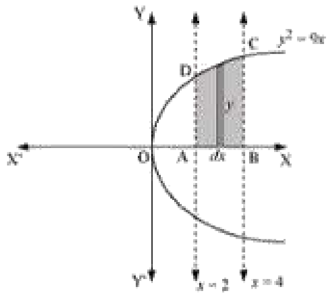


$$ar(ABCD) = \int_1^4 y\,dx$$
$$= \int_1^4 \sqrt{x}\,dx$$
$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$
$$= \frac{2}{3}[8-1]$$
$$= \frac{14}{3}$$

Question 2:

Find the area of the region bounded by $y^2 - 9x, x = 2, x = 4$ and the x-axis in the first quadrant.

Solution:



$$ar(ABCD)=\int_2^4 y\,dx$$

$$=\int_2^4 3\sqrt{x}\,dx$$

$$=3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$=2\left[x^{\frac{3}{2}}\right]_2^4$$

$$=2\left[(4)^{\frac{3}{2}}-(2)^{\frac{3}{2}}\right]$$

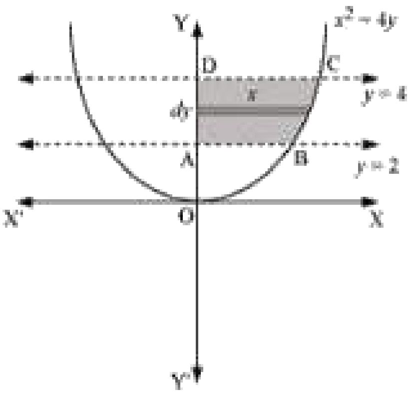
$$=2\left[8-2\sqrt{2}\right]$$

$$=\left(16-4\sqrt{2}\right)$$

Question 3:

Find the area of the region bounded by $x^2=4y, y=2, y=4$ and the *y*-axis in the first quadrant.

Solution:



$$ar(ABCD) = \int_2^4 x\,dy$$

$$= \int_2^4 2\sqrt{y}\,dy = 2\int_2^4 \sqrt{y}\,dy$$

$$= 2\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

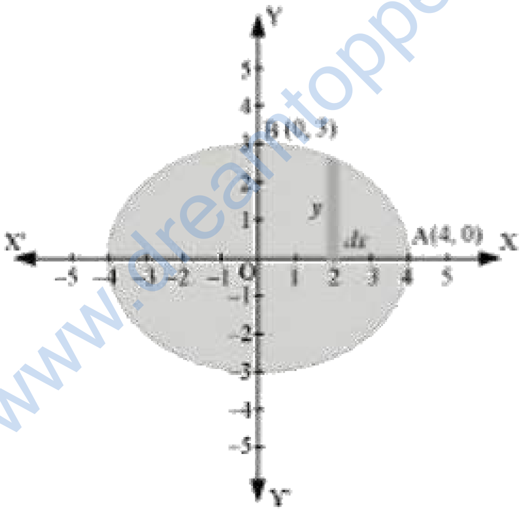
$$= \frac{4}{3}\left[(4)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right] = \frac{4}{3}\left[8 - 2\sqrt{2}\right]$$

$$= \left(\frac{32 - 8\sqrt{2}}{3}\right)$$

Question 4:

Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution:



It is given that

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\Rightarrow \frac{y^2}{9} = 1 - \frac{x^2}{16}$$

$$\Rightarrow y = 3\sqrt{1 - \frac{x^2}{16}}$$

Area of ellipse $= 4 \times ar(OAB)$

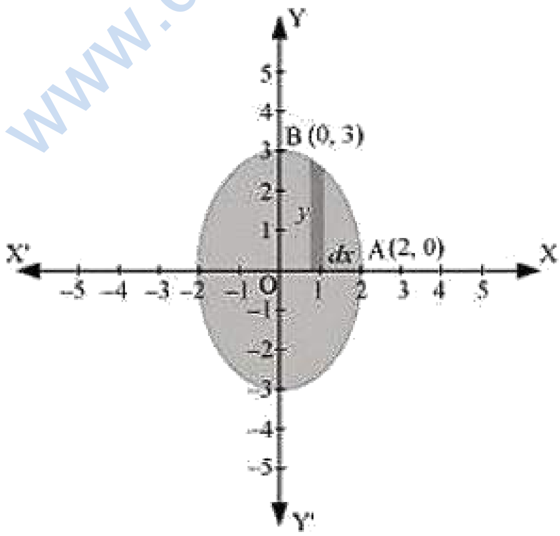
$$ar(OAB)=\int_0^4 y\,dx$$
$$=\int_0^4 3\sqrt{1-\frac{x^2}{16}}dx$$
$$=\frac{3}{4}\int_0^4\sqrt{16-x^2}dx$$
$$=\frac{3}{4}\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$
$$=\frac{3}{4}\left[2\sqrt{16-16}+8\sin^{-1}(1)-0-8\sin^{-1}(0)\right]$$
$$=\frac{3}{4}\left[\frac{8\pi}{2}\right]$$
$$=\frac{3}{4}[4\pi]$$
$$=3\pi$$

Area of ellipse $=4\times3\pi=12\pi$ units

Question 5:

Find the area of the region bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

Solution:



It is given that

$$\Rightarrow\frac{x^2}{4}+\frac{y^2}{9}=1$$
$$\Rightarrow y=3\sqrt{1-\frac{x^2}{4}}$$

Area of ellipse $=4\times ar(OAB)$

$$ar(OAB)=\int_0^2 ydx$$

$$=\int_0^2 3\sqrt{1-\frac{x^2}{4}}dx$$

$$=\frac{3}{2}\int_0^2\sqrt{4-x^2}dx$$

$$=\frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$=\frac{3}{2}\left[\frac{2\pi}{2}\right]$$

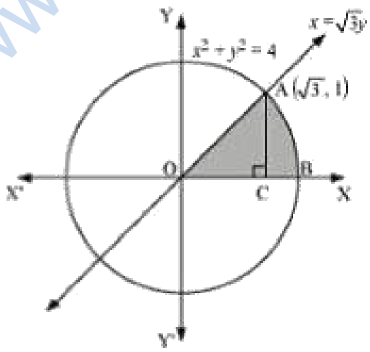
$$=\frac{3\pi}{2}$$

Area of ellipse $=4\times\frac{3\pi}{2}=6\pi$ units.

Question 6:

Find the area of the region in the first quadrant enclosed by x-axis, line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$

Solution:



$$ar(OAB)=ar(\Delta OAC)+ar(ABC)$$

$$ar(\Delta OAC)=\frac{1}{2}\times OC\times AC$$

$$=\frac{1}{2}\times\sqrt{3}\times 1$$

$$=\frac{\sqrt{3}}{2}$$

$$ar(ABC) = \int_{\sqrt{3}}^{2} y\,dx$$

$$= \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= \left[2\times\frac{\pi}{2} - \frac{\sqrt{3}}{2}\sqrt{4-3} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \left[\pi - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{3}\right)\right]$$

$$= \left[\pi - \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= \left[\frac{3\pi - 2\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

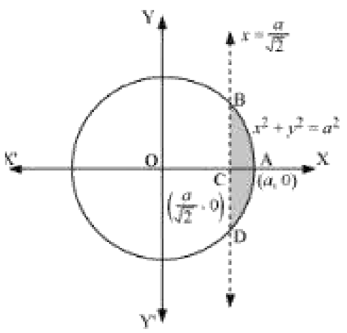
Therefore, required area enclosed $= \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ square units.

Question 7:

Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$.

Solution:

The area of the smaller part of the circle, $x^2 + y^2 = a^2$ cut off by the line, $x = \frac{a}{\sqrt{2}}$, is the area ABCD.



It can be observed that the area ABCD is symmetrical about x-axis.

$$ar(ABCD) = 2 \times ar(ABC)$$

$$\begin{aligned} ar(ABC) &= \int_{\frac{a}{\sqrt{2}}}^{a} y\,dx \\ &= \int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}\,dx \\ &= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{\frac{a}{\sqrt{2}}}^{a} \\ &= \left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{a}{2\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} - \frac{a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] \\ &= \frac{a^2\pi}{4} - \frac{a}{2\sqrt{2}}.\frac{a}{\sqrt{2}} - \frac{a^2}{2}\left(\frac{\pi}{4}\right) \\ &= \frac{a^2\pi}{4} - \frac{a^2}{4} - \frac{a^2\pi}{8} \\ &= \frac{a^2}{4}\left[\pi - 1 - \frac{\pi}{2}\right] \\ &= \frac{a^2}{4}\left[\frac{\pi}{2} - 1\right] \end{aligned}$$

$$\begin{aligned} ar(ABCD) &= 2\left[\frac{a^2}{4}\left(\frac{\pi}{2} - 1\right)\right] \\ &= \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right) \end{aligned}$$

Therefore, the required area is $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$ square units.

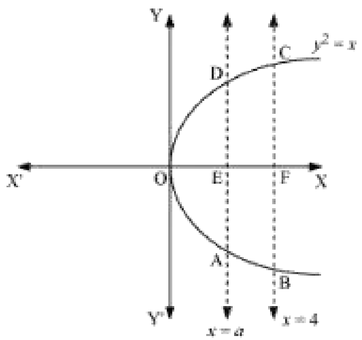
Question 8:

The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

Solution:

The line $x = a$ divides the area bounded by the parabola and $x = 4$ into two equal parts.

Therefore, $ar(OAD) = ar(ABCD)$



It can be observed that the given area is symmetrical about *x*-axis.

Hence, $ar(OED) = ar(EFCD)$

$$ar(OED) = \int_0^a y\,dx$$

$$= \int_0^a \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$= \frac{2}{3}a^{\frac{3}{2}} \qquad \ldots(1)$$

$$ar(EFCD) = \int_a^4 \sqrt{x}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_a^4$$

$$= \frac{2}{3}\left[8 - a^{\frac{3}{2}}\right] \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\Rightarrow \frac{2}{3}(a)^{\frac{3}{2}} = \frac{2}{3}\left[8-(a)^{\frac{3}{2}}\right]$$
$$\Rightarrow 2(a)^{\frac{3}{2}} = 8$$
$$\Rightarrow (a)^{\frac{3}{2}} = 4$$
$$\Rightarrow a = (4)^{\frac{2}{3}}$$

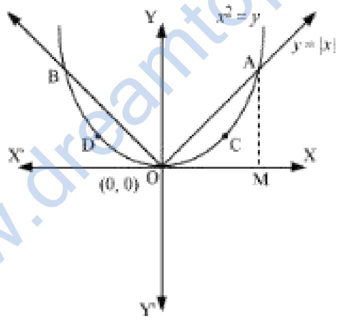
Therefore, the value of $a = (4)^{\frac{2}{3}}$.

Question 9:

Find the area of the region bounded by the parabola $y = x^2$ and the line $y = |x|$.

Solution:

The area bounded by the parabola $y = x^2$ and the line $y = |x|$, can be represented as



The given area is symmetrical about y-axis.

Therefore, $ar(OACO) = ar(ODBO)$

The point of intersection of parabola $y = x^2$ and the line $y = |x|$, is $A(1,1)$.

$ar(OACO) = ar(\Delta OAM) - ar(OMACO)$

$$ar(\Delta OAM) = \frac{1}{2} \times OM \times AM$$
$$= \frac{1}{2} \times 1 \times 1$$
$$= \frac{1}{2}$$

$$ar(OMACO) = \int_0^1 y\,dx$$
$$= \int_0^1 x^2\,dx$$
$$= \left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{3}$$

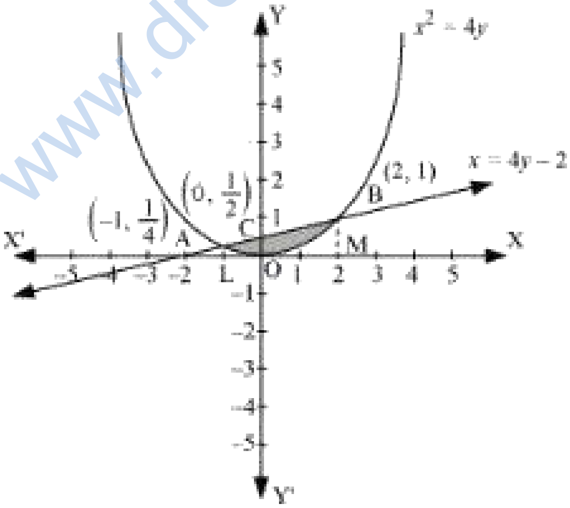
$$ar(OACO) = ar(\Delta OAM) - ar(OMACO)$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}$$

Therefore, the required area $= 2\left[\frac{1}{6}\right] = \frac{1}{3}$ units.

Question 10:

Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution:



Coordinates of point $A\left(-1, \frac{1}{4}\right)$.

Coordinates of point $B(2,1)$.

Draw AL and BM perpendicular to x-axis.

$$ar(OBAO) = ar(OBCO) + ar(OACO)$$

$$ar(OBCO) = ar(OMBC) - ar(OMBO)$$

$$= \int_0^2 \frac{x+2}{4}dx - \int_0^2 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}[2+4] - \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{3}{2} - \frac{2}{3}$$

$$= \frac{5}{6}$$

$$ar(OACO) = ar(OLAC) - ar(OLAO)$$

$$= \int_{-1}^0 \frac{x+2}{4}dx - \int_{-1}^0 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^0 - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$$

$$= -\frac{1}{4}\left[\frac{(-1)^2}{2} + 2(-1)\right] - \left[-\frac{1}{4}\left(\frac{(-1)^3}{3}\right)\right]$$

$$= -\frac{1}{4}\left[\frac{1}{2} - 2\right] - \frac{1}{12}$$

$$= -\frac{1}{8} + \frac{1}{2} - \frac{1}{12}$$

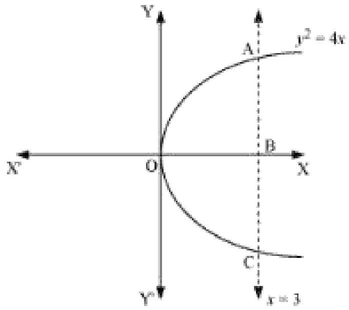
$$= \frac{7}{24}$$

Required area $= \left(\frac{5}{6} + \frac{7}{24}\right) = \frac{9}{8}$ units.

Question 11:

Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Solution:



OACO is symmetrical about *x*-axis.

Therefore, $ar(OACO) = 2 \times ar(AOB)$

$$ar(OACO) = 2\left[\int_0^3 y\,dx\right]$$
$$= 2\left[\int_0^3 2\sqrt{x}\,dx\right]$$
$$= 4\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$
$$= \frac{8}{3}\left[(3)^{\frac{3}{2}}\right]$$
$$= 8\sqrt{3}$$

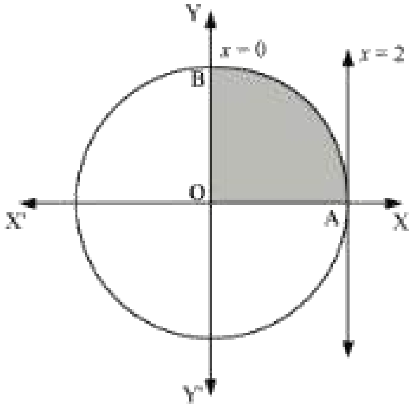
Required area is $8\sqrt{3}$ units.

Question 12:

Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(A) π (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Solution:



$$ar(OAB) = \int_0^2 y\,dx$$

$$= \int_0^2 \sqrt{4-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

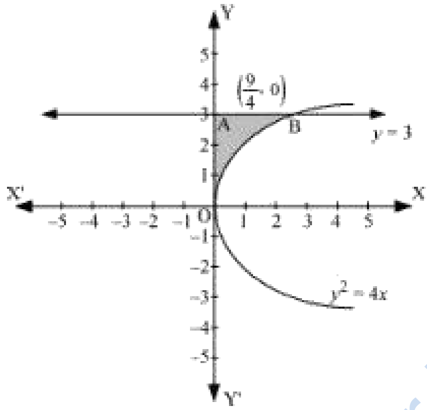
Correct answer is A.

Question 13:

Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(A) 2 (B) $\frac{9}{4}$ (C) $\frac{9}{3}$ (D) $\frac{9}{2}$

Solution:



$$ar(OAB) = \int_0^3 x\,dy$$
$$= \int_0^3 \frac{y^2}{4}\,dy$$
$$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3$$
$$= \frac{1}{12}(27)$$
$$= \frac{9}{4}$$

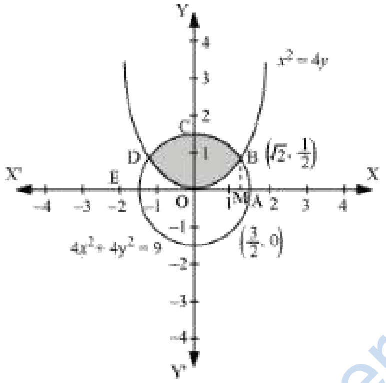
Correct answer is B.

EXERCISE 8.2

Question 1:

Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $x^2=4y$.

Solution:



Solving $4x^2+4y^2=9$ and $x^2=4y$, point of intersection $B\left(\sqrt{2},\frac{1}{2}\right)$ and $D\left(-\sqrt{2},\frac{1}{2}\right)$.
Required area is symmetrical about y-axis.

$$ar(OBCDO)=2\times ar(OBCO)$$

Draw BM perpendicular to OA

Coordinates of M are $\left(\sqrt{2},0\right)$

$$ar(OBCO)=ar(OMBCO)-ar(OMBO)$$

$$=\int_0^{\sqrt{2}}\sqrt{\frac{\left(9-4x^2\right)}{4}}dx-\int_0^{\sqrt{2}}\frac{x^2}{4}dx$$

$$=\frac{1}{2}\int_0^{\sqrt{2}}\sqrt{9-4x^2}\,dx-\frac{1}{4}\int_0^{\sqrt{2}}x^2dx$$

$$=\frac{1}{4}\left[x\sqrt{9-4x^2}+\frac{9}{2}\sin^{-1}\frac{2x}{3}\right]_0^{\sqrt{2}}-\frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$

$$=\frac{1}{4}\left[\sqrt{2}\sqrt{9-8}+\frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right]-\frac{1}{12}\left(\sqrt{2}\right)^3$$

$$=\frac{\sqrt{2}}{4}+\frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}-\frac{\sqrt{2}}{6}$$

$$=\frac{\sqrt{2}}{12}+\frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}$$

$$=\frac{1}{4}\left(\frac{\sqrt{2}}{3}+\frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right)$$

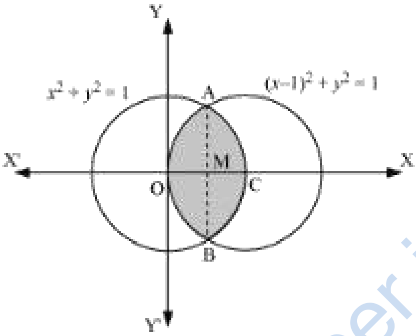
Required area OBCDO

$=\left(2\times\frac{1}{4}\left[\frac{\sqrt{2}}{3}+\frac{9}{2}\sin^{-1}\frac{2\sqrt{2}}{3}\right]\right)=\left[\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right]$ units.

Question 2:

Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.

Solution:



Solving $(x-1)^2+y^2=1$ and $x^2+y^2=1$, point of intersection $A\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $B\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

Required area is symmetrical about x-axis.

$ar(OBCAO)=2\times ar(OCAO)$

Join AB, intersects OC at M

AM is perpendicular to OC

Coordinates of $M\left(\frac{1}{2},0\right)$

$ar(OCAO) = ar(OMAO) + ar(MCAM)$

$$= \left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\,dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\,dx\right]$$

$$= \left[\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= \left[-\frac{1}{4}\sqrt{1-\left(\frac{1}{2}\right)^2} + \frac{1}{2}\sin^{-1}\left(\frac{1}{2}-1\right) - \frac{1}{2}\sin^{-1}(-1)\right] + \left[+\frac{1}{2}\sin^{-1}(-1) - \frac{1}{4}\sqrt{1-\left(\frac{1}{2}\right)^2} - \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{8} + \frac{1}{2}\left(-\frac{\pi}{6}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] + \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}}{8} - \frac{1}{2}\left(\frac{\pi}{6}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{12}\right]$$

$$= \left[-\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2}\right]$$

$$= \left[\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

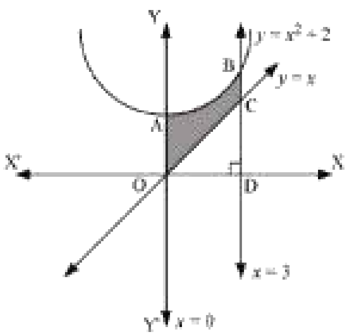
Required Area OBCAO is

$\left(2\times\left[\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right]\right) = \left[\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right]$ units.

Question 3:

Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$

Solution:



$$ar(OCBAO) = ar(ODBAO) - ar(ODCO)$$

$$= \int_0^3 (x^2+2)\,dx - \int_0^3 x\,dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3$$

$$= [9+6] - \left[\frac{9}{2}\right]$$

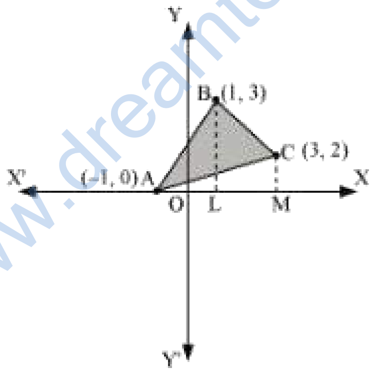
$$= 15 - \frac{9}{2}$$

$$= \frac{21}{2}$$

Question 4:

Using integration finds the area of the region bounded by the triangle whose vertices are $(-1,0),(1,3)$ and $(3,2)$.

Solution:



BL and CM are perpendicular to x-axis.

$$ar(\Delta ACB) = ar(ALBA) + ar(BLMCB) - ar(AMCA)$$

Equation of AB is

$$y - 0 = \frac{3-0}{1+1}(x+1)$$

$$y = \frac{3}{2}(x+1)$$

$$ar(ALBA) = \int_{-1}^{1} \frac{3}{2}(x+1)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{3}{2}\left[\frac{1}{2} + 1 - \frac{1}{2} + 1\right]$$

$$= 3$$

Equation of BC is

$$y - 3 = \frac{2-3}{3-1}(x-1)$$

$$y = \frac{1}{2}(-x+7)$$

$$ar(BLMCB) = \int_{1}^{3} \frac{1}{2}(-x+7)dx$$

$$= \frac{1}{2}\left[-\frac{x^2}{2} + 7x\right]_{1}^{3}$$

$$= \frac{1}{2}\left[-\frac{9}{2} + 21 + \frac{1}{2} - 7\right]$$

$$= 5$$

Equation of AC is

$$y - 0 = \frac{2-0}{3+1}(x+1)$$

$$y = \frac{1}{2}(x+1)$$

$$ar(AMCA) = \frac{1}{2}\int_{-1}^{3} (x+1)dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= \frac{1}{2}\left[\frac{9}{2} + 3 - \frac{1}{2} + 1\right]$$

$$= 4$$

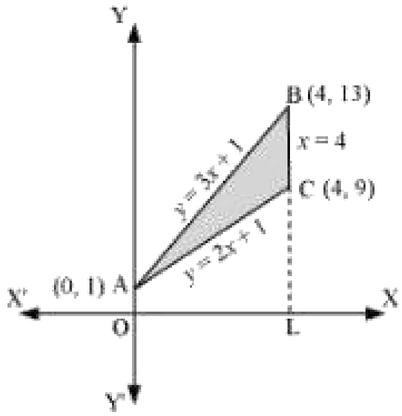
Therefore, $ar(\Delta ABC) = (3+5-4) = 4 units$

Question 5:

Using integration find the area of the triangular region whose sides have the equations $y = 2x+1, y = 3x+1$ and $x = 4$.

Solution:

Vertices of triangle are $A(0,1), B(4,13)$ and $C(4,9)$.



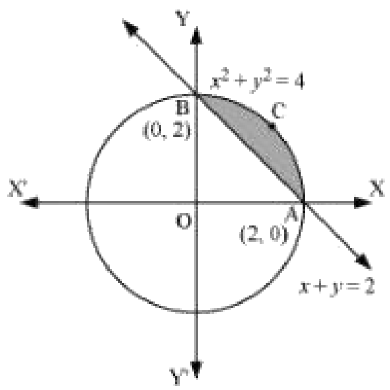
$$
\begin{aligned}
ar(\Delta ACB) &= ar(OLBAO) - ar(OLCAO) \\
&= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx \\
&= \left[\frac{3x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4 \\
&= (24+4) - (16+4) \\
&= 28 - 20 \\
&= 8
\end{aligned}
$$

Question 6:

Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(A) $2(\pi - 2)$ (B) $\pi - 2$ (C) $2\pi - 1$ (D) $2(\pi + 2)$

Solution:



$$ar(ACBA) = ar(OACBO) - ar(\Delta OAB)$$
$$= \int_0^2 \sqrt{4-x^2}\,dx - \int_0^2 (2-x)\,dx$$
$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$
$$= \left[2\times\frac{\pi}{2}\right] - [4-2]$$
$$= (\pi - 2)$$

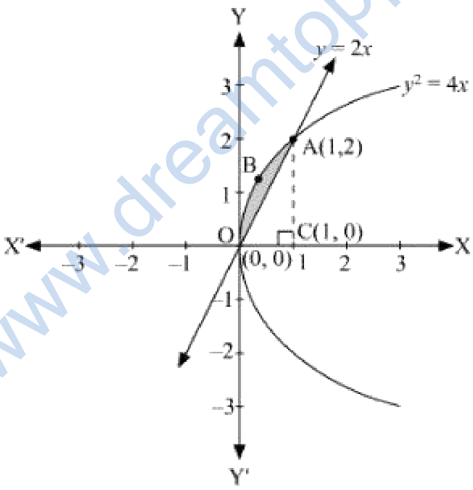
Correct answer is B.

Question 7:

Area lying between the curve $y^2 = 4x$ and $y = 2x$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

Solution:



Points of intersection of curve $y^2 = 4x$ and $y = 2x$ are $O(0,0)$ and $A(1,2)$.

Draw AC perpendicular to x-axis.

Coordinates of C are (1,0)

$$
\begin{aligned}
ar(OBAO) &= ar(\Delta OCA) - ar(OCABO) \\
&= \int_0^1 2x\,dx - \int_0^1 2\sqrt{x}\,dx \\
&= 2\left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 \\
&= \left|1 - \frac{4}{3}\right| \\
&= \left|-\frac{1}{3}\right| \\
&= \frac{1}{3}
\end{aligned}
$$

Correct answer is B.

MISCELLANEOUS EXERCISE

Question 1:

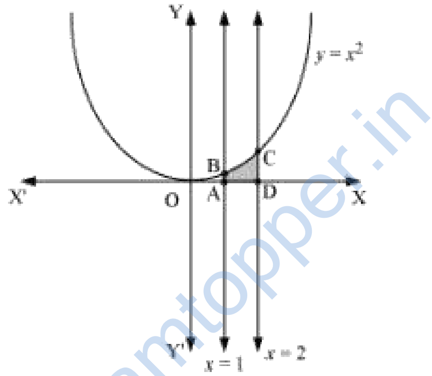
Find the area under the given curves and given lines:

(i) $y = x^2, x = 1, x = 2$ and x-axis

(ii) $y = x^4, x = 1, x = 5$ and x-axis

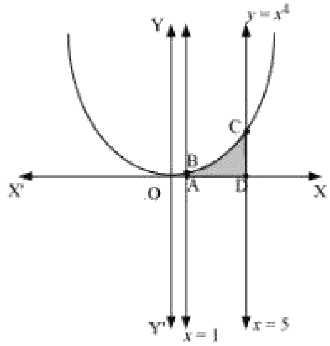
Solution:

(i) $y = x^2, x = 1, x = 2$ and x-axis



$$ar(ADCBA) = \int_1^2 y\,dx$$
$$= \int_1^2 x^2 dx$$
$$= \left[\frac{x^3}{3}\right]_1^2$$
$$= \frac{8}{3} - \frac{1}{3}$$
$$= \frac{7}{3}$$

(ii) $y = x^4, x = 1, x = 5$ and x-axis

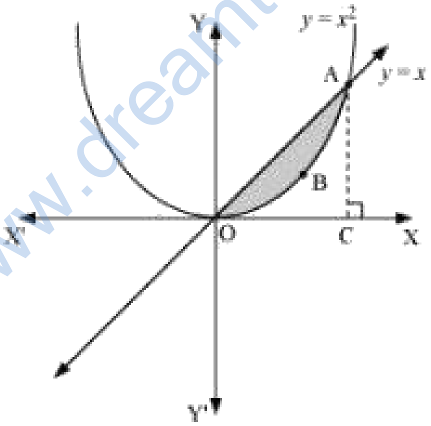


$$ar(ADCBA) = \int_1^5 y\,dx$$
$$= \int_1^5 x^4\,dx$$
$$= \left[\frac{x^5}{5}\right]_1^5$$
$$= \frac{(5)^5}{5} - \frac{1}{5}$$
$$= (5)^4 - \frac{1}{5}$$
$$= 625 - \frac{1}{5}$$
$$= 624.8$$

Question 2:

Find the area between the curves $y = x$ and $y = x^2$.

Solution:



Point of intersection of $y = x$ and $y = x^2$ is A (1,1).

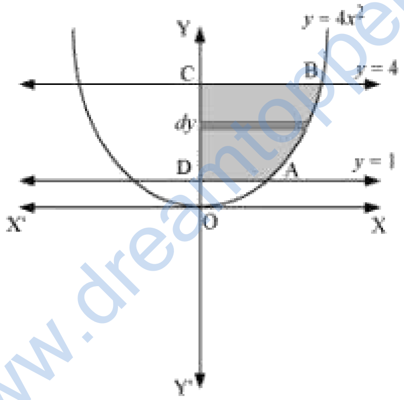
Draw AC perpendicular to x-axis.

$$ar(OBAO) = ar(\Delta OCA) - ar(OCABO)$$
$$= \int_0^1 x dx - \int_0^1 x^2 dx$$
$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}$$

Question 3:

Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and $y = 4$.

Solution:



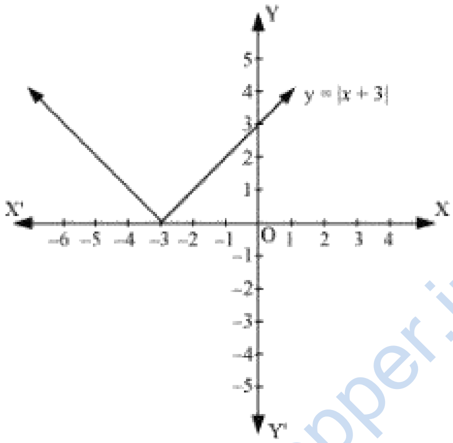
$$ar(ABCD) = \int_1^4 x dy$$
$$= \int_1^4 \frac{\sqrt{y}}{2} dy$$
$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$
$$= \frac{1}{3}\left[(4)^{\frac{3}{2}} - 1\right]$$
$$= \frac{1}{3}[8-1]$$
$$= \frac{7}{3}$$

Question 4:

Sketch the graph of $y=|x+3|$ and evaluate $\int_{-6}^{0}|x+3|\,dx$.

Solution:

X	– 6	– 5	– 4	– 3	– 2	– 1	0
Y	3	2	1	0	1	2	3



$(x+3)\le 0$ for $-6\le x\le -3$ and $(x+3)\ge 0$ for $-3\le x\le 0$

$$\int_{-6}^{0}|(x+3)|\,dx=-\int_{-6}^{-3}(x+3)\,dx+\int_{-3}^{0}(x+3)dx$$

$$=-\left[\frac{x^2}{2}+3x\right]_{-6}^{-3}+\left[\frac{x^2}{2}+3x\right]_{-3}^{0}$$

$$=-\left[\left(\frac{(-3)^2}{2}+3(-3)\right)-\left(\frac{(-6)^2}{2}+3(-6)\right)\right]+\left[0-\left(\frac{(-3)^2}{2}+3(-3)\right)\right]$$

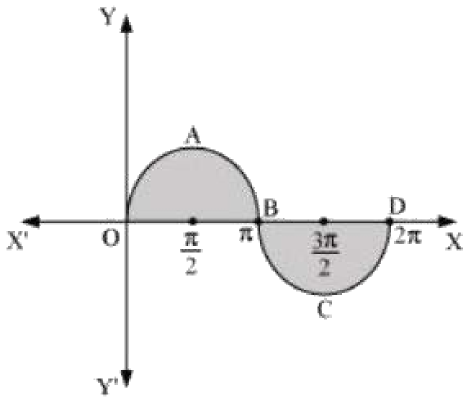
$$=-\left[-\frac{9}{2}\right]-\left[-\frac{9}{2}\right]$$

$$=9$$

Question 5:

Find the area bounded by the curve $y=\sin x$ between $x=0$ and $x=2\pi$.

Solution:



Area bounded by the curve = Area OABO + Area BCDB

$$ar(OABO)+ar(BCDB)=\int_0^{\pi}\sin x dx+\left|\int_{\pi}^{2\pi}\sin x dx\right|$$

$$=\left[-\cos x\right]_0^{\pi}+\left|\left[-\cos x\right]_{\pi}^{2\pi}\right|$$

$$=\left[-\cos\pi+\cos 0\right]+\left|-\cos 2\pi+\cos\pi\right|$$

$$=1+1+\left|(-1-1)\right|$$

$$=2+\left|-2\right|$$

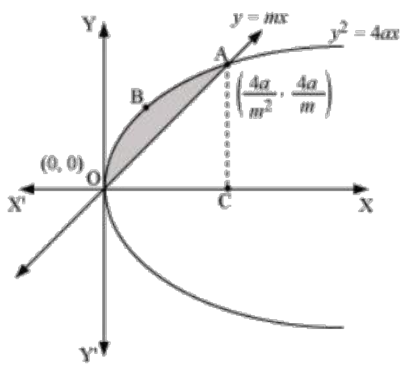
$$=2+2$$

$$=4$$

Question 6:

Find the area enclosed between the parabola $y^2=4ax$ and the line $y=mx$.

Solution:



Points of intersection of curves are $(0,0)$ and $\left(\frac{4a}{m^2},\frac{4a}{m}\right)$

Draw AC perpendicular to x-axis.

$$ar(OABO) = ar(OCABO) - ar(\Delta OCA)$$

$$= \int_0^{\frac{4a}{m^2}} 2\sqrt{ax}dx - \int_0^{\frac{4a}{m^2}} mxdx$$

$$= 2\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{4a}{m^2}} - m\left[\frac{x^2}{2}\right]_0^{\frac{4a}{m^2}}$$

$$= \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}} - \frac{m}{2}\left[\left(\frac{4a}{m^2}\right)^2\right]$$

$$= \frac{32a^2}{3m^3} - \frac{m}{2}\left(\frac{16a^2}{m^4}\right)$$

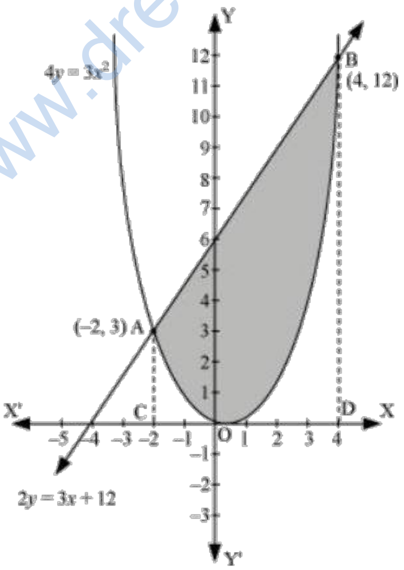
$$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$$

$$= \frac{8a^2}{3m^3}$$

Question 7:

Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

Solution:



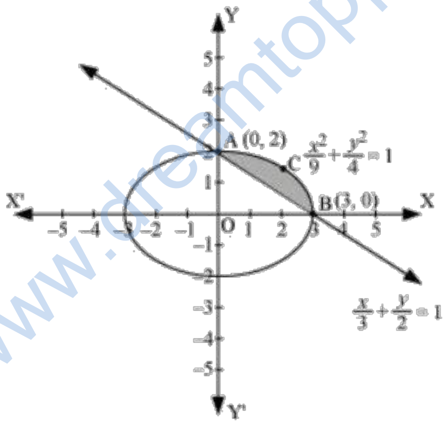
Points of intersection of curves are $A(-2,3)$ and $B(4,12)$.
Draw AC and BD perpendicular to x-axis.

$$ar(OBAO) = ar(CDBA) - ar(ODBO + OACO)$$
$$= \int_{-2}^{4} \frac{1}{2}(3x+12)\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx$$
$$= \frac{1}{2}\left[\frac{3x^2}{2} + 12x\right]_{-2}^{4} - \frac{3}{4}\left[\frac{x^3}{3}\right]_{-2}^{4}$$
$$= \frac{1}{2}[24 + 48 - 6 + 24] - \frac{1}{4}[64 + 8]$$
$$= \frac{1}{2}[90] - \frac{1}{4}[72]$$
$$= 45 - 18$$
$$= 27$$

Question 8:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

Solution:



$$ar(BCAB) = ar(OBCAO) - ar(OBAO)$$

$$= \int_0^3 2\sqrt{1-\frac{x^2}{9}}dx - \int_0^3 2\left(1-\frac{x}{3}\right)dx$$

$$= \frac{2}{3}\left[\int_0^3 \sqrt{9-x^2}dx\right] - \frac{2}{3}\int_0^3 (3-x)dx$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - \frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9}{2}\left(\frac{\pi}{2}\right)\right] - \frac{2}{3}\left[9-\frac{9}{2}\right]$$

$$= \frac{2}{3}\left[\frac{9\pi}{4}-\frac{9}{2}\right]$$

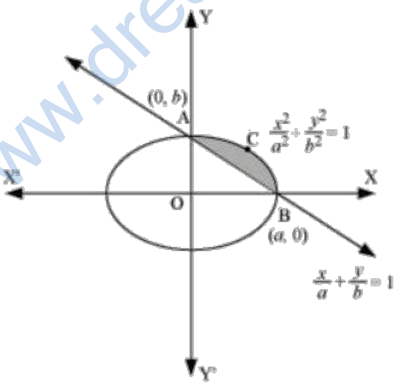
$$= \frac{2}{3}\times\frac{9}{4}(\pi-2)$$

$$= \frac{3}{2}(\pi-2)$$

Question 9:

Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

Solution:



.

$$ar(CBA) = ar(OBCAO) - ar(OBAO)$$

$$= \int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx - \int_0^a b\left(1-\frac{x}{a}\right)dx$$

$$= \frac{b}{a}\left[\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}_0^a - \left\{ax-\frac{x^2}{2}\right\}_0^a\right]$$

$$= \frac{b}{a}\left[\left\{\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right\} - \left\{a^2-\frac{a^2}{2}\right\}\right]$$

$$= \frac{b}{a}\left[\frac{a^2\pi}{4}-\frac{a^2}{2}\right]$$

$$= \frac{ba^2}{2a}\left[\frac{\pi}{2}-1\right]$$

$$= \frac{ab}{2}\left[\frac{\pi}{2}-1\right]$$

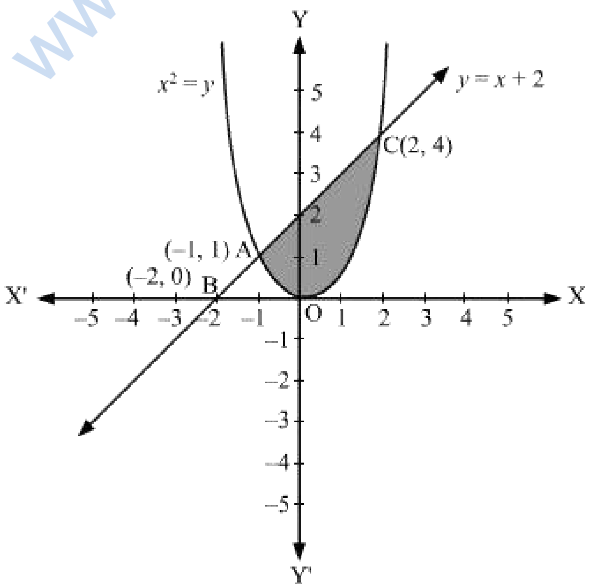
$$= \frac{ab}{4}(\pi-2)$$

Question 10:

Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x+2$ and x-axis.

Solution:

Point of intersection of $x^2 = y$ and $y = x+2$, is $A(-1,1)$ and $C(2,4)$.



Now required Area = Area of trapezium ALMB- Area of ALODBM

$$ar(trap.ALMB)-ar(ALODBM)=\int_{-1}^{2}(x+2)dx-\int_{-1}^{2}x^2dx$$

$$=\left[\frac{x^2}{2}+2x\right]_{-1}^{2}-\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$=\left[2+4-\frac{1}{2}+2\right]-\left[\frac{8}{3}+\frac{1}{3}\right]$$

$$=\frac{15}{2}-3$$

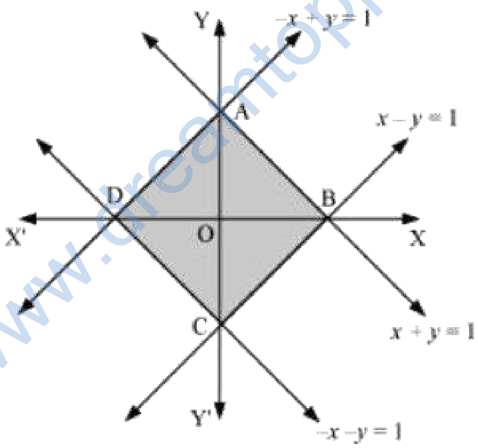
$$=\frac{9}{2}$$

Question 11:

Using the method of integration find the area bounded by the curve $|x|+|y|=1$

[Hint: The required region is bounded by lines $x+y=1, x-y=1, -x+y=1$ and $-x-y=1$]

Solution:



Curve intersects axis at points $A(0,1), B(1,0), C(0,-1)$ and $D(-1,0)$.

Curve is symmetrical about x-axis and y-axis.

$$ar(ADCB)=4\times ar(OBAO)$$

$$=4\int_0^1(1-x)dx$$

$$=4\left(x-\frac{x^2}{2}\right)_0^1$$

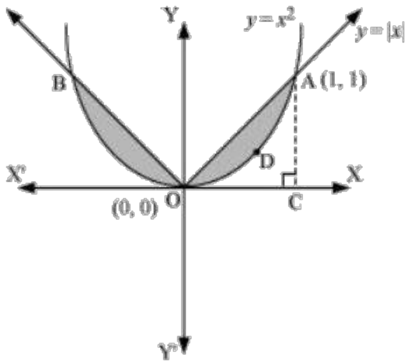
$$=4\left[1-\frac{1}{2}\right]$$

$$=2$$

Question 12:

Find the area bounded by curves $\{(x,y): y \geq x^2 \text{ and } y = |x|\}$

Solution:



Required area is symmetrical about y-axis.

Required area = 2[Area (OCAO) – Area (OCADO)]

$$2[ar(OCAO) - ar(OCADO)] = 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right]$$

$$= 2\left[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

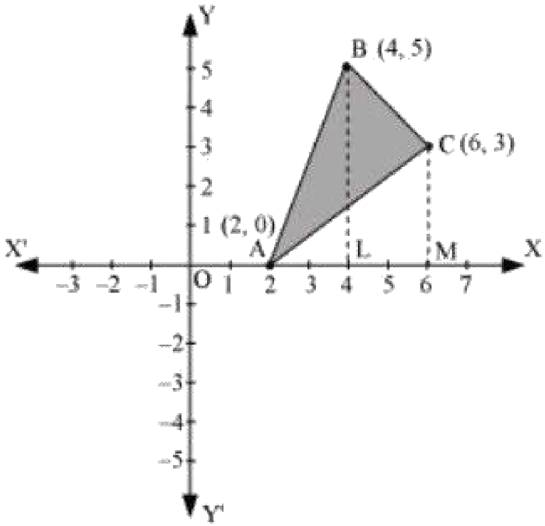
$$= 2\left[\frac{1}{6}\right]$$

$$= \frac{1}{3}$$

Question 13:

Using the method of integration find the area of the triangle ABC, coordinates of whose vertices are $A(2,0), B(4,5)$ and $C(6,3)$.

Solution:



Equation of AB is

$$y-0=\frac{5-0}{4-2}(x-2)$$
$$2y=5x-10$$
$$y=\frac{5}{2}(x-2)$$

Equation of BC is

$$y-5=\frac{3-5}{6-4}(x-4)$$
$$2y-10=-2x+8$$
$$2y=-2x+18$$
$$y=-x+9$$

Equation of CA is

$$y-3=\frac{0-3}{2-6}(x-6)$$
$$-4y+12=-3x+18$$
$$4y=3x-6$$
$$y=\frac{3}{4}(x-2)$$

$$ar(\Delta ABC) = ar(ABLA) + ar(BLMCB) - ar(ACMA)$$

$$= \int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (-x+9)\,dx - \int_2^6 \frac{3}{4}(x-2)\,dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2} - 2x\right]_2^4 + \left[\frac{-x^2}{2} + 9x\right]_4^6 - \frac{3}{4}\left[\frac{x^2}{2} - 2x\right]_2^6$$

$$= \frac{5}{2}[8-8-2+4] + [-18+54+8-36] - \frac{3}{4}[18-12-2+4]$$

$$= 5 + 8 - \frac{3}{4}(8)$$

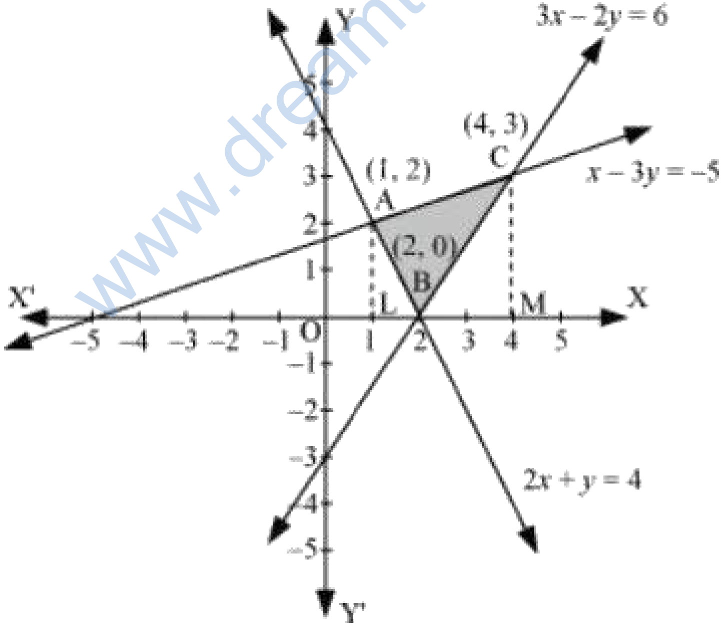
$$= 13 - 6$$

$$= 7\,units.$$

Question 14:

Using the method of integration find the area of the region bounded by lines: $2x + y = 4, 3x - 2y = 6$ and $x - 3y + 5 = 0$.

Solution:



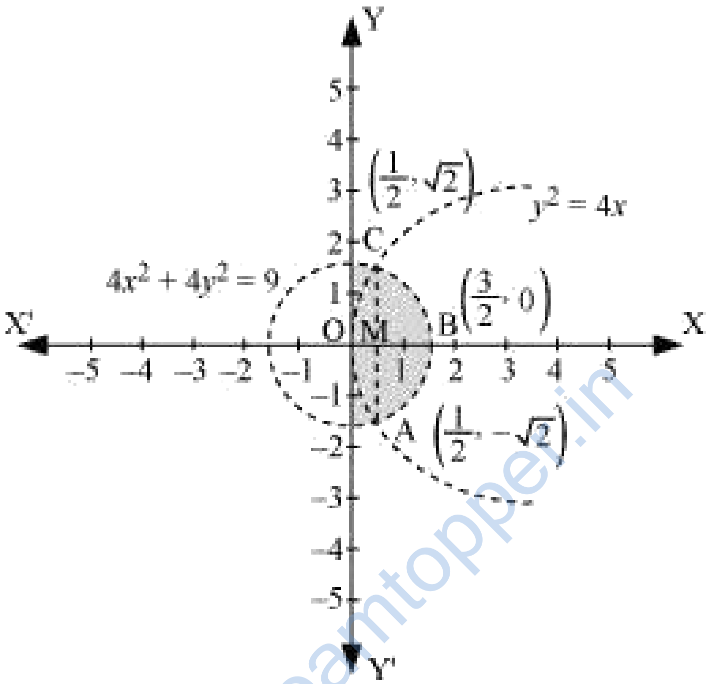
AL and CM are perpendicular on *x*-axis.

$$
\begin{aligned}
ar(\Delta ABC) &= ar(ALMCA) - ar(ALB) - ar(CMB) \\
&= \int_1^4 \left(\frac{x+5}{3}dx\right) - \int_1^2 (4-2x)dx - \int_2^4 \left(\frac{3x-6}{2}\right)dx \\
&= \frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4 - \left[4x-x^2\right]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4 \\
&= \frac{1}{3}\left[8+20-\frac{1}{2}-5\right] - [8-4-4+1] - \frac{1}{2}[24-24-6+12] \\
&= \left(\frac{1}{3}\times\frac{45}{2}\right) - (1) - \frac{1}{2}(6) \\
&= \frac{15}{2}-1-3 \\
&= \frac{15}{2}-4 \\
&= \frac{15-8}{2} \\
&= \frac{7}{2}
\end{aligned}
$$

Question 15:

Find the area of the region $\{(x,y): y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$

Solution:



Points of intersection of curves are $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$.

Required area is OABCO.

Area OABCO is symmetrical about x-axis.

$$Area\ OABCO = 2 \times Area\ OBC$$

$$ar(OBCO) = ar(OMC) + ar(MBC)$$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{9 - 4x^2}\,dx$$

$$= \int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}\sqrt{(3)^2 - (2x)^2}\,dx$$

$$put\ 2x = t \Rightarrow dx = \frac{dt}{2}$$

$$When\ x = \frac{3}{2}, t = 3\ and\ when\ x = \frac{1}{2}, t = 1$$

$$ar(OBCO) = \int_0^{\frac{1}{2}} 2\sqrt{x}dx + \frac{1}{4}\int_1^3 \sqrt{(3)^2 - (t)^2}\,dt$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\frac{1}{2}} + \frac{1}{4}\left[\frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_1^3$$

$$= 2\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right] + \frac{1}{4}\left[\left\{\frac{3}{2}\sqrt{9-(3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right\} - \left\{\frac{1}{2}\sqrt{9-(1)^2} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{2}{3\sqrt{2}} + \frac{1}{4}\left[\left\{0 + \frac{9}{2}\sin^{-1}(1)\right\} - \left\{\frac{1}{2}\sqrt{8} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{\sqrt{2}}{3} + \frac{1}{4}\left[\frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{3} + \frac{9\pi}{16} - \sqrt{2} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}$$

$$ar(OABCO) = 2 \times ar(OBC)$$

$$= 2 \times \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}$$

$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{6}$$

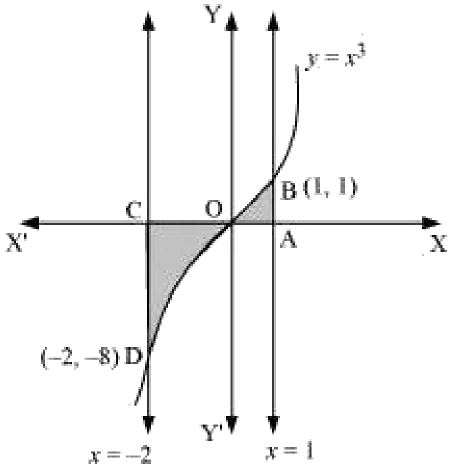
$$= \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}$$

Question 16:

Area bounded by the curve $y = x^3$, the x-axis and the coordinates $x = -2$ and $x = 1$ is

(A) -9 (B) $-\frac{15}{4}$ (C) $\frac{15}{4}$ (D) $\frac{17}{4}$

Solution:



$$required\ area = \int_{-2}^{0} y dx + \int_{0}^{1} y dx$$

$$= \int_{-2}^{0} x^3 dx + \int_{0}^{1} x^3 dx$$

$$= \left[\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_{0}^{1}$$

$$= \left[\frac{(-2)^4}{4} + \frac{1}{4}\right]$$

$$= \left(4 + \frac{1}{4}\right) = \frac{17}{4}$$

Correct answer is D

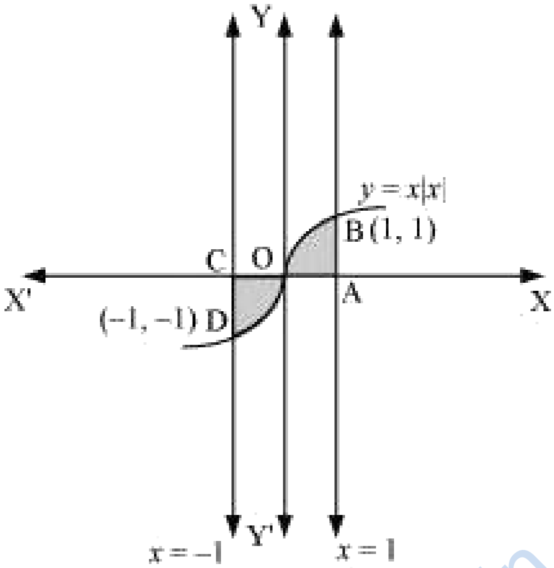
Question17:

The area bounded by the curve $y = x|x|$, x-axis and the coordinates $x = -1$ and $x = 1$ is given by

[Hint: $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$]

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{3}$

Solution:



$$\begin{aligned} required\ area &= \int_{-1}^{1} y\,dx \\ &= \int_{-1}^{1} x|x|\,dx \\ &= \int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx \\ &= \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1} \\ &= -\left(-\frac{1}{3}\right) + \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

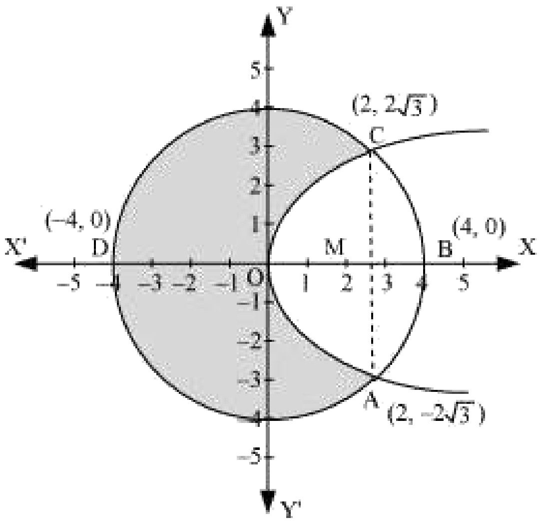
Correct answer is C.

Question 18:

The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.

(A) $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$ (B) $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$ (C) $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$ (D) $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

Solution:



Required area = 2[Area (OADO) + Area (ADBA)]

$$2\left[ar(OADO)+ar(ADBA)\right]=2\left[\int_0^2\sqrt{6x}dx+\int_2^4\sqrt{16-x^2}\,dx\right]$$

$$=2\int_0^2\sqrt{6x}dx+2\int_2^4\sqrt{16-x^2}\,dx$$

$$=2\sqrt{6}\int_0^2\sqrt{x}dx+2\int_2^4\sqrt{16-x^2}\,dx$$

$$=2\sqrt{6}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2+2\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$$

$$=\frac{4\sqrt{6}}{3}\left(2\sqrt{2}-0\right)+2\left[\left\{0+8\sin^{-1}(1)\right\}-\left\{2\sqrt{3}+8\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$=\frac{16\sqrt{3}}{3}+2\left[8\times\frac{\pi}{2}-2\sqrt{3}-8\times\frac{\pi}{6}\right]$$

$$=\frac{16\sqrt{3}}{3}+8\pi-4\sqrt{3}-\frac{8\pi}{3}$$

$$=\frac{16\sqrt{3}+24\pi-12\sqrt{3}-8\pi}{3}$$

$$=\frac{4\sqrt{3}+16\pi}{3}$$

$$=\frac{4}{3}\left[4\pi+\sqrt{3}\right]\ units$$

$$\begin{aligned}\text{Area of circle} &= \pi (r)^2 \\ &= \pi (4)^2 \\ &= 16\pi \end{aligned}$$

$$\begin{aligned}\text{required area} &= 16\pi - \frac{4}{3}\left[4\pi + \sqrt{3}\right] \\ &= \frac{4}{3}\left[4\times 3\pi - 4\pi - \sqrt{3}\right] \\ &= \frac{4}{3}\left(8\pi - \sqrt{3}\right)\end{aligned}$$

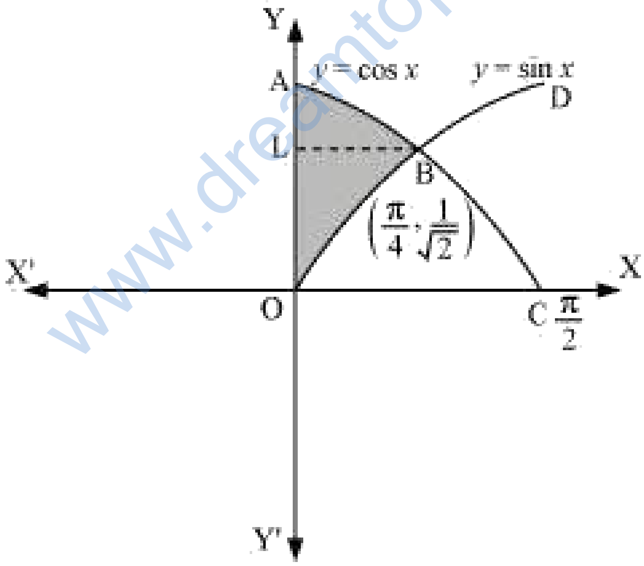
Correct answer is C.

Question 19:

The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$.

(A) $2\left(\sqrt{2}-1\right)$ (B) $\sqrt{2}-1$ (C) $\sqrt{2}+1$ (D) $\sqrt{2}$

Solution:



Required area = Area (ABLA) + Area (OBLO)

$$ar\left(ABLA\right)+ar\left(OBLO\right)=\int_{\frac{1}{\sqrt{2}}}^{1}xdy+\int_{0}^{\frac{1}{\sqrt{2}}}xdy$$

$$=\int_{\frac{1}{\sqrt{2}}}^{1}\cos^{-1}ydy+\int_{0}^{\frac{1}{\sqrt{2}}}\sin^{-1}xdy$$

$$=\left[y\cos^{-1}y-\sqrt{1-y^2}\right]_{\frac{1}{\sqrt{2}}}^{1}+\left[x\sin^{-1}x+\sqrt{1-x^2}\right]_{0}^{\frac{1}{\sqrt{2}}}$$

$$=\left[\cos^{-1}(1)-\frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sqrt{1-\frac{1}{2}}\right]+\left[\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sqrt{1-\frac{1}{2}}-1\right]$$

$$=\frac{-\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}-1$$

$$=\frac{2}{\sqrt{2}}-1$$

$$=\sqrt{2}-1$$

Correct answer is B.