8

Application of Integrals

Short Answer Type Questions

Q. 1 Find the area of the region bounded by the curves $y^2 = 9x$ and $y = 3x$.

Thinking Process

On solving both the equation of curves, get the values of x and then at those values, find the area of the shaded region.

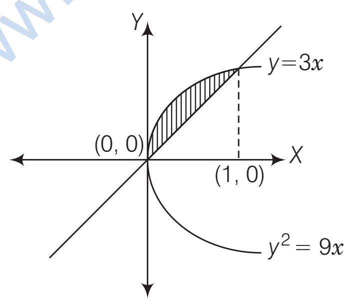
Sol. We have, $y^2 = 9x$ and $y = 3x$

$\Rightarrow$ $(3x)^2 = 9x$

$\Rightarrow$ $9x^2 - 9x = 0$

$\Rightarrow$ $9x(x-1) = 0$

$\Rightarrow$ $x = 1, 0$



$\therefore$ Required area, $A = \int_0^1 \sqrt{9x}\,dx - \int_0^1 3x\,dx$

$$= 3\int_0^1 x^{1/2}\,dx - 3\int_0^1 x\,dx$$

$$= 3\left[\frac{x^{3/2}}{3/2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$$

$$= 3\left(\frac{2}{3} - 0\right) - 3\left(\frac{1}{2} - 0\right)$$

$$= 2 - \frac{3}{2} = \frac{1}{2} \text{ sq units}$$

Q. 2 Find the area of the region bounded by the parabola $y^2 = 2px$ $x^2 = 2py$.

Sol. We have, $y^2 = 2px$ and $x^2 = 2py$

$\therefore$ $y = \sqrt{2px}$

$\Rightarrow$ $x^2 = 2p \cdot \sqrt{2px}$

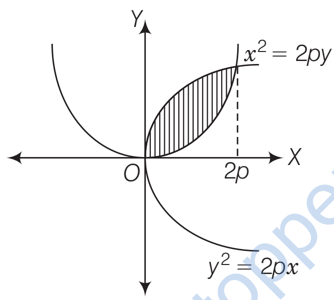
$\Rightarrow$ $x^4 = 4p^2 \cdot (2px)$

$\Rightarrow$ $x^4 = 8p^3 x$

$\Rightarrow$ $x^4 - 8p^3 x = 0$

$\Rightarrow$ $x^3(x - 8p^3) = 0$

$\Rightarrow$ $x = 0, 2p$



$\therefore$ Required area $= \int_0^{2p} \sqrt{2px}\, dx - \int_0^{2p} \frac{x^2}{2p} dx$

$= \sqrt{2p} \int_0^{2p} x^{1/2} dx - \frac{1}{2p} \int_0^{2p} x^2 dx$

$= \sqrt{2p} \left[\frac{2(x)^{3/2}}{3} \right]_0^{2p} - \frac{1}{2p} \left[\frac{x^3}{3} \right]_0^{2p}$

$= \sqrt{2p} \left[\frac{2}{3} \cdot (2p)^{3/2} - 0 \right] - \frac{1}{2p} \left[\frac{1}{3} (2p)^3 - 0 \right]$

$= \sqrt{2p} \left(\frac{2}{3} \cdot 2\sqrt{2} p^{3/2} \right) - \frac{1}{2p} \left(\frac{1}{3} 8p^3 \right)$

$= \sqrt{2p} \left(\frac{4\sqrt{2}}{3} p^{3/2} \right) - \frac{1}{2p} \left(\frac{8}{3} p^3 \right)$

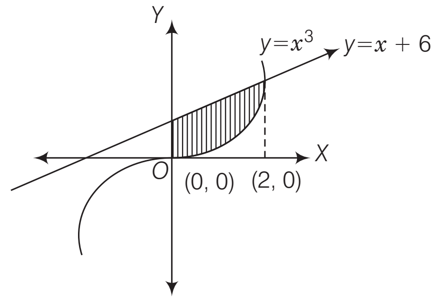
$= \frac{4\sqrt{2}}{3} \cdot \sqrt{2} p^2 - \frac{8}{6} p^2$

$= \frac{(16 - 8)p^2}{6} = \frac{8p^2}{6}$

$= \frac{4p^2}{3}$ sq units

Q. 3 Find the area of the region bounded by the curve $y = x^3$, $y = x + 6$ and $x = 0$.

Sol. We have, $y = x^3$, $y = x + 6$ and $x = 0$



$$\therefore \quad x^3 = x + 6$$
$$\Rightarrow \quad x^3 - x = 6$$
$$\Rightarrow \quad x^3 - x - 6 = 0$$
$$\Rightarrow \quad x^2(x-2) + 2x(x-2) + 3(x-2) = 0$$
$$\Rightarrow \quad (x-2)(x^2+2x+3) = 0$$
$\Rightarrow \quad x = 2$, with two imaginary points

$\therefore$ Required area of shaded region $= \int_0^2 (x + 6 - x^3)\,dx$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2$$
$$= \left[\frac{4}{2} + 12 - \frac{16}{4} - 0\right]$$
$$= [2 + 12 - 4] = 10 \text{ sq units}$$

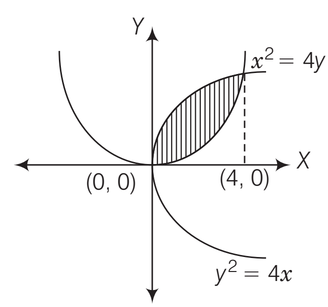
Q. 4 Find the area of the region bounded by the curve $y^2 = 4x$ and $x^2 = 4y$.

Thinking Process

First, by using both the equation get the values of x and then find the shaded region by using these value of x in the equation of curve in x only.

Sol. Given equation of curves are

$$y^2 = 4x \quad \ldots\text{(i)}$$
and
$$x^2 = 4y \quad \ldots\text{(ii)}$$
$$\Rightarrow \quad \left(\frac{x^2}{4}\right)^2 = 4x$$
$$\Rightarrow \quad \frac{x^4}{4 \cdot 4} = 4x$$
$$\Rightarrow \quad x^4 = 64x$$
$$\Rightarrow \quad x^4 - 64x = 0$$
$$\Rightarrow \quad x(x^3 - 4^3) = 0$$
$$\Rightarrow \quad x = 4, 0$$



$\therefore$ Area of shaded region, $A = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4} \right) dx$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx = \left[\frac{2x^{3/2} \cdot 2}{3} - \frac{1}{4} \cdot \frac{x^3}{3} \right]_0^4$$

$$= \frac{2 \cdot 2}{3} \cdot 8 - \frac{1}{4} \cdot \frac{64}{3} - 0 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq units}$$

Q. 5 **Find the area of the region included between $y^2 = 9x$ and $y = x$.**

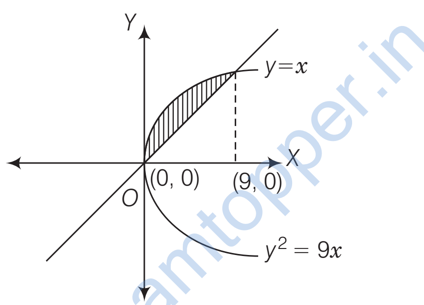
Sol. We have, $y^2 = 9x$ and $y = x$

$$\Rightarrow \quad x^2 = 9x$$

$$\Rightarrow \quad x^2 - 9x = 0$$

$$\Rightarrow \quad x(x - 9) = 0$$

$$\Rightarrow \quad x = 0, 9$$



$\therefore$ Area of shaded region, $A = \int_0^9 (\sqrt{9x} - x)\, dx = \int_0^9 3x^{1/2} dx - \int_0^9 x\, dx$

$$= \left[3 \cdot \frac{x^{3/2}}{3} \cdot 2 \right]_0^9 - \left[\frac{x^2}{2} \right]_0^9$$

$$= \left[\frac{3 \cdot 3^{\frac{3}{2} \times 2}}{3} \cdot 2 - 0 \right] - \left[\frac{81}{2} - 0 \right]$$

$$= 54 - \frac{81}{2} = \frac{108 - 81}{2} = \frac{27}{2} \text{ sq units}$$

Q. 6 **Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.**

Sol. We have, $x^2 = y$ and $y = x + 2$

$$\Rightarrow \quad x^2 = x + 2$$

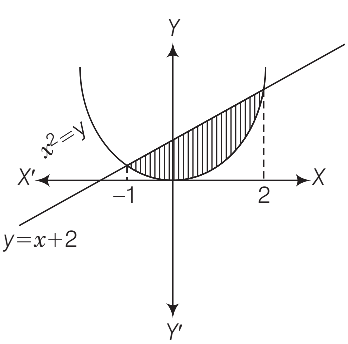
$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 - 2x + x - 2 = 0$$

$$\Rightarrow \quad x(x - 2) + 1(x - 2) = 0$$

$$\Rightarrow \quad (x + 1)(x - 2) = 0$$

$$\Rightarrow \quad x = -1, 2$$



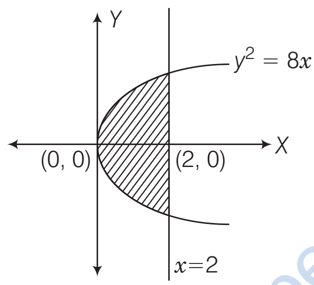
$\therefore$ Required area of shaded region $= \int_{-1}^{2}(x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$

$$= \left[\frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right]$$

$$= 6 + \frac{3}{2} - \frac{9}{3} = \frac{36 + 9 - 18}{6} = \frac{27}{6} = \frac{9}{2} \text{ sq units}$$

Q. 7 Find the area of the region bounded by line $x = 2$ and parabola $y^2 = 8x$.

Sol. We have, $y^2 = 8x$ and $x = 2$



$\therefore$ Area of shaded region, $A = 2\int_0^2 \sqrt{8x}\,dx = 2 \cdot 2\sqrt{2}\int_0^2 x^{1/2}dx$

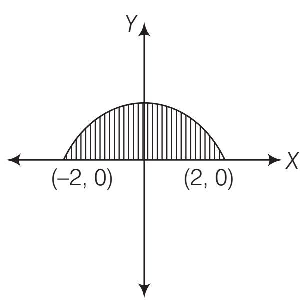
$$= 4 \cdot \sqrt{2} \cdot \left[2 \cdot \frac{x^{3/2}}{3}\right]_0^2 = 4\sqrt{2}\left[\frac{2}{3} \cdot 2\sqrt{2} - 0\right]$$

$$= \frac{32}{3} \text{ sq units}$$

Q. 8 Sketch the region $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and X-axis. Find the area of the region using integration.

Sol. Given region is $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and X-axis.

We have, $y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4$



$\therefore$ Area of shaded region, $A = \int_{-2}^{2}\sqrt{4 - x^2}\,dx = \int_{-2}^{2}\sqrt{2^2 - x^2}\,dx$

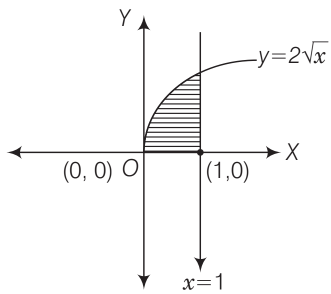
$$= \left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2} \cdot \sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= \frac{2}{2} \cdot 0 + 2 . \frac{\pi}{2} + \frac{2}{2} \cdot 0 - 2\sin^{-1}(-1) = 2 . \frac{\pi}{2} + 2 . \frac{\pi}{2}$$

$$= 2\pi \text{ sq units}$$

Q. 9 Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

Sol. We have, $y = 2\sqrt{x}$, $x = 0$ and $x = 1$



$\therefore$ Area of shaded region, $A = \int_0^1 (2\sqrt{x})\, dx$

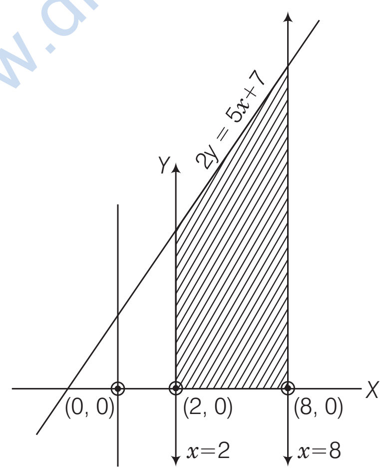
$$= 2 \cdot \left[\frac{x^{3/2}}{3} \cdot 2 \right]_0^1$$

$$= 2\left(\frac{2}{3} \cdot 1 - 0\right) = \frac{4}{3} \text{ sq units}$$

Q. 10 Using integration, find the area of the region bounded by the line $2y = 5x + 7$, X-axis and the lines $x = 2$ and $x = 8$.

Sol. We have, $2y = 5x + 7$

$\Rightarrow$ $y = \frac{5x}{2} + \frac{7}{2}$



$\therefore$ Area of shaded region $= \frac{1}{2}\int_2^8 (5x + 7)\, dx = \frac{1}{2}\left[5 \cdot \frac{x^2}{2} + 7x\right]_2^8$

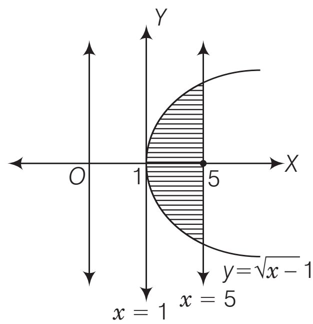
$$= \frac{1}{2}[5 \cdot 32 + 7 \cdot 8 - 10 - 14] = \frac{1}{2}[160 + 56 - 24]$$

$$= \frac{192}{2} = 96 \text{ sq units}$$

Q. 11 Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval [1, 5]. Find the area under the curve and between the lines $x = 1$ and $x = 5$.

Sol. Given equation of the curve is $y = \sqrt{x - 1}$.

$\Rightarrow \quad y^2 = x - 1$



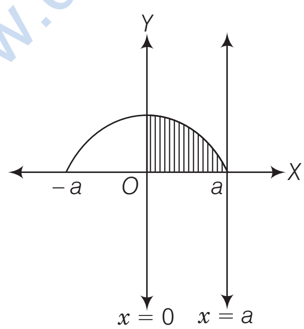
$\therefore$ Area of shaded region, $A = \int_1^5 (x-1)^{1/2}\, dx = \left[\dfrac{2 \cdot (x-1)^{3/2}}{3}\right]_1^5$

$$= \left[\frac{2}{3} \cdot (5-1)^{3/2} - 0\right] = \frac{16}{3} \text{ sq units}$$

Q. 12 Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

Sol. Given equation of the curve is $y = \sqrt{a^2 - x^2}$.

$\Rightarrow \quad y^2 = a^2 - x^2 \quad \Rightarrow \quad y^2 + x^2 = a^2$



$\therefore$ Required area of shaded region, $A = \int_0^a \sqrt{a^2 - x^2}\, dx$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0 - \frac{a^2}{2}\sin^{-1} 0\right]$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4} \text{ sq units}$$

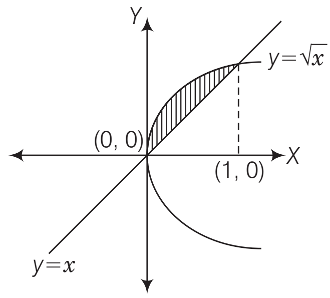
Q. 13 Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

Sol. Given equation of curves are $y = \sqrt{x}$ and $y = x$.

$\Rightarrow \quad x = \sqrt{x} \Rightarrow x^2 = x$

$\Rightarrow \quad x^2 - x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow \quad x = 0, 1$



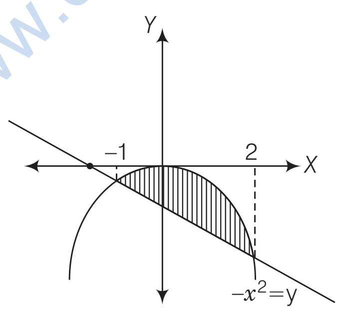
$\therefore$ Required area of shaded region, $A = \int_0^1 (\sqrt{x})dx - \int_0^1 x dx$

$$= \left[2 \cdot \frac{x^{3/2}}{3}\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{2}{3} \cdot 1 - \frac{1}{2} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ sq units}$$

Q. 14 Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.

Sol. We have, $y = -x^2$ and $x + y + 2 = 0$



$\Rightarrow \quad -x - 2 = -x^2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow \quad x^2 + x - 2x - 2 = 0 \Rightarrow x(x+1) - 2(x+1) = 0$

$\Rightarrow \quad (x-2)(x+1) = 0 \Rightarrow x = 2, -1$

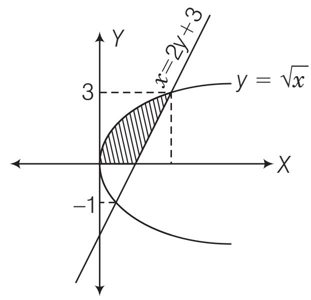
$\therefore$ Area of shaded region, $A = \left|\int_{-1}^{2} (-x - 2 + x^2)\, dx\right| = \left|\int_{-1}^{2} (x^2 - x - 2)\, dx\right|$

$$= \left|\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2}\right| = \left|\left[\frac{8}{3} - \frac{4}{2} - 4 + \frac{1}{3} + \frac{1}{2} - 2\right]\right|$$

$$= \left|\frac{16 - 12 - 24 + 2 + 3 - 12}{6}\right| = \left|-\frac{27}{6}\right| = \frac{9}{2} \text{ sq units}$$

Q. 15 Find the area bounded by the curve $y=\sqrt{x}$, $x=2y+3$ in the first quadrant and X-axis.

Sol. Given equation of the curves are $y=\sqrt{x}$ and $x=2y+3$ in the first quadrant.



On solving both the equations for y, we get

$$y=\sqrt{2y+3}$$

$\Rightarrow$ $$y^2=2y+3$$

$\Rightarrow$ $$y^2-2y-3=0$$

$\Rightarrow$ $$y^2-3y+y-3=0$$

$\Rightarrow$ $$y(y-3)+1(y-3)=0$$

$\Rightarrow$ $$(y+1)(y-3)=0$$

$\Rightarrow$ $$y=-1,3$$

$\therefore$ Required area of shaded region,

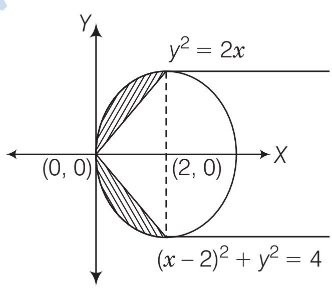
$$A=\int_0^3(2y+3-y^2)dy=\left[\frac{2y^2}{2}+3y-\frac{y^3}{3}\right]_0^3$$

$$=\left[\frac{18}{2}+9-9-0\right]=9 \text{ sq units}$$

Long Answer Type Questions

Q. 16 Find the area of the region bounded by the curve $y^2=2x$ and $x^2+y^2=4x$.

Sol. We have, $y^2=2x$ and $x^2+y^2=4x$



$\Rightarrow$ $$x^2+2x=4x$$

$\Rightarrow$ $$x^2-2x=0$$

$\Rightarrow$ $$x(x-2)=0$$

$\Rightarrow$ $$x=0,2$$

Also, $$x^2+y^2=4x$$

$\Rightarrow$ $$x^2-4x=-y^2$$

$\Rightarrow$ $$x^2-4x+4=-y^2+4$$

$\Rightarrow$ $$(x-2)^2-2^2=-y^2$$

$\therefore$ Required area $= 2 \cdot \int_0^2 \left[\sqrt{2^2 - (x-2)^2} - \sqrt{2x} \right] dx$

$$= 2 \left[\left[\frac{x-2}{2} \cdot \sqrt{2^2 - (x-2)^2} + \frac{2^2}{2} \sin^{-1}\left(\frac{x-2}{2}\right) \right]_0^2 - \left[\sqrt{2} \cdot \frac{x^{3/2}}{3/2} \right]_0^2 \right]$$

$$= 2\left[\left(0 + 0 - 1 \cdot 0 + 2 \cdot \frac{\pi}{2} \right) - \frac{2\sqrt{2}}{3}(2^{3/2} - 0) \right]$$

$$= \frac{4\pi}{2} - \frac{8 \cdot 2}{3} = 2\pi - \frac{16}{3} = 2\left(\pi - \frac{8}{3}\right) \text{ sq units}$$

Q. 17 Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

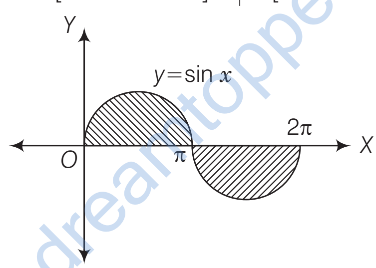
Thinking Process

We know that, $\sin x$ curve has positive region from $[0, \pi]$ and negative region in $[\pi, 2\pi]$.

Sol. Required area $= \int_0^{2\pi} \sin x \, dx = \int_0^{\pi} \sin x dx + \left| \int_{\pi}^{2\pi} \sin x dx \right|$

$$= -[\cos x]_0^{\pi} + \left| [-\cos x]_{\pi}^{2\pi} \right|$$

$$= -[\cos \pi - \cos 0] + \left| -[\cos 2\pi - \cos \pi] \right|$$

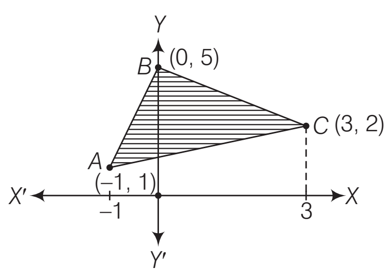


$$= -[-1-1] + \left| -(1+1) \right|$$

$$= 2 + 2 = 4 \text{ sq units}$$

Q. 18 Find the area of region bounded by the triangle whose vertices are $(-1, 1)$, $(0, 5)$ and $(3, 2)$, using integration.

Sol. Let we have the vertices of a ΔABC as $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$.



$\therefore$ Equation of AB is $y - 1 = \left(\frac{5-1}{0+1}\right)(x+1)$

$\Rightarrow$ $y - 1 = 4x + 4$

$\Rightarrow$ $y = 4x + 5$ ...(i)

and equation of BC is $y - 5 = \left(\frac{2-5}{3-0}\right)(x - 0)$

$$\Rightarrow \quad y - 5 = \frac{-3}{3}(x)$$

$$\Rightarrow \quad y = 5 - x \qquad \text{...(ii)}$$

Similarly, equation of AC is $y - 1 = \left(\frac{2-1}{3+1}\right)(x+1)$

$$\Rightarrow \quad y - 1 = \frac{1}{4}(x+1)$$

$$\Rightarrow \quad 4y = x + 5 \qquad \text{...(iii)}$$

$\therefore$ Area of shaded region $= \int_{-1}^{0}(y_1 - y_2)dx + \int_{0}^{3}(y_1 - y_2)dx$

$$= \int_{-1}^{0}\left[4x + 5 - \frac{x+5}{4}\right]dx + \int_{0}^{3}\left[5 - x - \frac{x+5}{4}\right]dx$$

$$= \left[\frac{4x^2}{2} + 5x - \frac{x^2}{8} - \frac{5x}{4}\right]_{-1}^{0} + \left[5x - \frac{x^2}{2} - \frac{x^2}{8} - \frac{5x}{4}\right]_{0}^{3}$$

$$= \left[0 - \left(4\cdot\frac{1}{2} + 5(-1) - \frac{1}{8} + \frac{5}{4}\right)\right] + \left[\left(15 - \frac{9}{2} - \frac{9}{8} - \frac{15}{4}\right) - 0\right]$$

$$= \left[-2 + 5 + \frac{1}{8} - \frac{5}{4} + 15 - \frac{9}{2} - \frac{9}{8} - \frac{15}{4}\right]$$

$$= 18 + \left(\frac{1 - 10 - 36 - 9 - 30}{8}\right)$$

$$= 18 + \left(-\frac{84}{8}\right) = 18 - \frac{21}{2} = \frac{15}{2} \text{ sq units}$$

Q. 19 Draw a rough sketch of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$. Also, find the area of the region sketched using method of integration.

Sol. We have, $y^2 = 6ax$ and $x^2 + y^2 = 16a^2$

$$\Rightarrow \quad x^2 + 6ax = 16a^2$$

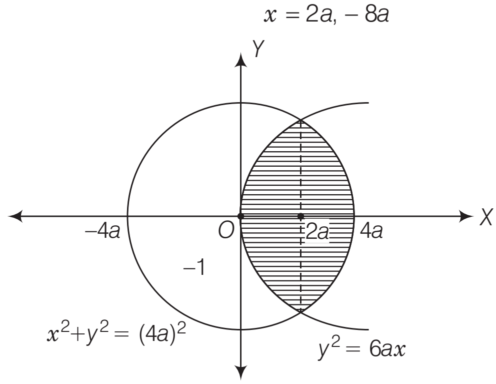
$$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0$$

$$\Rightarrow \quad x^2 + 8ax - 2ax - 16a^2 = 0$$

$$\Rightarrow \quad x(x + 8a) - 2a(x + 8a) = 0$$

$$\Rightarrow \quad (x - 2a)(x + 8a) = 0$$

$$\Rightarrow \quad x = 2a, -8a$$



$\therefore$ Area of required region $= 2\left[\int_0^{2a} \sqrt{6ax}\, dx + \int_{2a}^{4a} \sqrt{(4a)^2 - x^2} dx\right]$

$$= 2\left[\int_0^{2a} \sqrt{6a}\, x^{1/2} dx + \int_{2a}^{4a} \sqrt{(4a)^2 - x^2} dx\right]$$

$$= 2\left[\sqrt{6a}\left[\frac{x^{3/2}}{3/2}\right]_0^{2a} + \left(\frac{x}{2}\sqrt{(4a)^2 - x^2} + \frac{(4a)^2}{2}\sin^{-1}\frac{x}{4a}\right)_{2a}^{4a}\right]$$

$$= 2\left[\sqrt{6a}\cdot\frac{2}{3}((2a)^{3/2} - 0) + \frac{4a}{2}\cdot 0 + \frac{16a^2}{2}\cdot\frac{\pi}{2} - \frac{2a}{2}\sqrt{16a^2 - 4a^2} - \frac{16a^2}{2}\cdot\sin^{-1}\frac{2a}{4a}\right]$$

$$= 2\left[\sqrt{6a}\ \frac{2}{3}\cdot 2\sqrt{2}\ a^{3/2} + 0 + 4\pi a^2 - \frac{2a}{2}\cdot 2\sqrt{3}a - 8a^2\cdot\frac{\pi}{6}\right]$$

$$= 2\left[\sqrt{12}\cdot\frac{4}{3}a^2 + 4\pi a^2 - 2\sqrt{3}a^2 - \frac{4a^2\pi}{3}\right]$$

$$= 2\left[\frac{8\sqrt{3}a^2 + 12\pi a^2 - 6\sqrt{3}a^2 - 4a^2\pi}{3}\right]$$

$$= \frac{2}{3}a^2[8\sqrt{3} + 12\pi - 6\sqrt{3} - 4\pi]$$

$$= \frac{2}{3}a^2[2\sqrt{3} + 8\pi] = \frac{4}{3}a^2[\sqrt{3} + 4\pi]$$

Q. 20 Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

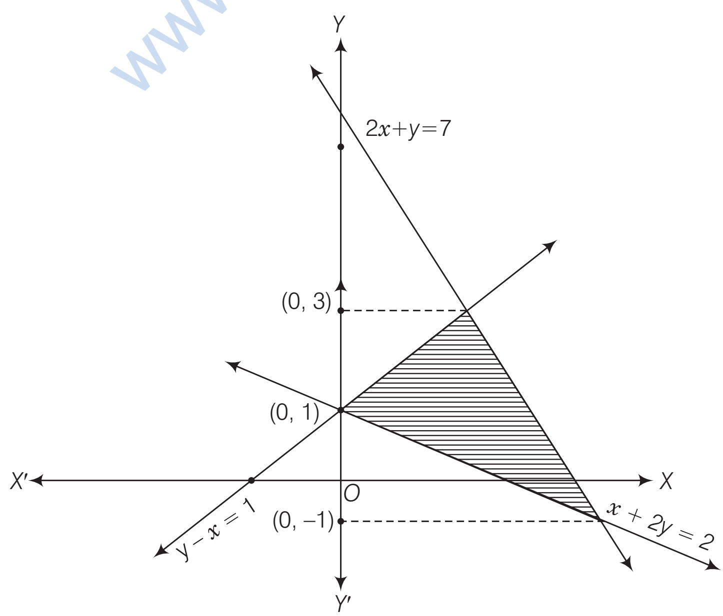
Sol. We have, $x + 2y = 2$ ...(i)

$y - x = 1$ ...(ii)

and $2x + y = 7$ ...(iii)

On solving Eqs. (i) and (ii), we get

$$y - (2 - 2y) = 1 \Rightarrow 3y - 2 = 1 \Rightarrow y = 1$$



$$2(y - 1) + y = 7$$

On solving Eqs. (ii) and (iii), we get

$$\Rightarrow \quad 2y - 2 + y = 7$$
$$\Rightarrow \quad y = 3$$

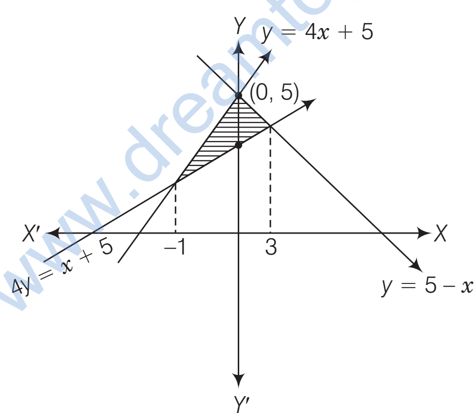
On solving Eqs. (i) and (iii), we get

$$2(2 - 2y) + y = 7$$
$$\Rightarrow \quad 4 - 4y + y = 7$$
$$\Rightarrow \quad -3y = 3$$
$$\Rightarrow \quad y = -1$$

$$\therefore \text{ Required area} = \int_{-1}^{1} (2 - 2y)\,dy + \int_{-1}^{3} \frac{(7 - y)}{2}\,dy - \int_{1}^{3} (y - 1)\,dy$$
$$= \left[-2y + \frac{2y^2}{2}\right]_{-1}^{1} + \left[\frac{7y}{2} - \frac{y^2}{2\cdot 2}\right]_{-1}^{3} - \left[\frac{y^2}{2} - y\right]_{1}^{3}$$
$$= \left[-2 + \frac{2}{2} - 2 - \frac{2}{2}\right] + \left[\frac{21}{2} - \frac{9}{4} + \frac{7}{2} + \frac{1}{4}\right] - \left[\frac{9}{2} - 3 - \frac{1}{2} + 1\right]$$
$$= [-4] + \left[\frac{42 - 9 + 14 + 1}{4}\right] - \left[\frac{9 - 6 - 1 + 2}{2}\right]$$
$$= -4 + 12 - 2 = 6 \text{ sq units}$$

Q. 21 Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

Sol.



Given equations of lines are

$$y = 4x + 5 \quad \ldots\text{(i)}$$
$$y = 5 - x \quad \ldots\text{(ii)}$$
and
$$4y = x + 5 \quad \ldots\text{(iii)}$$

On solving Eqs. (i) and (ii), we get

$$4x + 5 = 5 - x$$
$$\Rightarrow \quad x = 0$$

On solving Eqs. (i) and (iii), we get

$$4(4x + 5) = x + 5$$
$$\Rightarrow \quad 16x + 20 = x + 5$$
$$\Rightarrow \quad 15x = -15$$
$$\Rightarrow \quad x = -1$$

On solving Eqs. (ii) and (iii), we get

$$4(5 - x) = x + 5$$
$$\Rightarrow \quad 20 - 4x = x + 5$$

$\Rightarrow \qquad x = 3$

$\therefore$ Required area $= \int_{-1}^{0}(4x + 5)dx + \int_{0}^{3}(5 - x)dx - \frac{1}{4}\int_{-1}^{3}(x + 5)dx$

$$= \left[\frac{4x^2}{2} + 5x\right]_{-1}^{0} + \left[5x - \frac{x^2}{2}\right]_{0}^{3} - \frac{1}{4}\left[\frac{x^2}{2} + 5x\right]_{-1}^{3}$$

$$= [0 - 2 + 5] + \left[15 - \frac{9}{2} - 0\right] - \frac{1}{4}\left[\frac{9}{2} + 15 - \frac{1}{2} + 5\right]$$

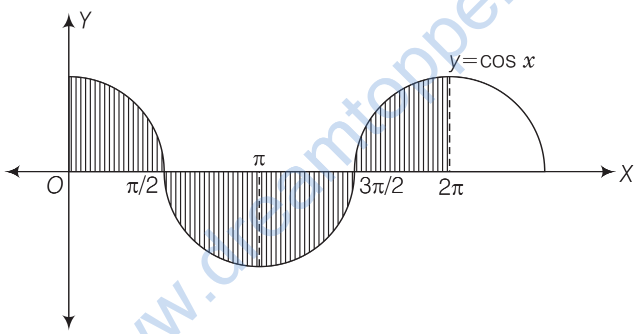
$$= 3 + \frac{21}{2} - \frac{1}{4}\cdot 24$$

$$= -3 + \frac{21}{2} = \frac{15}{2} \text{ sq units}$$

Q. 22 Find the area bounded by the curve $y = 2\cos x$ and the X-axis from $x = 0$ to $x = 2\pi$.

Sol. Required area of shaded region $= \int_{0}^{2\pi} 2\cos x dx$

$$= \int_{0}^{\pi/2} 2\cos x\, dx + \left|\int_{\pi/2}^{3\pi/2} 2\cos x dx\right| + \int_{3\pi/2}^{2\pi} 2\cos x\, dx$$

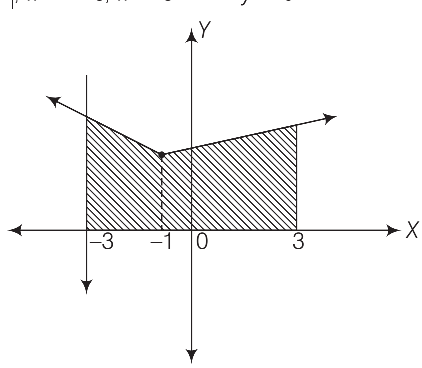


$$= 2[\sin x]_{0}^{\pi/2} + \left|2(\sin x)_{\pi/2}^{3\pi/2}\right| + 2[\sin x]_{3\pi/2}^{2\pi}$$

$$= 2 + 4 + 2 = 8 \text{ sq units}$$

Q. 23 Draw a rough sketch of the given curve $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$ and find the area of the region bounded by them, using integration.

Sol. We have, $y = 1 + |x + 1|$, $x = -3$, $x = 3$ and $y = 0$



$$\because \qquad y = \begin{cases} -x, & \text{if } x < -1 \\ x + 2, & \text{if } x \geq -1 \end{cases}$$

$\therefore$ Area of shaded region, $A = \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x + 2)\,dx$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$= -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\frac{9}{2} + 6 - \frac{1}{2} + 2\right]$$

$$= -[-4] + [8 + 4]$$

$$= 12 + 4 = 16 \text{ sq units}$$

Objective Type Questions

Q. 24 The area of the region bounded by the Y-axis $y = \cos x$ and $y = \sin x$, where $0 \leq x \leq \frac{\pi}{2}$, is

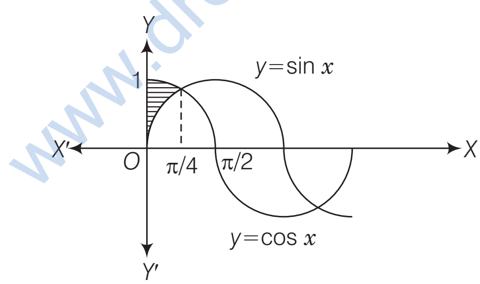
(a) $\sqrt{2}$ sq units

(b) $(\sqrt{2} + 1)$ sq units

(c) $(\sqrt{2} - 1)$ sq units

(d) $(2\sqrt{2} - 1)$ sq units

Sol. *(c)* We have, Y-axis *i.e.*, $x = 0$, $y = \cos x$ and $y = \sin x$, where $0 \leq x \leq \frac{\pi}{2}$.



$\therefore$ Required area $= \int_0^{\pi/4} (\cos x - \sin x)dx$

$$= [\sin x]_0^{\pi/4} + [\cos x]_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} - \sin 0\right) + \left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

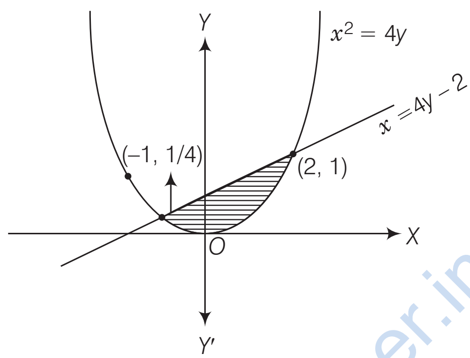
$$= -1 + \frac{2}{\sqrt{2}} = \frac{-\sqrt{2} + 2}{\sqrt{2}}$$

$$= \frac{-2 + 2\sqrt{2}}{2} = (\sqrt{2} - 1) \text{ sq units}$$

Q. 25 The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(a) $\frac{3}{8}$ sq unit
(b) $\frac{5}{8}$ sq unit
(c) $\frac{7}{8}$ sq unit
(d) $\frac{9}{8}$ sq units

Sol. *(d)* Given equation of curve is $x^2 = 4y$ and the straight line $x = 4y - 2$.



For intersection point, put $x = 4y - 2$ in equation of curve, we get

$$(4y-2)^2 = 4y$$
$$\Rightarrow \quad 16y^2 + 4 - 16y = 4y$$
$$\Rightarrow \quad 16y^2 - 20y + 4 = 0$$
$$\Rightarrow \quad 4y^2 - 5y + 1 = 0$$
$$\Rightarrow \quad 4y^2 - 4y - y + 1 = 0$$
$$\Rightarrow \quad 4y(y-1) - 1(y-1) = 0$$
$$\Rightarrow \quad (4y-1)(y-1) = 0$$
$$\therefore \quad y = 1, \frac{1}{4}$$

For $y = 1$, $x = \sqrt{4 \cdot 1} = 2$ [since, negative value does not satisfy the equation of line]

For $y = \frac{1}{4}$, $x = \sqrt{4 \cdot \frac{1}{4}} = -1$ [positive value does not satisfy the equation of line]

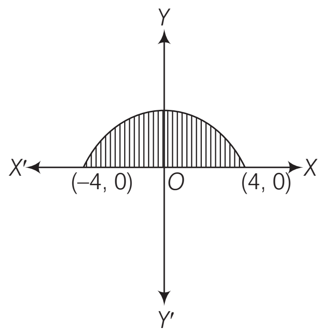
So, the intersection points are (2, 1) and $\left(-1, \frac{1}{4}\right)$.

$$\therefore \text{ Area of shaded region} = \int_{-1}^{2}\left(\frac{x+2}{4}\right)dx - \int_{-1}^{2}\frac{x^2}{4}dx$$
$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left|\frac{x^3}{3}\right|_{-1}^{2}$$
$$= -\frac{1}{4}\left[\frac{4}{2} + 4 - -\frac{1}{2} + 2\right] - \frac{1}{4}\left[\frac{8}{3} + \frac{1}{3}\right]$$
$$= \frac{1}{4} \cdot \frac{15}{2} - \frac{1}{4} \cdot \frac{9}{3} = \frac{45-18}{24}$$
$$= \frac{27}{24} = \frac{9}{8} \text{ sq units}$$

Q. 26 The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and X-axis is

(a) 8π sq units (b) 20π sq units
(c) 16π sq units (d) 256π sq units

Sol. *(a)* Given equation of curve is $y = \sqrt{16 - x^2}$ and the equation of line is X-axis *i.e.,* $y = 0$.



$$\therefore \quad \sqrt{16 - x^2} = 0 \quad \ldots\text{(i)}$$
$$\Rightarrow \quad 16 - x^2 = 0$$
$$\Rightarrow \quad x^2 = 16$$
$$\Rightarrow \quad x = \pm 4$$

So, the intersection points are (4, 0) and (–4, 0).

$\therefore$ Area of curve, $A = \int_{-4}^{4} (16 - x^2)^{1/2} dx$

$$= \int_{-4}^{4} \sqrt{(4^2 - x^2)}\, dx$$
$$= \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4} \right]_{-4}^{4}$$
$$= \left[\frac{4}{2}\sqrt{4^2 - 4^2} + 8\sin^{-1}\frac{4}{4} \right] - \left[-\frac{4}{2}\sqrt{4^2 - (-4)^2} + 8\sin^{-1}\left(-\frac{4}{4}\right) \right]$$
$$= \left[2 \cdot 0 + 8 \cdot \frac{\pi}{2} - 0 + 8 \cdot \frac{\pi}{2} \right] = 8\pi \text{ sq units}$$

Q. 27 Area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

(a) 16π sq units (b) 4π sq units (c) 32π sq units (d) 24π sq units

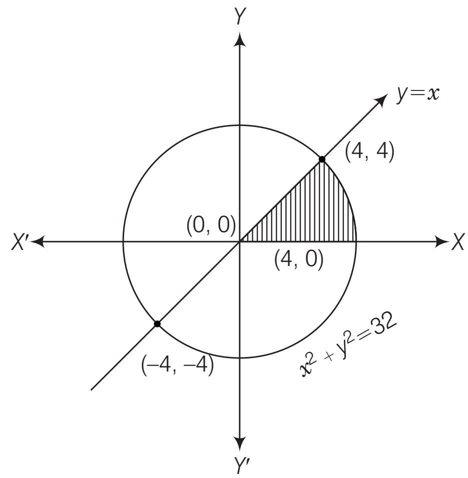
Sol. *(b)* We have, area enclosed by X-axis *i.e.,* $y = 0$, $y = x$ and the circle $x^2 + y^2 = 32$ in first quadrant.

$$\text{Since,} \quad x^2 + (x)^2 = 32 \quad [\because y = x]$$
$$\Rightarrow \quad 2x^2 = 32$$
$$\Rightarrow \quad x = \pm 4$$

So, the intersection point of circle $x^2 + y^2 = 32$ and line $y = x$ are (4, 4) or (– 4, 4).

$$\text{and} \quad x^2 + y^2 = (4\sqrt{2})^2$$
$$\text{Since,} \quad y = 0$$
$$\therefore \quad x^2 + (0)^2 = 32$$
$$\Rightarrow \quad x = \pm 4\sqrt{2}$$

So, the circle intersects the X-axis at $(\pm 4\sqrt{2}, 0)$.



$$\text{Area of shaded region} = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^4 + \left|\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right|_4^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left[\frac{4\sqrt{2}}{2}\cdot 0 + 16\sin^{-1}\frac{(4\sqrt{2})}{(4\sqrt{2})} - \frac{4}{2}\sqrt{(4\sqrt{2})^2 - 16} - 16\sin^{-1}\frac{4}{4\sqrt{2}}\right]$$

$$= 8 + \left[16\cdot\frac{\pi}{2} - 2\cdot\sqrt{16} - 16\cdot\frac{\pi}{4}\right]$$

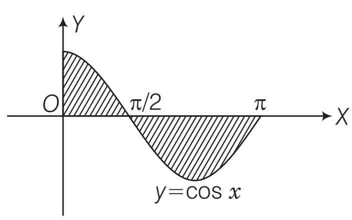
$$= 8 + [8\pi - 8 - 4\pi] = 4\pi \text{ sq units}$$

Q. 28 Area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is

(a) 2 sq units (b) 4 sq units

(c) 3 sq units (d) 1 sq unit

Sol. *(a)* Required area enclosed by the curve $y = \cos x$, $x = 0$ and $x = \pi$ is



$$A = \int_0^{\pi/2} \cos x\,dx + \left|\int_{\pi/2}^{\pi} \cos x\,dx\right|$$

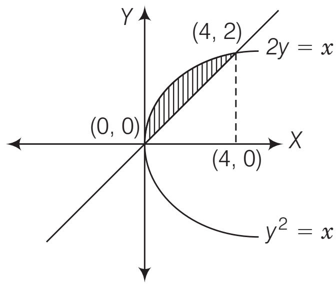
$$= \left[\sin\frac{\pi}{2} - \sin 0\right] + \left|\sin\frac{\pi}{2} - \sin\pi\right|$$

$$= 1 + 1 = 2 \text{ sq units}$$

Q. 29 The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is

(a) $\frac{4}{3}$ sq units (b) 1 sq unit (c) $\frac{2}{3}$ sq unit (d) $\frac{1}{3}$ sq unit

Sol. *(a)* We have to find the area enclosed by parabola $y^2 = x$ and the straight line $2y = x$.



$$\therefore \qquad \left(\frac{x}{2}\right)^2 = x$$

$$\Rightarrow \qquad x^2 = 4x \Rightarrow x(x-4) = 0$$

$$\Rightarrow \qquad x = 4 \Rightarrow y = 2 \text{ and } x = 0 \Rightarrow y = 0$$

So, the intersection points are (0, 0) and (4, 2).

Area enclosed by shaded region,

$$A = \int_0^4 \left[\sqrt{x} - \frac{x}{2}\right] dx$$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{1}{2}\cdot\frac{x^2}{2}\right]_0^4 = \left[2\cdot\frac{x^{3/2}}{3} - \frac{x^2}{4}\right]_0^4$$

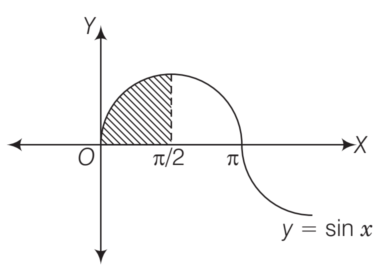
$$= \frac{2}{3}4^{3/2} - \frac{16}{4} - \frac{2}{3}\cdot 0 + \frac{1}{4}\cdot 0$$

$$= \frac{16}{3} - \frac{16}{4} = \frac{64-48}{12} = \frac{16}{12} = \frac{4}{3} \text{ sq units}$$

Q. 30 The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the X-axis is

(a) 2 sq units (b) 4 sq units (c) 3 sq units (d) 1 sq unit

Sol. *(d)* Area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the X-axis is



$$A = \int_0^{\pi/2} \sin x\, dx$$

$$= -[\cos x]_0^{\pi/2} = -\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -[0-1] = 1 \text{ sq unit}$$

Q. 31 The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

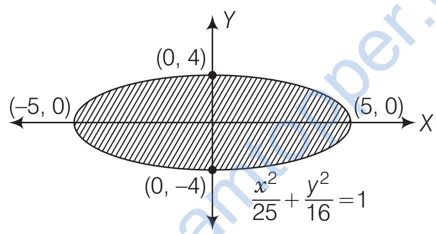
(a) 20π sq units (b) $20\pi^2$ sq units (c) $16\pi^2$ sq units (d) 25π sq units

Sol. *(a)* We have, $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$

Here, $a = \pm 5$ and $b = \pm 4$

and $\frac{y^2}{4^2} = 1 - \frac{x^2}{5^2}$

$\Rightarrow$ $y^2 = 16\left(1 - \frac{x^2}{25}\right)$



$\Rightarrow$ $y = \sqrt{\frac{16}{25}(25 - x^2)}$

$\Rightarrow$ $y = \frac{4}{5}\sqrt{(5^2 - x^2)}$

$\therefore$ Area enclosed by ellipse, $A = 2 \cdot \frac{4}{5}\int_{-5}^{5}\sqrt{5^2 - x^2}\, dx$

$$= 2 \cdot \frac{8}{5}\int_0^5 \sqrt{5^2 - x^2}\, dx$$

$$= 2 \cdot \frac{8}{5}\left[\frac{x}{2}\sqrt{5^2 - x^2} + \frac{5^2}{2}\sin^{-1}\frac{x}{5}\right]_0^5$$

$$= 2 \cdot \frac{8}{5}\left[\frac{5}{2}\sqrt{5^2 - 5^2} + \frac{5^2}{2}\sin^{-1}\frac{5}{5} - 0 - \frac{25}{2}\cdot 0\right]$$

$$= 2 \cdot \frac{8}{5}\left[\frac{25}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{16}{5}\cdot\frac{25\pi}{4}$$

$$= 20\pi \text{ sq units}$$

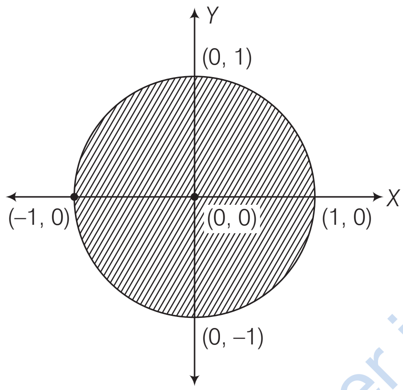
Q. 32 The area of the region bounded by the circle $x^2 + y^2 = 1$ is

(a) 2π sq units (b) π sq units

(c) 3π sq units (d) 4π sq units

Sol. *(b)* We have, $x^2 + y^2 = 1^2$ $[\because r = \pm 1]$

$\Rightarrow$ $y^2 = 1 - x^2 \Rightarrow y = \sqrt{1 - x^2}$



$\therefore$ Area enclosed by circle $= 2\int_{-1}^{1} \sqrt{1^2 - x^2}\, dx = 2 \cdot 2\int_0^1 \sqrt{1^2 - x^2}\, dx$

$$= 2 \cdot 2\left[\frac{x}{2}\sqrt{1^2 - x^2} + \frac{1^2}{2}\sin^{-1}\frac{x}{1}\right]_0^1$$

$$= 4\left[\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{\pi}{2} - 0 - \frac{1}{2} \cdot 0\right]$$

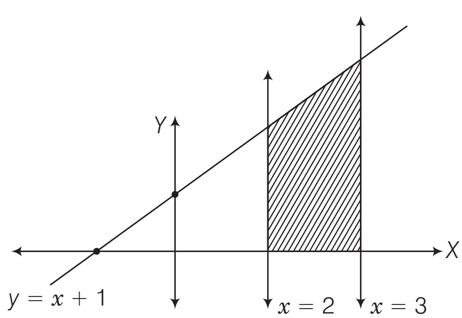
$$= 4 \cdot \frac{\pi}{4} = \pi \text{ sq units}$$

Q. 33 The area of the region bounded by the curve $y = x + 1$ and the lines $x = 2,\ x = 3$, is

(a) $\frac{7}{2}$ sq units (b) $\frac{9}{2}$ sq units (c) $\frac{11}{2}$ sq units (d) $\frac{13}{2}$ sq units

Sol. *(a)* Required area, $A = \int_2^3 (x + 1)dx = \left[\frac{x^2}{2} + x\right]_2^3$

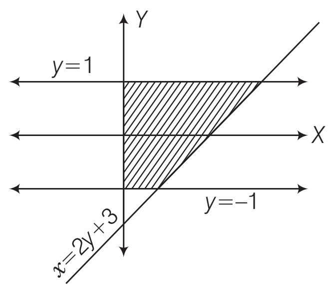
$$= \left[\frac{9}{2} + 3 - \frac{4}{2} - 2\right] = \left[\frac{5}{2} + 1\right] = \frac{7}{2} \text{ sq units}$$



Q. 34 The area of the region bounded by the curve $x = 2y + 3$ and the lines $y = 1,\ y = -1$ is

(a) 4 sq units
(b) $\frac{3}{2}$ sq units
(c) 6 sq units
(d) 8 sq units

Sol. *(c)* Required area, $A = \int_{-1}^{1} (2y + 3)dy$



$$= \left[\frac{2y^2}{2} + 3y\right]_{-1}^{1}$$

$$= [y^2 + 3y]_{-1}^{1}$$

$$= [1 + 3 - 1 + 3] = 6 \text{ sq units}$$